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On Some Properties of Liouville Numbers in the non-Archimedean Case

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Abstract. We study Liouville numbers in the non-archimedean case. We give the analogues of the Erdös theorem in the non-archimedean case, both in the p-adic numbers field $\mathbb{Q}_p$ and the functions field $K\langle x\rangle$.

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1. Introduction

The classical Liouville's theorem states that if $\alpha \in \mathbb{R}$ is an algebraic number of degree $n \geq 2$, then there exists a positive constant $C(\alpha)$ depending only on α such that

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{C(\alpha)}{b^n}$$

for all $a, b \in \mathbb{Z}^+$. The existence of transcendental numbers has been usually shown using the Liouville's theorem. For instance, the transcendence of the number $\xi = \sum_{n=1}^{\infty} 10^{-n!}$ can be easily proved from the Liouville's theorem [see 3]. A real number $\xi \in \mathbb{R}$ is called a (real) *Liouville number* if for every positive integer n, there exist integer a and $b(> 1)$ such that

$$0 < \left|\xi - \frac{a}{b}\right| < \frac{1}{b^n}.$$

Real Liouville numbers have many interesting properties and investigated by many authors [see 2, 7, 9, 10, 12, 14]. We note that Liouville numbers are real numbers that can be rapidly approximated by algebraic numbers with degree one. A general theory of approximation by algebraic numbers is given in [5]. Here we mainly focus on the Erdös theorem:

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Theorem 1. *[P. Erdös, [8]] Let $a_1 < a_2 < a_3 < \ldots$ be an infinite sequence of integers satisfying*

$$\lim_{n\to\infty} \sup a_n^{\frac{1}{t^n}} = \infty$$

for every $t > 0$, and

$$a_n > n^{1+\varepsilon}$$

for fixed $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Then

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$$

is a Liouville number.

It is well known that real numbers field $\mathbb{R}$ is archimedean. There are interesting non-archimedean fields as the p-adic numbers field $\mathbb{Q}_p$ and the functions field.

Let p be a fixed prime number. By $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

In the present work we investigate some properties of Liouville numbers in non-archimedean case. Mainly, we give the analogues of the Erdös theorem in the non-archimedean case, both in p-adic numbers field $\mathbb{Q}_p$ and the functions field $K\langle x \rangle$.

Although the classical Liouville numbers are real numbers that can be rapidly approximated by rational numbers, the p-adic Liouville numbers are those numbers that can be rapidly approximated by positive integers in the p-adic norm. The p-adic Liouville numbers are defined as follows:

Definition 1 ([6, 21])**.** *Let α be a p-adic integer. If*

$$\lim_{n\to\infty} \inf \sqrt[n]{|n-\alpha|_p} = 0,$$

then the number α is called p-adic Liouville number.

Example 1. *Let consider the series $\alpha = \sum_{n=0}^{\infty} p^{n!}$. It is easy to see that the sum is a p-adic Liouville number.*

The definition above is first introduced by D. Clark [6] and it is better adapted to differential equations. In fact, consider the differential equation

$$xf'(x) - \lambda f(x) = \frac{1}{1-x}$$

on a neighborhood D of 0 in $\mathbb{Z}_p$ where $\lambda \in \mathbb{Z}_p \backslash \{0, 1, 2, \ldots\}$. This equation has an unique formal solution, namely, $f(x) = \sum_{n=1}^{\infty} \frac{1}{n-\lambda} x^n$. It is clear that this solution divergent if only if λ is a p-adic Liouville number (for details see [20]).

It is well known that the set $\mathscr{L}$ of p-adic Liouville numbers have the following basic properties:

1. $\mathscr{L} \subset \mathbb{Z}_p$

2. $\mathscr{L}$ has measure 0 for the real Haar measure on $\mathbb{Z}_p$

3. If $\alpha \in \mathscr{L}$ and $n, m \in \mathbb{Z}$ with $m > 0$, the $n + m\alpha \in \mathscr{L}$

4. $\mathscr{L} \neq -\mathscr{L}$ and $\mathscr{L} \cap -\mathscr{L} \neq \emptyset$

5. $\mathscr{L}$ forms a dense subset of $\mathbb{Z}_p$

6. Every $\alpha \in \mathscr{L}$ is transcendental over $\mathbb{Q}$.

In general case the p-adic transcendental numbers have been studied by K. Mahler [15], W. W. Adams [1], X. X. Long [13], K. Nishioka [19] and others. As a special case the p-adic Liouville numbers have been studied in [4, 11, 17, 18] and others.

2. The Erdös Theorem in the p-adic Numbers Field $\mathbb{Q}_p$.

We prove the following result as an analogue of the Erdös theorem in the p-adic numbers field $\mathbb{Q}_p$.

Theorem 2. *Let (a_n) be a sequence of p-adic integers such that*

$$v_p(a_n) < v_p(a_{n+1}) \tag{1}$$

for every n, and

$$v_p(a_{n+1}) \geq n^{1+\varepsilon} \tag{2}$$

for fixed $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Then

$$\alpha = \sum_{n=1}^{\infty} a_n$$

is a p-adic Liouville number.

Proof. First we show that the series $\sum_{n=1}^{\infty} a_n$ is convergent. It follows from the condition (2) that

$$v_p(a_{n+1}) \geq n^{1+\varepsilon}$$

for fixed $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Then, we have

$$\left|a_{n+1}\right|_p = p^{-v_p(a_{n+1})} \leq p^{-n^{1+\varepsilon}} \to 0, (n \to \infty).$$

Hence, $\lim\limits_{n\to 0} a_n = 0$, so the series $\sum_{n=1}^{\infty} a_n$ is convergent. By the property $\left|\sum_{n=1}^{\infty} a_n\right|_p \leq \max\limits_{n\in\mathbb{N}} \left|a_n\right|_p$, we obtain that $\alpha = \sum_{n=1}^{\infty} a_n \in \mathbb{Z}_p$. Also, by the condition (1) $\alpha \in \mathbb{Z}_p \backslash \mathbb{Z}$.
Let $\varepsilon > 0$ be an arbitrary real number. Then,

$$0 < \left|\alpha - S_n\right|_p^{\frac{1}{n}} = \left|\sum_{i=1}^{\infty} a_{n+i}\right|_p^{\frac{1}{n}} = \left[\max\left\{\left|a_{n+1}\right|_p, \left|a_{n+2}\right|_p, \ldots\right\}\right]^{\frac{1}{n}}$$

where $S_n = \sum_{i=1}^{n} a_i$. Hence, from the condition (1) we obtain

$$0 < \left|\alpha - S_n\right|_p^{\frac{1}{n}} = \left|a_{n+1}\right|_p^{\frac{1}{n}}.$$

Thus,

$$0 < \left|\alpha - S_n\right|_p^{\frac{1}{n}} = \left[p^{-\nu_p\left(a_{n+1}\right)}\right]^{\frac{1}{n}}$$

and by the inequality (2) we get

$$0 < \left|\alpha - S_n\right|_p^{\frac{1}{n}} = \left[p^{-\nu_p\left(a_{n+1}\right)}\right]^{\frac{1}{n}} \leq p^{-\frac{n^{1+\varepsilon}}{n}} = p^{-n^{\varepsilon}} \left(n \geq n_0\right).$$

Thus we have

$$\left|\alpha - S_n\right|_p^{\frac{1}{n}} \to 0 (n \to \infty).$$

Since $S_n \in \mathbb{Z}_p$ for every $n \in \mathbb{N}$, and the set of natural numbers $\mathbb{N}$ is dense in $\mathbb{Z}_p$, there exists a sequence b_n from $\mathbb{N}$ such that

$$\left|S_n - b_n\right|_p < \left|\alpha - S_n\right|_p$$

for every $n \in \mathbb{N}$. By the ultrametric inequality we can write

$$0 < \left|\alpha - b_n\right|_p \leq \max\left\{\left|\alpha - S_n\right|_p, \left|S_n - b_n\right|_p\right\} = \left|\alpha - S_n\right|_p$$

for every $n \in \mathbb{N}$. Hence, we can obtain a positive integer sequence b_n such that

$$0 < \left|\alpha - b_n\right|_p^{\frac{1}{n}} \leq \left|\alpha - S_n\right|_p^{\frac{1}{n}} = p^{-n^{\varepsilon}} \to 0 \left(n \to \infty\right).$$

So, the theorem is proved.

Remark 1. *Since $\nu_p\left(a_n\right) \in \mathbb{N}$ for all $a_n \in \mathbb{Z}_p$, in Theorem 2, the condition (2) can be replaced by the condition*

$$\nu_p\left(a_{n+1}\right) \geq n^2.$$

In similar way, we can give the following result.

Corollary 1. *Let $\left(a_n\right)$ be a sequence of positive integers such that*

$$\nu_p\left(a_n\right) < \nu_p\left(a_{n+1}\right) \tag{3}$$

for every n, and

$$\nu_p\left(a_{n+1}\right) \geq n^2 \tag{4}$$

for $n > n_0$. Then

$$\alpha = \sum_{n=1}^{\infty} a_n$$

is a p-adic Liouville number.

Proof. By the relations (3) and (4) we have, $\lim_{n\to\infty} a_n = 0$, and so, the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\alpha \in \mathbb{Z}_p$. Similarly, we can obtain that

$$0 < \left|\alpha - S_n\right|_p^{\frac{1}{n}} = \left[p^{-\nu_p\left(a_{n+1}\right)}\right]^{\frac{1}{n}} \leq p^{-\frac{n^2}{n}} = p^{-n} \to 0\,(n \to \infty)$$

where $S_n = \sum_{i=1}^{n} a_i$. Also, since $S_n \in \mathbb{N}$ for all $n \in \mathbb{N}$, the number α is a p-adic Liouville number.

3. The Erdös Theorem in the Functions Field $K\langle x\rangle$

Let K be an arbitrary field, x an indeterminate, $K[x]$ the ring of all polynomials in x with coefficients in K, $K(x)$ the field of all rational functions in x with coefficients in K, and $K\langle x\rangle$ the field of all formal series

$$z = a_k x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \ldots$$

in x where the coefficients $a_k, a_{k-1}, a_{k-2}, \ldots$ are in K. Thus $K(x)$ is the quotient field of $K[x]$ and a subfield of $K\langle x\rangle$.

A valuation $|z|$ in $K\langle x\rangle$ is now defined by putting $|0| = 0$; but $|z| = e^k$ if

$$z = a_k x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \ldots$$

and $a_k \neq 0$.

If z lies in $K[x]$, then $\log|z| = \deg z$.

It is clear that this norm is a non-archimedean and so, $K\langle x\rangle$ is a non-archimedean field with this norm.

The analogue of Liouville's theorem states that if $\alpha \in K\langle x\rangle$ is an algebraic number of degree $n \geq 2$ over $K(x)$, then there exists a positive constant $C(\alpha)$ depending only on α such that

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{C(\alpha)}{b^n}$$

for all $a, b \in K[x]\,(b \neq 0)$ [see 16]. Some investigations involve the Liouville numbers in the functions field was done in [11]. Now we recall the definition of Liouville numbers in this field.

Definition 2. *An element $\xi \in K\langle x\rangle$ is called a Liouville number if for every $\omega \in \mathbb{R}^+$, there exist integer $a, b \in K[x]\setminus\{0\}$ with $|b| > 1$ such that*

$$0 < \left|\xi - \frac{a}{b}\right| < \frac{1}{b^{\omega}}.$$

We can give an analogue of the Erdös theorem in the functions field as follows

Theorem 3. *Let* (z_n) *be a sequence of formal series in* $K\langle x\rangle$ *such that*

$$\deg(z_{n+1}) < \deg(z_n) < 0 \tag{5}$$

for every n *and*

$$\deg(z_{n+1}) \le -n^{1+\varepsilon} \tag{6}$$

for fixed $\varepsilon > 0$ *and* $n > n_0(\varepsilon)$*. Then,*

$$\alpha = \sum_{n=1}^{\infty} z_n$$

is a Liouville number in $K\langle x\rangle$*.*

Proof. First we show that the series $\sum_{n=1}^{\infty} z_n$ is convergent. It follows from the condition (6) that

$$\left|z_{n+1}\right| = e^{\deg(z_{n+1})} \le e^{-n^{1+\varepsilon}}$$

for fixed $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Then, we get

$$\lim_{n\to\infty} z_n = 0.$$

Thus, the series $\sum_{n=1}^{\infty} z_n$ is convergent.
Let $\varepsilon > 0$ be an arbitrary real number. Then,

$$0 < \left|\alpha - S_n\right|^{\frac{1}{n}} = \left|\sum_{i=1}^{\infty} z_{n+i}\right|^{\frac{1}{n}} = \left[\max\left\{\left|z_{n+1}\right|, \left|a_{n+2}\right|, \ldots\right\}\right]^{\frac{1}{n}}$$

where $S_n = \sum_{i=1}^{n} a_i$. Hence, from the condition (5) we obtain

$$0 < \left|\alpha - S_n\right|^{\frac{1}{n}} = \left|z_{n+1}\right|^{\frac{1}{n}}.$$

Thus,

$$0 < \left|\alpha - S_n\right|^{\frac{1}{n}} = \left[e^{\deg(z_{n+1})}\right]^{\frac{1}{n}}$$

and by the inequality (6) we get

$$0 < \left|\alpha - S_n\right|^{\frac{1}{n}} = \left[e^{\deg(z_{n+1})}\right]^{\frac{1}{n}} \le e^{-\frac{n^{1+\varepsilon}}{n}} = e^{-n^{\varepsilon}}$$

for $n > n_0(\varepsilon)$. Thus, we have

$$\left|\alpha - S_n\right|^{\frac{1}{n}} \to 0 (n \to \infty).$$

Since $S_n \in K\langle x\rangle$ for every $n \in \mathbb{N}$, and the rational polynomials field set $K(x)$ is dense in $K\langle x\rangle$ with respect the non-archimedean norm, there exists a sequence $\frac{a_n}{b_n} \in K(x)$ $(a_n, b_n \in K[x])$ such that

$$\left|S_n - \frac{a_n}{b_n}\right| < \left|\alpha - S_n\right|$$

for every $n \in \mathbb{N}$. By the ultrametric inequality we can write

$$\left|\alpha - \frac{a_n}{b_n}\right| \leq \max\left\{\left|\alpha - S_n\right|, \left|S_n - b_n\right|\right\} = \left|\alpha - S_n\right|$$

for every $n \in \mathbb{N}$. Hence, we can obtain $\frac{a_n}{b_n} \in K(x)$ such that

$$\left|\alpha - \frac{a_n}{b_n}\right|^{\frac{1}{n}} \leq \left|\alpha - S_n\right|^{\frac{1}{n}} = e^{-n^{\varepsilon}} \to 0\,(n \to \infty).$$

So, $\alpha \in K\langle x\rangle$ is a Liouville number.

Example 2. *Consider the element* $\xi = \sum_{n=1}^{\infty} x^{-n!}$ *in* $K\langle x\rangle$. *Let* $z_n = x^{-n!}$. *It is clear that* z_n *satisfy the conditions* (5) *and* (6). *By Theorem 3,* ξ *is a Liouville number in the functions field.*

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