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# Statistically Almost $\lambda$-convergence of Sequences of Sets

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**Abstract.** The concept of Wijsman statistical convergence was defined by Nuray and Rhoades [9]. In this paper we define statistically almost $\lambda$- convergence for sequences for sets in sense of Wijsman and study some properties of this concept.

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## 1. Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. We shall define Wijsman statistically almost $\lambda$-convergence for sequences of sets and establish some basic results regarding this notions.

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [13]. The concept was formally introduced by Steinhaus [11] and Fast [2] and later was introduced by Schoenberg [10], and also independently by Buck [1]. A lot of developments have been made in this areas after the works of Šalát [12] and Fridy [4]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In the recent years, generalization of statistical

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convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers.

A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to $L$ if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k \le n : \left|x_k - L\right| \ge \varepsilon\right\}\right| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and $S$ denotes the set of all statistically convergent sequences.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V,\lambda)-$summable to number $L$ [5] if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V,\lambda)-$summability reduces to $(C,1)$-summability.

Mursaleen [8] defined $\lambda-$statistically convergent sequence as follows: A sequence $x = (x_k)$ is said to be $\lambda-$ *statistically convergent* to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|x_k - L\right| \ge \varepsilon\right\}\right| = 0.$$

Let $S_\lambda$ denotes the set of all $\lambda-$statistically convergent sequences. If $\lambda_n = n$, then $S_\lambda$ is the same as $S$.

The idea of almost convergence of sequences of points was introduced by Lorentz [6]. A sequence $x = (x_k)$ is said to be *almost convergent* to $L$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{k+m} = L \text{ uniformly in } m.$$

Maddox [7] and Freedman et al. [3] introduced the notion of strong almost convergence of sequences of points independently. A sequence $x = (x_k)$ is said to be *strongly almost convergent* to $L$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{k+m} - L| = 0 \text{ uniformly in } m.$$

Let $\ell_\infty, c, ac$ and $|ac|$ denote the sets of all bounded , convergent, almost convergent and strongly almost convergent sequences, respectively. It is known [7] that

$$c \subset ac \subset |ac| \subset \ell_\infty.$$

## 2. Wijsman Convergence and Preliminaries

Let $(X,\rho)$ be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x,A) = \inf_{y\in A} \rho(x,y).$$

**Definition 1** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence $(A_k)$ is Wijsman convergent to $A$ if $\lim_k d(x,A_k) = d(x,A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$.*

The concepts of Wijsman statistical convergence and boundedless for the sequence $(A_k)$ were given by Nuray and Rhoades [9] as follows:

**Definition 2.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence $(A_k)$ is Wijsman statistical convergent to $A$ if the sequence $(d(x,A_k))$ is statistically convergent to $d(x,A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$*

$$\lim_n \frac{1}{n}\left|\left\{k \le n : \left|d(x,A_k) - d(x,A)\right| \ge \varepsilon\right\}\right| = 0.$$

*In this case, we write $st - \lim_k A_k = A$ or $A_k \to A(WS)$.*

The sequence $(A_k)$ is bounded if $\sup_k d(x,A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by $L_\infty$.

**Definition 3** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman Cesaro summable to $A$ if $\{d(x,A_k)\}$ is Cesaro summable to $d(x,A)$, i.e. for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = d(x,A).$$

**Definition 4** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly Cesaro summable to $A$ if $\{d(x,A_k)\}$ is Cesaro summable to $d(x,A)$, i.e. for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| = 0.$$

**Definition 5** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman almost convergent to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_{k+m}) = d(x,A) \text{ uniformly in } m.$$

**Definition 6** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly almost convergent to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x,A_{k+m}) - d(x,A)| = 0 \text{ uniformly in } m.$$

*Let $L_\infty, C, AC$ and $|AC|$ denote the sets of all bounded , Wijsman convergent, Wijsman almost convergent and Wijsman strongly almost convergent sequences, respectively. It is known [9] that*

$$C \subset AC \subset |AC| \subset L_\infty.$$

**Definition 7** ([9]). *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman almost statistically convergent to $A$ if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |d(x,A_{k+m}) - d(x,A)| \geq \varepsilon\}| = 0 \text{ uniformly in } m.$$

## 3. Wijsman Statistically Almost $\lambda$-convergence

In this section, we will define Wijsman strongly $\lambda$-summable and Wijsman statistically almost $\lambda$-convergence of sequences of sets and will give the relations between Wijsman strongly $\lambda$-summable and Wisjman statistically almost $\lambda-$ convergence of sequences of sets.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

**Definition 8.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman $\lambda$-summable to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} d(x,A_k) = d(x,A).$$

If $\lambda_n = n$, then Wijsman $\lambda$-summable reduces to Wijsman Cesaro summable.

**Definition 9.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly $\lambda$-summable to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |d(x,A_k) - d(x,A)| = 0.$$

*In this case, we write $w_\lambda^W - \lim_k A_k = A$ or $A_k \to A \left(w_\lambda^W\right)$ and*

$$w_\lambda^W = \left\{ (A_k) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left|d(x,A_k) - d(x,A)\right| = 0 \right\}.$$

*If $\lambda_n = n$, then Wijsman strongly $\lambda$-summable reduces to Wijsman strongly Cesaro summable, i.e.*

$$w^W = \left\{ (A_k) : \lim_n \frac{1}{n} \sum_{k\in\mathbb{N}} \left|d(x,A_k) - d(x,A)\right| = 0 \right\}.$$

**Definition 10.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman almost $\lambda$-convergent to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} d(x,A_{k+m}) = d(x,A) \text{ uniformly in } m.$$

If $\lambda_n = n$, then Wijsman almost $\lambda$-convergent reduces to Wijsman almost convergent. In special case $m = 0$, then Wijsman almost $\lambda$-convergent reduces to Wijsman $\lambda$-summable.

**Definition 11.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly almost $\lambda$-convergent to $A$ if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |d(x,A_{k+m}) - d(x,A)| = 0 \text{ uniformly in } m.$$

*In this case, we write $\overline{w}_{\lambda}^{W} - \lim_k A_k = A$ or $A_k \to A\left(\overline{w}_{\lambda}^{W}\right)$.*

If $\lambda_n = n$, then Wijsman strongly almost $\lambda$-convergent reduces to Wijsman strongly almost convergent. In special case $m = 0$, then Wijsman strongly almost $\lambda$-convergent reduces to Wijsman strongly $\lambda$-summable.

**Definition 12.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence $(A_k)$ is Wijsman statistically $\lambda$-convergent to $A$ if the sequence $(d(x,A_k))$ is statistically $\lambda-$ convergent to $d(x,A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$*

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| d(x,A_k) - d(x,A) \right| \geq \varepsilon \right\} \right| = 0.$$

*In this case, we write $s_{\lambda}^{W} - \lim_k A_k = A$ or $A_k \to A\left(s_{\lambda}^{W}\right)$.*

If $\lambda_n = n$, then Wijsman statistical $\lambda$-convergent reduces to Wijsman statistical convergent.

**Definition 13.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman almost statistically $\lambda$-convergent to $A$ if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |d(x,A_{k+m}) - d(x,A)| \geq \varepsilon\}| = 0 \text{ uniformly in } m.$$

*In this case, we write $\bar{s}_{\lambda}^{W} - \lim_k A_k = A$ or $A_k \to A\left(\bar{s}_{\lambda}^{W}\right)$.*

If $\lambda_n = n$, then Wijsman almost statistically $\lambda$-convergent reduces to Wijsman almost statistically convergent. In special case $m = 0$, then Wijsman almost statistically $\lambda$-convergent reduces to Wijsman statistically $\lambda$-convergent.

**Example 1.** *Let $X = \mathbb{R}^2$ and the sequence $(A_k)$ is defined as follows:*

$$A_k = \begin{cases} \left\{ (x,y) : x^2 + (y-1)^2 = k^{-1} \right\}, & \text{if } n - \left[ |\lambda_n| \right] + 1 \leq k \leq n, k \text{ is square integer} \\ \{(0,0)\}, & \text{otherwise} \end{cases}.$$

*Then the sequence $(A_k)$ is Wijsman $\lambda-$statistical convergent to $A = \{(0,0)\}$ since*

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| d(x,A_k) - d(x,\{(0,0)\}) \right| \geq \varepsilon \right\} \right| = 0.$$

*But it is not Wijsman convergent.*

**Theorem 1.** *Let $(X,\rho)$ be a metric space and $A, A_k \subset X$ $(k \in \mathbb{N})$ be non-empty closed subsets of $X$. Then*

*a)* $\overline{w}_\lambda^W \subset \bar{s}_\lambda^W$ *and the inclusion is proper.*

*b) Let* $(A_k) \in \overline{L}_\infty$*, then* $\bar{s}_\lambda^W \subset \overline{w}_\lambda^W$.

*c)* $\bar{s}_\lambda^W \cap \overline{L}_\infty = \overline{w}_\lambda^W \cap \overline{L}_\infty$*, where*

$$\overline{L}_\infty = \{(A_k) : \sup_{k,m} |d(x, A_{k+m}) - d(x, A)| < \infty\}.$$

*Proof.*

a) Let $\varepsilon > 0$ and $(A_k) \in \overline{w}_\lambda^W$. Then for all $m \in \mathbb{N}$ we can write

$$\sum_{k \in I_n} \left| d(x, A_{k+m}) - d(x, A) \right| \geq \sum_{\substack{k \in I_n \\ \left| d(x, A_{k+m}) - d(x,A) \right| \geq \varepsilon}} \left| d(x, A_{k+m}) - d(x, A) \right|$$
$$\geq \varepsilon \left| \left\{ k \in I_n : \left| d(x, A_{k+m}) - d(x, A) \right| \geq \varepsilon \right\} \right|$$

which gives the result. To show that the inclusion is strict, we define the sequence $(A_k)$ as follows:

$$A_k = \begin{cases} \{k\}, & \text{if } n - \left[ \left| \lambda_n \right| \right] + 1 \leq k \leq n; \\ \{0\}, & \text{otherwise} \end{cases}$$

It is clear that $(A_k) \notin L_\infty$ and for $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| d(x, A_{k+m}) - d(x, \{0\}) \right| \geq \varepsilon \right\} \right| = \lim_n \frac{1}{\lambda_n} \left[ \left| \lambda_n \right| \right] = 0.$$

So $(A_k) \in \bar{s}_\lambda^W$, but

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| d(x, A_{k+m}) - d(x, \{0\}) \right| = \lim_n \frac{1}{\lambda_n} \frac{\left( \left[ \left| \lambda_n \right| \right] \left( \left[ \left| \lambda_n \right| \right] + 1 \right) \right)}{2} = \frac{1}{2} \neq 0.$$

Therefore $(A_k) \notin \overline{w}_\lambda^W$. This completes the proof of (a).

b) Suppose that $(A_k) \in \bar{s}_\lambda^W$ and $(A_k) \in L_\infty$, say $\left| d(x, A_{k+m}) - d(x, A) \right| \leq M$ for each $x \in X$ and for all $k, m \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left| d(x, A_{k+m}) - d(x, A) \right| = \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left| d(x, A_{k+m}) - d(x,A) \right| \geq \varepsilon}} \left| d(x, A_{k+m}) - d(x, A) \right|$$

$$+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \left|d(x,A_{k+m})-d(x,A)\right|<\varepsilon}}\left|d(x,A_{k+m})-d(x,A)\right|$$
$$\leq\frac{M}{\lambda_n}\left|\left\{k\in I_n:\left|d(x,A_{k+m})-d(x,A)\right|\geq\varepsilon\right\}\right|+\varepsilon$$

from which the result follows.

c) It follows from (a) and (b).

If we let $\lambda_n = n$ in Theorem 1, then we have the following corollary.

**Corollary 1.** *Let $(X,\rho)$ be a metric space and $A, A_k \subset X$ $(k\in\mathbb{N})$ be non-empty closed subsets of $X$. Then*

*a) $\overline{w}^W \subset \bar{s}^W$ and the inclusion is proper.*

*b) Let $(A_k)\in\overline{L}_\infty$, then $\bar{s}^W \subset \overline{w}^W$.*

*c) $\bar{s}^W \cap \overline{L}_\infty = \overline{w}^W \cap \overline{L}_\infty$.*

**Theorem 2.** $\bar{s}^W \subset \bar{s}^W_\lambda$ *if and only if* $\liminf\frac{\lambda_n}{n} > 0$.

*Proof.* Suppose that $\liminf\frac{\lambda_n}{n} > 0$. For given $\varepsilon > 0$, for all $m\in\mathbb{N}$, we have

$$\left\{k\leq n:\left|d(x,A_{km})-d(x,A)\right|\geq\varepsilon\right\}\supset\left\{k\in I_n:\left|d(x,A_{k+m})-d(x,A)\right|\geq\varepsilon\right\}.$$

Therefore

$$\frac{1}{n}\left|\left\{k\leq n:\left|d(x,A_{k+m})-d(x,A)\right|\geq\varepsilon\right\}\right|\geq\frac{1}{n}\left|\left\{k\in I_n:\left|d(x,A_{k+m})-d(x,A)\right|\geq\varepsilon\right\}\right|$$
$$\geq\frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|d(x,A_{k+m})-d(x,A)\right|\geq\varepsilon\right\}\right|.$$

Taking the limit as $n\to\infty$ and using $\liminf\frac{\lambda_n}{n} > 0$, we get the desired result.

Conversely, suppose that $\liminf_n\frac{\lambda_n}{n} = 0$. Then we can select a subsequence $(n(i))_{i=1}^{\infty}$ such that

$$\frac{\lambda_{n(i)}}{n(i)} < \frac{1}{i}.$$

We define a sequence $(A_k)$ as follows:

$$A_k=\begin{cases}\{1\}, & \text{if } n(i)-\left[\left|\lambda_{n(i)}\right|\right]+1\leq k\leq n(i), i=1,2,3,\ldots;\\ \{0\}, & \text{otherwise}\end{cases}.$$

Then $(A_k)$ is Wijsman-statistically convergent, so $(A_k)\in\bar{s}^W$. But $(A_k)\notin\overline{w}^W_\lambda$.

Therefore the Theorem 1 (b) implies that $(A_k)\notin\bar{s}^W_\lambda$. This completes the proof.

**Theorem 3.** *$\bar{s}^W_\lambda \subset \bar{s}^W$ if $\liminf \frac{\lambda_n}{n} = 1$.*

*Proof.* Since $\lim_n \frac{\lambda_n}{n} = 1$, then for $\varepsilon > 0$, for all $m \in \mathbb{N}$, we observe that

$$\begin{aligned}\frac{1}{n}|\{k \le n : |d(x,A_{k+m}) - d(x,A)| \ge \varepsilon\}| \le & \frac{1}{n}|\{k \le n - \lambda_n : |d(x,A_{k+m}) - d(x,A)| \ge \varepsilon\}| \\ & + \frac{1}{n}|\{k \in I_n : |d(x,A_{k+m}) - d(x,A)| \ge \varepsilon\}| \\ \le & \frac{n-\lambda_n}{n} + \frac{1}{n}|\{k \in I_n : |d(x,A_{k+m} - d(x,A)| \ge \varepsilon\}| \\ = & \frac{n-\lambda_n}{n} + \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : |d(x,A_{k+m} - d(x,A)| \ge \varepsilon\}|.\end{aligned}$$

This implies that $(A_k)$ Wijsman almost statistically convergent, if $(A_k)$ is Wijsman almost statistically $\lambda$-convergent. Thus $\bar{s}^W_\lambda \subset \bar{s}^W$.

**Remark 1.** *Since $\lim_n \frac{\lambda_n}{n} = 1$, implies that $\liminf_n \frac{\lambda_n}{n} > 0$, then from Theorem 2, we have $\bar{s}^W \subset \bar{s}^W_\lambda$. Hence $\bar{s}^W_\lambda = \bar{s}^W$.*

**Definition 14** ([9])**.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly p-almost convergent to A if for each $x \in X$, $p \in (0,\infty)$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x,A_{k+m}) - d(x,A)|^p = 0 \textit{ uniformly in } m.$$

We introduced the following definition.

**Definition 15.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is Wijsman strongly almost $\lambda_p$-summable to A if for each $x \in X$, $p \in (0,\infty)$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |d(x,A_{k+m}) - d(x,A)|^p = 0 \textit{ uniformly in } m.$$

If $\lambda_n = n$, Wijsman strongly almost $\lambda_p$-summable reduces to Wijsman strongly almost $p$-Cesaro summable defined as follows:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x,A_{k+m}) - d(x,A)|^p = 0 \text{ uniformly in } m.$$

**Theorem 4.** *Let $(X,\rho)$ be a metric space and $A, A_k \subset X$ $(k \in \mathbb{N})$ be non-empty closed subsets of $X$. If $(A_k)$ is Wijsman strongly almost $\lambda_p$-summable to A, then it is Wijsman statistically almost $\lambda$-convergent to A.*

*Proof.* For any $(A_k,)$ fix an $\varepsilon > 0$ and for all $m \in \mathbb{N}$, we have

$$\sum_{k\in I_n} |d(x,A_{k+m}) - d(x,A)|^p \geq \varepsilon |\{k \in I_n : |d(x,A_{k+m}) - d(x,A)|^p \geq \varepsilon\}|,$$

and it follows that if $(A_k)$ is Wijsman strongly almost $\lambda_p$-summable to $A$, then it is Wijsman statistically almost $\lambda$-convergent to $A$.

**Theorem 5.** *Let $(X,\rho)$ be a metric space and $A, A_k \subset X$ $(k \in \mathbb{N})$ be non-empty closed subsets of $X$. If $(A_k)$ is bounded and Wijsman statistically almost $\lambda$-convergent to $A$, then it is Wijsman strongly almost $\lambda_p$-summable to $A$ and hence $(A_k)$ is Wijsman strongly almost p-Cesaro summable to $A$.*

*Proof.* Let $(A_k)$ is bounded and Wijsman statistically almost $\lambda$-convergent to $A$. Since $(A_k)$ is bounded,then there exists $M > 0$ such that $|d(x,A_{k+m}) - d(x,A)| \leq M$ for all $k,m \in \mathbb{N}$. Let $\varepsilon > 0$ be given and for all $m \in \mathbb{N}$, we select $n_0 = n_0(\varepsilon)$ such that

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : |d(x,A_{k+m}) - d(x,A)|^p \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right| < \frac{\varepsilon}{2M^p} \text{ for all } n > n_0.$$

We put

$$K(\varepsilon) = \left\{k \in I_n : |d(x,A_{k+m}) - d(x,A)| \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}.$$

For all $m \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} |d(x,A_{k+m}) - d(x,A)|^p =& \frac{1}{\lambda_n}\sum_{k\in I_n, k\in K(\varepsilon)} |d(x,A_{k+m}) - d(x,A)|^p \\ &+ \frac{1}{\lambda_n}\sum_{k\in I_n, k\notin K(\varepsilon)} |d(x,A_{k+m}) - d(x,A)|^p = T_1 + T_2\end{aligned}$$

where

$$T_1 = \frac{1}{\lambda_n}\sum_{k\in I_n, k\in K(\varepsilon)} |d(x,A_{k+m}) - d(x,A)|^p$$

and

$$T_2 = \frac{1}{\lambda_n}\sum_{k\in I_n, k\notin K(\varepsilon)} |d(x,A_{k+m}) - d(x,A)|^p.$$

If $k \in K(\varepsilon)$, then $T_2 < \frac{\varepsilon}{2}$.

If $k \notin K(\varepsilon)$, then

$$T_1 \leq (\sup_{k,m} |d(x,A_{k+m}) - d(x,A)|^p)\frac{1}{\lambda_n}|K(\varepsilon)| \leq \frac{1}{\lambda_n}\frac{\lambda_n \varepsilon}{2M^p}M^p = \frac{\varepsilon}{2}.$$

Therefore for all $m \in \mathbb{N}$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} |d(x,A_{k+m}) - d(x,A)|^p < \varepsilon.$$

Hence $(A_k)$ is Wijsman strongly almost $\lambda_p$-summable to $A$.

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