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On solving Partial Differential Equations of Fractional Order by Using the Variational Iteration Method and Multivariate Padé Approximations

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Abstract. In this article, multivariate Padé approximation and variational iteration method proposed by He is adopted for solving linear and nonlinear fractional partial differential equations. The fractional derivatives are described in the Caputo sense. Numerical illustrations that include nonlinear time-fractional hyperbolic equation and linear fractional Klein-Gordon equation are investigated to show efficiency of multivariate Padé approximation. Comparison of the results obtained by the variational iteration method with those obtained by multivariate Padé approximation reveals that the present methods are very effective and convenient.

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1. Introduction

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance, physical and biological processes and systems [4, 10, 11, 18, 19, 28–30, 43–45]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Recently, the Adomian decomposition method [2, 3, 31, 32, 34–36, 46, 48, 49] and variational iteration [14, 16, 17, 20–24, 33, 37, 40] method have been used for solving a wide range of problems.

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Many approximation and numerical techniques have been used to solve fractional differential equations. The variational iteration method is relatively new approach to provide an analytical approximation to linear and nonlinear problems and it is particularly valuable as tool for scientists and applied mathematicians, because it provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to fractional differential equations. In the literature, the unvariate Padé approximation has been used to obtain approximate solutions of fractional order [38, 39]. So the objective of the present paper is to show the application of the multivariate Padé approximation to provide approximate solutions for initial value problems of linear and nonlinear partial differential equations of fractional order and to make comparison with variational iteration method.

2. Basic Definitions

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.*

Definition 2. *The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as*

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \ , \quad \alpha > 0, \qquad x > 0 \\ J^0 f(x) &= f(x). \end{aligned} \tag{1}$$

Properties of the operator J^α can be found in [28, 43, 44]. For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\alpha J^\beta f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity will be introduced [4].

Definition 3. *The fractional derivative off (x) in the Caputo sense is defined as*

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)dt, \tag{2}$$

for $m-1<\alpha\leq m$, $m\in\mathbb{N}$, $x>0$, $f\in C_{-1}^{m}$.

Definition 4. *For m to be the smallest integer that exceeds* α, *the Caputo time-fractional derivative operator of order* $\alpha>0$ *is defined as*

$$D_t^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}d\tau & m-1<\alpha<m\\ \frac{\partial^m u(x,t)}{\partial t^m} & \alpha=m\in\mathbb{N}\end{cases}\tag{3}$$

Lemma 1. *If* $m-1<\alpha\leq m$, $m\in\mathbb{N}$, *and* $f\in C_{\mu}^{m}$, $\mu\geq-1$ *then*

$$D_*^{\alpha}J^{\alpha}f(x)=f(x)\tag{4}$$

$$J^{\alpha}D_*^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^k(0^+)\frac{x^k}{k!},\quad x>0\tag{5}$$

3. Multivariate Padé Approximation

The principles and theory of the multivariate Padé approximation and its applicability for various of differential equations are given in [1, 5–9, 12, 13, 47, 50, 51]. Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y)=\sum_{i,j=0}^{\infty}c_{ij}x^iy^j\tag{6}$$

around the origin. We know that a solution of unvariate Padé approximation problem for

$$f(x)=\sum_{i=0}^{\infty}c_ix^i\tag{7}$$

is given by

$$p(x)=\begin{vmatrix}\sum_{i=0}^{m}c_ix^i & x\sum_{i=0}^{m-1}c_ix^i & \cdots & x^n\sum_{i=0}^{m-n}c_ix^i\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m\end{vmatrix}\tag{8}$$

and

$$q(x)=\begin{vmatrix}1 & x & \cdots & x^n\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m\end{vmatrix}\tag{9}$$

Let us now multiply jth row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j=2,\ldots,n+1)$ and afterwards divide jth column in $p(x)$ and $q(x)$ by $x^{j-1}(j=2,\ldots,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{10}$$

if $(D = \det D_{m,n} \neq 0)$.

This quotient of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced kth partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in (x,y). Here a bivariate term $c_{ij}x^i y^j$ is said to be of degree $i+j$. If we define

$$p(x,y) = \begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i y^j & \sum_{i+j=0}^{m-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{11}$$

and

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{12}$$

Then it is easy to see that $p(x,y)$ and $q(x,y)$ are of the form

$$\begin{aligned} p(x,y) &= \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j \\ q(x,y) &= \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j \end{aligned} \tag{13}$$

We know that $p(x,y)$ and $q(x,y)$ are called Padé equations [8]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{14}$$

4. Variational Iteration Method

The principles of the variational iteration method are given in [14–26]. Ji-Huan He applied the variational iteration method to obtain analytical solution for the fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,t), \quad u(a) = b, 1 < \alpha < 2. \tag{15}$$

The application of the variational iteration method has been extended in [42] to solve the time fractional differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} u(x,t) = R[x]\,u(x,t) + q(x,t), \quad t > 0, x \in R, \tag{16}$$

where $R[x]$ is a differential operator in x, subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) =& f(x), \quad 0 < \alpha \leq 1, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, \quad t > 0, \end{aligned} \tag{17}$$

and

$$\begin{aligned} u(x,0) =& f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), 1 < \alpha \leq 2, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, t > 0, \end{aligned} \tag{18}$$

where $f(x)$, $g(x)$ and $q(x,t)$ all are continuous functions and α, $m-1 < \alpha \leq m$ is a parameter describing the order of the time-fractional derivative in the Caputo sense. According to the variational iteration method, the correction functional for Eq. (16) has been constructed in [42] as:

$$\begin{aligned} u_{k+1}(x,t) =& u_k(x,t) + J_t^\beta \left[\lambda \left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,t) - R[x]\,\tilde{u}_k(x,t) - q(x,t) \right) \right] \\ =& u_k(x,t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \lambda(\tau) \left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,\tau) - R[x]\,\tilde{u}_k(x,\tau) - q(x,\tau) \right) d\tau. \end{aligned} \tag{19}$$

where J_t^β is the Riemann-Liouville fractional integral operator of order $\beta = \alpha - floor(\alpha)$ that is $\beta = \alpha + 1 - m$, with respect to the variable t and λ is a general Lagrange multiplier, which can be identified optimally via variational theory [27]. To identify approximately Lagrange multiplier, some approximation has been made in [42]. The correction functional (19) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\tau) \left(\frac{\partial^m}{\partial \tau^m} u_k(x,\tau) - R[x]\,\tilde{u}_k(x,\tau) - q(x,\tau) \right) \right] d\tau. \tag{20}$$

Here restricted variations are applied to the nonlinear term $R[x]u$, in this case the multiplier can be easily determined. Making the above functional stationary, noticing that $\delta\tilde{u}_k = 0$,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \left[\lambda(\tau) \left(\frac{\partial^m}{\partial \tau^m} u_k(x,\tau) - q(x,\tau) \right) \right] d\tau, \tag{21}$$

yields the following multipliers

$$\lambda = -1, \text{ for } m = 1 \tag{22}$$

$$\lambda = \tau - t, \text{ for } m = 2 \tag{23}$$

Therefore, for $m = 1$ $(0 < \alpha \leq 1)$, $\lambda = -1$ is substituted into the functional (19) to obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,t) - R[x]\, u_k(x,t) - q(x,t) \right]. \tag{24}$$

For $m = 2$, $(1 < \alpha \leq 2)$, $\lambda = \tau - t$ is substituted into the functional (19) to get

$$\begin{aligned} u_{k+1}(x,t) =& u_k(x,t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2}(\tau - t) \\ & \times \left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,\tau) - R[x]\, u_k(x,\tau) - q(x,\tau) \right) d\tau \\ =& u_k(x,t) - \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}(\tau - t) \\ & \times \left(\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,\tau) - R[x]\, u_k(x,\tau) - q(x,\tau) \right) d\tau. \end{aligned} \tag{25}$$

So, the following iteration formula is obtained in [42]

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1) J_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x,t) - R[x]\, u_k(x,t) - q(x,t) \right]. \tag{26}$$

The initial approximation (trial function) u_0 can be freely chosen if it satisfies the initial and boundary conditions of the problem. However the success of the method depends on the proper selection of the initial approximation u_0. Finally, the solution $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ is approximated by the Nth term $u_N(x,t)$.

4.1. Nonlinear Time-fractional Partial Differential Equation

The following nonlinear time-fractional partial differential equation is considered in [41]

$$D_{*t}^\alpha u(x,t) = f(u, u_x, u_{xx}) + g(x,t), \quad m - 1 < \alpha \leq m, \tag{27}$$

where $D_{*t}^{\alpha}u(x,t)=\frac{\partial^\alpha}{\partial t^\alpha}$, is the Caputo fractional derivative of order α, $m\in\mathbb{N}$, f is a nonlinear function and $g$is the source function. The initial and boundary conditions associated with (27) are of the from

$$\begin{aligned} u(x,0)=&h(x), \quad 0<\alpha\leq 1,\\ u(x,t)\to&0 \text{ as } |x|\to\infty, \quad t>0, \end{aligned} \tag{28}$$

and

$$\begin{aligned} u(x,0)=&h(x), \quad \frac{\partial u(x,0)}{\partial t}=k(x), 1<\alpha\leq 2,\\ u(x,t)\to&0 \text{ as } |x|\to\infty, \quad t>0. \end{aligned} \tag{29}$$

The correction functional for Eq. (27) has been approximately expressed in [41] as follows:

$$u_{k+1}(x,t)=u_k(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi)-f(\tilde{u}_k,(\tilde{u}_k)_x,(\tilde{u}_k)_{xx})-g(x,\xi)\right)d\xi, \tag{30}$$

where λ is a general Lagrange multiplier [27], which can be identified optimally via variational theory [14, 22–24, 27], here $\tilde{u}_k,(\tilde{u}_k)_x,(\tilde{u}_k)_{xx}$ are considered as restricted variations, i.e.,$\delta\tilde{u}_n=0$. Making the above functional stationary,

$$\delta u_{k+1}(x,t)=\delta u_k(x,t)+\delta\int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi)-g(x,\xi)\right)d\xi, \tag{31}$$

yields the following Lagrange multipliers

$$\begin{gathered} \lambda=-1 \text{ for } m=1,\\ \lambda=\xi-t, \text{ for } m=2. \end{gathered}$$

Therefore, for $m=1$, the following iteration formula has been obtained in [41]:

$$u_{k+1}(x,t)=u_k(x,t)+\int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi)-f(u_k,(u_k)_x,(u_k)_{xx})-g(x,\xi)\right)d\xi. \tag{32}$$

In this case, it can be begun with the initial approximation

$$u_0(x,t)=h(x). \tag{33}$$

For $m=2$, the following iteration formula is obtained [41]:

$$u_{k+1}(x,t)=u_k(x,t)+\int_0^t (\xi-t)\left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi)-f(u_k,(u_k)_x,(u_k)_{xx})-g(x,\xi)\right)d\xi. \tag{34}$$

In this case, it can be begun with the initial approximation

$$u_0(x,t)=h(x)+tk(x). \tag{35}$$

The correction functional (30) will give several approximations, and therefore the exact solution is obtained as

$$u(x,t)=\lim_{k\to\infty}u_k(x,t) \tag{36}$$

5. Numerical Experiments

In this section two methods, VIM and MPA, shall be illustrated by two examples. All the numerical results are calculated by using the software Maple12.

Example 1. *Consider the one-dimensional linear inhomogeneous fractional Klein-Gordon equation [42]*

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, x \in R, 1 < \alpha \leq 2, \tag{37}$$

subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0. \tag{38}$$

According to the variational iteration method and to Eq. (26)*, the iteration formula for Eq.* (37) *is given by*

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1)J_t^\alpha \left[\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u - 6x^3 t - (x^3 - 6x)t^3 \right]. \tag{39}$$

By the above variational iteration formula, if it is begun with $u_0 = 0$*, so following approximations has been obtained in [42]*

$$u_1(x,t) = (\alpha - 1)\left[6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \right],$$

$$\begin{aligned} u_2(x,t) =& 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 6(x^3 - 6x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} \\ & - (\alpha-1)^2 \left[6(x^3 - 6x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 12x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right] + \ldots. \end{aligned} \tag{40}$$

the variational iteration method gives the solution for the classical Klein-Gordon Eq. (37) *(when* $\alpha = 2$ *) which is given by*

$$u(x,t) = x^3 t^3 + (x^3 - 6x)\frac{6t^5}{\Gamma(6)} + 36x\frac{t^5}{\Gamma(6)} - 36x\frac{6t^7}{\Gamma(8)} - 6x^3\frac{t^5}{\Gamma(6)} - (x^3 - 6x)\frac{6t^7}{\Gamma(8)} + \ldots \tag{41}$$

$$= x^3 t^3 - 0.001190476190 x^3 t^7 - 0.01428571428 x t^7 \tag{42}$$

the exact solution of (37)*, for the special case* $\alpha = 2$ *is given in [42]*

$$u(x,t) = x^3 t^3 \tag{43}$$

Now let us calculate the approximate solution of Eq. (42) *for* $m = 8$ *and* $n = 2$ *by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.* (42) *for* $m = 8$ *and* $n = 2$*, we use Eqs.* (11) *and* (12)*. By using Eqs.* (11) *and* (12) *we obtain,*

$$p(x,t) = \begin{vmatrix} x^3t^3 - 0.01428571428xt^7 & x^3t^3 & x^3t^3 \\ 0 & -0.01428571428xt^7 & 0 \\ -0.001190476190x^3t^7 & 0 & -0.01428571428xt^7 \end{vmatrix} \tag{44}$$

$$= -0.00001700680271(x^4 - 12.00000000x^2 + 0.1714285714t^4)x^3t^{17} \tag{45}$$

and

$$q(x,t) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -0.01428571428xt^7 & 0 \\ -0.001190476190x^3t^7 & 0 & -0.01428571428xt^7 \end{vmatrix} \tag{46}$$

$$= -0.00001700680271(-12.00000000 + x^2)x^2t^{14} \tag{47}$$

So the Multivariate Padé approximation of order $(8,2)$ *for eq.* (42)*, that is,*

$$[8,2]_{(x,t)} = \frac{(x^4 - 12.00000000x^2 + 0.1714285714t^4)xt^3}{-12.00000000 + x^2} \tag{48}$$

the variational iteration method gives the solution for the classical Klein-Gordon Eq. (37) *(when* $\alpha = 1.5$*) which is given by*

$$\begin{aligned} u(x,t) =& 1.805406668x^3t^{2.5} + 0.1146289948(x^3 - 6x)t^{4.5} \\ &- 0.06250000000(x^3 - 6x)t^{4.0} - 0.002083333334(x^3 - 12x)t^{6.0} \\ =& 1.805406668x^3t^{2.5} + 0.1146289948x^3t^{4.5} - 0.6877739688xt^{4.5} \\ &- 0.06250000000x^3t^{4.0} + 0.3750000000xt^{4.0} - 0.002083333334x^3t^{6.0} \\ &+ 0.02500000001xt^{6.0} \end{aligned} \tag{49}$$

For simplicity, let $t^{1/2} = a$*; then*

$$\begin{aligned} u(x,a) =& 1.805406668x^3a^5 + 0.1146289948x^3a^9 - 0.6877739688xa^9 - 0.06250000000x^3a^8 \\ &+ 0.3750000000xa^8 - 0.002083333334x^3a^{12} + 0.02500000001xa^{12} \end{aligned} \tag{50}$$

and let

$$\begin{aligned} K =& 1.805406668x^3a^5 + 0.1146289948x^3a^9 - 0.6877739688xa^9 - 0.06250000000x^3a^8 \\ &+ 0.3750000000xa^8 + 0.02500000001xa^{12} \end{aligned}$$

$$\begin{aligned} L =& 1.805406668x^3a^5 + 0.1146289948x^3a^9 - 0.6877739688xa^9 \\ &- 0.06250000000x^3a^8 + 0.3750000000xa^8 \end{aligned}$$

$$M = 1.805406668x^3a^5 - 0.6877739688xa^9 - 0.06250000000x^3a^8 + 0.3750000000xa^8$$

Then, using the Eqs. (11) *and* (12) *to calculate the multivariate Padé equations for Eq.* (50) *we get*

$$p(x,a) = \begin{vmatrix} K & L & M \\ 0 & 0.02500000001xa^{12} & 0.1146289948x^3a^9 \\ -0.002083333334x^3a^{12} & 0 & 0.02500000001xa^{12} \end{vmatrix} \tag{51}$$

$$\begin{aligned} =& -0.00005208333337(-1.805406668x^4a^5 - 0.6877739692x^4a^9 \\ &+ 0.06250000000x^4a^8 + 0.3750000000x^2a^8 - 21.66488002x^2a^5 \\ &+ 8.253287626a^9 - 4.500000000a^8 - 0.3000000001a^{12})x^3a^{24} \end{aligned} \tag{52}$$

and

$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.02500000001xa^{12} & 0.1146289948x^3a^9 \\ -0.002083333334x^3a^{12} & 0 & 0.02500000001xa^{12} \end{vmatrix} \tag{53}$$

$$= -0.00005208333337(-12.00000000 - x^2 + \frac{4.585159790}{a^3})x^2a^{24} \tag{54}$$

recalling that $t^{1/2} = a$, *we get multivariate Padé approximation of order*$(13,2)$ *for Eq.* (49), *that is;*

$$\begin{aligned} [13,2]_{(x,t)} =& (-1.805406668x^4t^{5/2} - 0.6877739692x^4t^{9/2} + 0.06250000000x^4t^4 \\ &+ 0.3750000000x^2t^4 - 21.66488002x^2t^{5/2} + 8.253287626t^{9/2} - 4.500000000t^4 \\ &- 0.3000000001t^6)x/(-12.00000000 - x^2 + \frac{4.585159790}{t^{1.5}}) \end{aligned} \tag{55}$$

the variational iteration method gives the solution for the classical Klein-Gordon Eq. (37) *(when* $\alpha = 1.75$ *) which is given by*

$$\begin{aligned} u(x,t) =& 1.356548886x^3t^{2.75} + 0.07615713042(x^3 - 6x)t^{4.75} \\ &- 0.06447880955(x^3 - 6x)t^{4.5} - 0.001803603064(x^3 - 12x)t^{6.5} \end{aligned} \tag{56}$$

$$\begin{aligned} =& 1.356548886x^3t^{2.75} + 0.07615713042x^3t^{4.75} - 0.4569427825xt^{4.75} \\ &- 0.06447880955x^3t^{4.5} + 0.3868728573xt^{4.5} - 0.001803603064x^3t^{6.5} \\ &+ 0.02164323677xt^{6.5} \end{aligned} \tag{57}$$

For simplicity, let $t^{1/4} = a$; *then*

$$\begin{aligned} u(x,a) =& 1.356548886x^3a^{11} + 0.07615713042x^3a^{19} - 0.4569427825xa^{19} \\ &- 0.06447880955x^3a^{18} + 0.3868728573xa^{18} - 0.001803603064x^3a^{26} \\ &+ 0.02164323677xa^{26} \end{aligned} \tag{58}$$

and let

$$
\begin{aligned}
N =& 1.356548886x^3a^{11} + 0.07615713042x^3a^{19} - 0.4569427825xa^{19} \\
& - 0.06447880955x^3a^{18} + 0.3868728573xa^{18} + 0.02164323677xa^{26}
\end{aligned}
$$

$$
\begin{aligned}
P =& 1.356548886x^3a^{11} + 0.07615713042x^3a^{19} - 0.4569427825xa^{19} \\
& - 0.06447880955x^3a^{18} + 0.3868728573xa^{18}
\end{aligned}
$$

$$
\begin{aligned}
R =& 1.356548886x^3a^{11} + 0.07615713042x^3a^{19} - 0.4569427825xa^{19} \\
& - 0.06447880955x^3a^{18} + 0.3868728573xa^{18}
\end{aligned}
$$

Then, using the Eqs. (11) *and* (12) *to calculate the multivariate Padé equations for Eq.* (58) *we get*

$$
p(x,a) = \begin{vmatrix} N & P & R \\ 0 & 0.02164323677xa^{26} & 0 \\ -0.001803603064x^3a^{26} & 0 & 0.02164323677xa^{26} \end{vmatrix} \tag{59}
$$

$$
\begin{aligned}
=& 0.00003903580815(1.356548886x^4 + 0.07615713042x^4a^8 \\
& + 0.4569427825x^2a^8 - 0.06447880955x^4a^7 - 0.3868728573x^2a^7 \\
& + 16.27858663x^2 - 5.483313390a^8 + 4.642474288a^7 + 0.2597188412a^{15})x^3a^{63}
\end{aligned} \tag{60}
$$

and

$$
q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.02164323677xa^{26} & 0 \\ -0.001803603064x^3a^{26} & 0 & 0.02164323677xa^{26} \end{vmatrix} \tag{61}
$$

$$
= 0.00003903580815(12.00000000 + x^2)x^2a^{52} \tag{62}
$$

recalling that $t^{1/4} = a$, *we get multivariate Padé approximation of order* $(27,2)$ *for Eq.* (57), *that is;*

$$
\begin{aligned}
[27,2]_{(x,t)} =& ((1.356548886x^4 + 0.07615713042x^4t^2 + 0.4569427825x^2t^2 \\
& - 0.06447880955x^4t^{7/4} - 0.3868728573x^2t^{7/4} + 16.27858663x^2 \\
& - 5.483313390t^2 + 4.642474288t^{7/4} + 0.2597188412t^{15/4})xt^{11/4} \\
& /(12.00000000 + x^2)
\end{aligned} \tag{63}
$$

Example 2. *Consider the nonlinear time-fractional hyperbolic equation [41]*

$$
D_{*t}^{\alpha}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t > 0, x \in R, 1 < \alpha \leq 2, \tag{64}
$$

subject to the initial condition

$$u(x,0)=x^2,\quad u_t(x,0)=-2x^2. \tag{65}$$

According to the formula (34)*, the iteration formula for Eq.* (64) *is given by*

$$u_{k+1}(x,t)=u_k(x,t)+\int_0^t(\xi-t)\left(\frac{\partial^\alpha}{\partial\xi^\alpha}u_k(x,\xi)-f(u_k,(u_k)_x,(u_k)_{xx})-g(x,\xi)\right)d\xi. \tag{66}$$

By the above iteration formula, if we begin with $u_0=x^2-2tx^2$*, following approximations has been obtained in [41]*

$$u_0(x,t)=x^2(1-2t)$$
$$u_1(x,t)=x^2(1-2t+3t^2-4t^3+2t^4),$$
$$\begin{aligned}u_2(x,t)=&x^2(1-2t+6t^2-8t^3+7t^4-6t^5+\frac{174}{30}t^6-\frac{192}{42}t^7+\frac{168}{56}t^8-\frac{96}{72}t^9+\frac{24}{90}t^{10})\\&+x^2(\frac{-6}{\Gamma(5-\alpha)}t^{4-\alpha}+\frac{24}{\Gamma(6-\alpha)}t^{5-\alpha}-\frac{48}{\Gamma(7-\alpha)}t^{6-\alpha})\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (66) *can be obtained using maple software. The variational iteration method gives the solution for the Eq.* (64) *(when* $\alpha=2$ *) which is given by*

$$\begin{aligned}u(x,t)=&x^2(1-2t+6t^2-8t^3+7t^4-6t^5+5.8t^6-4.571428571t^7+3t^8\\&-1.333333333t^9+0.2666666667t^{10})+x^2(-3t^2+4.000000001t^3-2t^4) \quad (67)\\=&x^2-2x^2t+3x^2t^2-3.999999999x^2t^3+5x^2t^4-6x^2t^5+5.8x^2t^6\\&-4.571428571x^2t^7+3x^2t^8-1.333333333x^2t^9+0.2666666667x^2t^{10} \quad (68)\end{aligned}$$

and let

$$A=x^2-2x^2t+3x^2t^2-3.999999999x^2t^3+5x^2t^4-6x^2t^5+5.8x^2t^6-4.571428571x^2t^7+3x^2t^8$$

$$B=x^2-2x^2t+3x^2t^2-3.999999999x^2t^3+5x^2t^4-6x^2t^5+5.8x^2t^6-4.571428571x^2t^7$$

$$C=x^2-2x^2t+3x^2t^2-3.999999999x^2t^3+5x^2t^4-6x^2t^5+5.8x^2t^6$$

Now let us calculate the approximate solution of Eq. (67) *for* $m=10$ *and* $n=2$ *by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.* (67) *for* $m=10$ *and* $n=2$*, we use Eqs.* (11) *and* (12)*. By using Eqs.* (11) *and* (12) *We obtain,*

$$\begin{aligned}p(x,t)=&\begin{vmatrix}A & B & C\\ -1.333333333x^2t^9 & 3x^2t^8 & -4.571428571x^2t^7\\ 0.2666666667x^2t^{10} & -1.333333333x^2t^9 & 3x^2t^8\end{vmatrix} \quad (69)\\=&0.3555555555t^{16}(4.70382656t^8-8.48265312t^7+14.20535725t^6\\&-20.91071441t^5+17.81250006t^4-14.71428577t^3+11.61607147t^2\end{aligned}$$

$$-8.51785717t+8.169642871)x^6$$

and

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ -1.333333333x^2t^9 & 3x^2t^8 & -4.571428571x^2t^7 \\ 0.2666666667x^2t^{10} & -1.333333333x^2t^9 & 3x^2t^8 \end{vmatrix} \tag{70}$$
$$=0.3555555555t^{16}(8.16964287+7.821428571t+2.749999997t^2)x^4$$

So the Multivariate Padé approximation of order $(10,2)$ *for eq.* (67)*, that is,*

$$\begin{aligned}[10,2]_{(x,t)} =&(4.70382656t^8-8.48265312t^7+14.20535725t^6-20.91071441t^5\\ &+17.81250006t^4-14.71428577t^3+11.61607147t^2-8.51785717t\\ &+8.169642871)x^2/(8.16964287+7.821428571t+2.749999997t^2)\end{aligned} \tag{71}$$

The variational iteration method gives the solution for the Eq. (64) *(when* $\alpha = 1.50$ *) which is given by*

$$\begin{aligned}u(x,t) =&x^2(1-2t+6t^2-8t^3+7t^4-6t^5+5.8t^6-4.571428571t^7+3t^8\\ &-1.333333333t^9+0.2666666667t^{10})+x^2(-1.805406668t^{2.5}\\ &+2.063321905t^{3.5}-0.9170319581t^{4.5})\end{aligned} \tag{72}$$

$$\begin{aligned}=&x^2-2x^2t+6x^2t^2-8x^2t^3+7x^2t^4-6x^2t^5+5.8x^2t^6-4.571428571x^2t^7\\ &+3x^2t^8-1.333333333x^2t^9+0.2666666667x^2t^{10}-1.805406668x^2t^{2.5}\\ &+2.063321905x^2t^{3.5}-0.9170319581x^2t^{4.5}\end{aligned} \tag{73}$$

For simplicity, let $t^{1/2}=a$*; then*

$$\begin{aligned}u(x,a) =&x^2-2x^2a^2+6x^2a^4-8x^2a^6+7x^2a^8-6x^2a^{10}+5.8x^2a^{12}-4.571428571x^2a^{14}\\ &+3x^2a^{16}-1.333333333x^2a^{18}+0.2666666667x^2a^{20}-1.805406668x^2a^5\\ &+2.063321905x^2a^7-0.9170319581x^2a^9\end{aligned} \tag{74}$$

and let

$$\begin{aligned}D =&x^2-2x^2a^2+6x^2a^4-8x^2a^6+7x^2a^8-6x^2a^{10}+5.8x^2a^{12}\\ &-4.571428571x^2a^{14}+3x^2a^{16}-1.333333333x^2a^{18}\\ &-1.805406668x^2a^5+2.063321905x^2a^7-0.9170319581x^2a^9\end{aligned}$$

$$\begin{aligned}E =&x^2-2x^2a^2+6x^2a^4-8x^2a^6+7x^2a^8\\ &-6x^2a^{10}+5.8x^2a^{12}-4.571428571x^2a^{14}+3x^2a^{16}\\ &-1.805406668x^2a^5+2.063321905x^2a^7-0.9170319581x^2a^9\end{aligned}$$

$$
\begin{aligned}
F =& x^2 - 2x^2a^2 + 6x^2a^4 - 8x^2a^6 + 7x^2a^8 - 6x^2a^{10} + 5.8x^2a^{12} \\
& - 4.571428571x^2a^{14} + 3x^2a^{16} - 1.805406668x^2a^5 \\
& + 2.063321905x^2a^7 - 0.9170319581x^2a^9
\end{aligned}
$$

Now let us calculate the approximate solution of Eq. (74) *for* $m = 20$ *and* $n = 2$ *by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.* (74) *for* $m = 20$ *and* $n = 2$*, we use Eqs.* (11) *and* (12)*. By using Eqs.* (11) *and* (12) *We obtain,*

$$
\begin{aligned}
p(x,a) =& \begin{vmatrix} D & E & F \\ 0 & -1.333333333x^2a^{18} & 0 \\ 0.2666666667x^2a^{20} & 0 & -1.333333333x^2a^{18} \end{vmatrix} \qquad (75) \\
=& -0.3555555555(2.521837884a^9 - 8.511202852a^7 + 9.027033336a^5 \\
& - 10.42857142a^{16} + 17.05714285a^{14} - 22.99999999a^{12} + 22.99999999a^{10} \\
& - 26.99999999a^8 + 33.99999998a^6 - 27.99999999a^4 + 8.999999996a^2 \\
& + 3.666666662a^{18} - 4.999999998 + 0.917031958a^{11})x^6a^{36}
\end{aligned}
$$

and

$$
\begin{aligned}
q(x,a) =& \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1.333333333x^2a^{18} & 0 \\ 0.2666666667x^2a^{20} & 0 & -1.333333333x^2a^{18} \end{vmatrix} \qquad (76) \\
=& -0.3555555555(4.999999998 + a^2)x^4a^{36}
\end{aligned}
$$

recalling that $t^{1/2} = a$*, we get multivariate Padé approximation of order* $(20,2)$ *for Eq.* (72)*, that is;*

$$
\begin{aligned}
[20,2]_{(x,t)} =& -(2.521837884t^{9/2} - 8.511202852t^{7/2} + 9.027033336t^{5/2} - 10.42857142t^8 \\
& + 17.05714285t^7 - 22.99999999t^6 + 22.99999999t^5 - 26.99999999t^4 \\
& + 33.99999998t^3 - 27.99999999t^2 + 8.999999996t + 3.666666662t^9 \\
& - 4.999999998 + 0.917031958t^{11/2})x^2/(4.999999998 + t) \qquad (77)
\end{aligned}
$$

The variational iteration method gives the solution for the Eq. (64) *(when* $\alpha = 1.75$ *) which is given by*

$$
\begin{aligned}
u(x,t) =& x^2(1 - 2t + 6t^2 - 8t^3 + 7t^4 - 6t^5 + 5.8t^6 - 4.571428571t^7 + 3t^8 \\
& - 1.333333333t^9 + 0.2666666667t^{10}) + x^2(-2.353626989t^{2.25} \\
& + 2.896771680t^{3.25} - 1.363186673t^{4.25}) \qquad (78) \\
=& x^2 - 2x^2t + 6x^2t^2 - 8x^2t^3 + 7x^2t^4 - 6x^2t^5 + 5.8x^2t^6 - 4.571428571x^2t^7 \\
& + 3x^2t^8 - 1.333333333x^2t^9 + 0.2666666667x^2t^{10} - 2.353626989x^2t^{2.25} \\
& + 2.896771680x^2t^{3.25} - 1.363186673x^2t^{4.25}
\end{aligned}
$$

For simplicity, let $t^{1/4} = a$; then

$$\begin{aligned} u(x,a) =& x^2 - 2x^2a^4 + 6x^2a^8 - 8x^2a^{12} + 7x^2a^{16} - 6x^2a^{20} + 5.8x^2a^{24} \\ & - 4.571428571x^2a^{28} + 3x^2a^{32} - 1.333333333x^2a^{36} + 0.2666666667x^2a^{40} \\ & - 2.353626989x^2a^9 + 2.896771680x^2a^{13} - 1.363186673x^2a^{17} \end{aligned} \tag{79}$$

and let,

$$\begin{aligned} G =& x^2 - 2x^2a^4 + 6x^2a^8 - 8x^2a^{12} + 7x^2a^{16} - 6x^2a^{20} + 5.8x^2a^{24} \\ & - 4.571428571x^2a^{28} + 3x^2a^{32} - 1.333333333x^2a^{36} - 2.353626989x^2a^9 \\ & + 2.896771680x^2a^{13} - 1.363186673x^2a^{17} \end{aligned}$$

$$\begin{aligned} H =& x^2 - 2x^2a^4 + 6x^2a^8 - 8x^2a^{12} + 7x^2a^{16} - 6x^2a^{20} + 5.8x^2a^{24} \\ & - 4.571428571x^2a^{28} + 3x^2a^{32} - 1.333333333x^2a^{36} - 2.353626989x^2a^9 \\ & + 2.896771680x^2a^{13} - 1.363186673x^2a^{17} \end{aligned}$$

$$\begin{aligned} I =& x^2 - 2x^2a^4 + 6x^2a^8 - 8x^2a^{12} + 7x^2a^{16} - 6x^2a^{20} + 5.8x^2a^{24} + 3x^2a^{32} \\ & - 4.571428571x^2a^{28} - 2.353626989x^2a^9 + 2.896771680x^2a^{13} - 1.363186673x^2a^{17} \end{aligned}$$

To obtain Multivariate Padé equations of Eq. (79) *for $m = 41$ and $n = 2$, we use Eqs.* (11) *and* (12). *By using Eqs.* (11) *and* (12) *We obtain,*

$$p(x,a) = \begin{vmatrix} G & H & I \\ 0.2666666667x^2a^{40} & 0 & 0 \\ 0 & 0.2666666667x^2a^{40} & 0 \end{vmatrix} \tag{80}$$

$$\begin{aligned} =& - 0.07111111113x^6a^{80}(-1 + 2a^4 - 6a^8 + 8a^{12} - 7a^{16} + 6a^{20} - 5.8a^{24} \\ & + 4.571428571a^{28} - 3a^{32} + 1.333333333a^{36} + 2.353626989a^9 \\ & - 2.896771680a^{13} + 1.363186673a^{17}) \end{aligned}$$

and

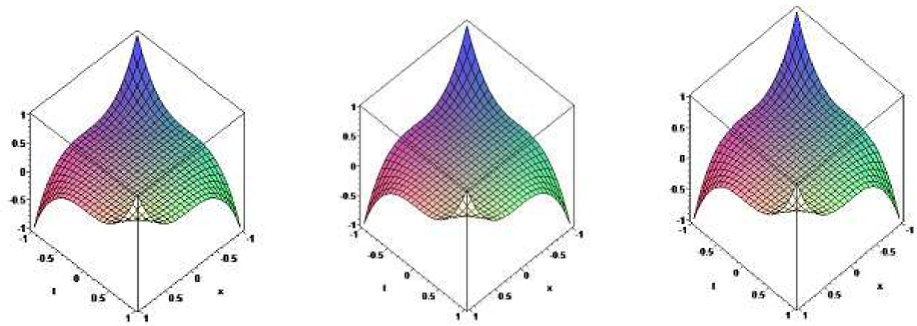
$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 0.2666666667x^2a^{40} & 0 & 0 \\ 0 & 0.2666666667x^2a^{40} & 0 \end{vmatrix} \tag{81}$$

$$= 0.07111111113x^4a^{80}$$

recalling that $t^{1/4} = a$, we get multivariate Padé approximation of order $(41,2)$ *for Eq.* (77), *that is;*

$$\begin{aligned} [41,2]_{(x,t)} =& - 0.07111111113x^6t^{20}(-1 + 2t - 6t^2 + 8t^3 - 7t^4 + 6t^5 - 5.8t^6 \\ & + 4.571428571t^7 - 3t^8 + 1.333333333t^9 + 2.353626989t^{9/4} \\ & - 2.896771680t^{13/4} + 1.363186673t^{17/4})/0.07111111113x^4t^{20} \end{aligned} \tag{82}$$

As it is presented above in Example 1 we obtained multivariate Padé approximations of variational iteration solution of Eq. (37) for values of $\alpha = 2.0$, $\alpha = 1.5$, and $\alpha = 1.75$. Tables 1 - 3, and Figures 1 - 3 show the approximate solutions for Eq. (37) obtained for the three different values of α using the variational iteration method (VIM) and the multivariate Padé approximation (MPA). The values of $\alpha = 2.0$ is the only case for which we know the exact solution $u(x,t) = x^3 t^3$ and the results of multivariate Padé approximation (MPA) are in excellent agreement with the exact solution and those obtained by the variational iteration method (VIM).

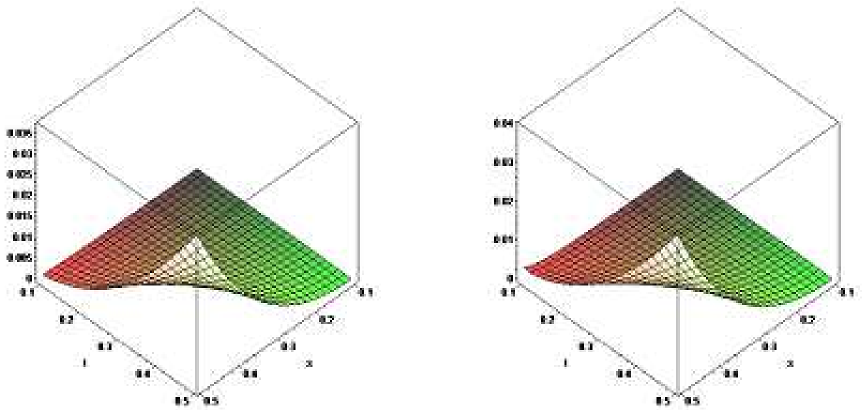


(a) Exact solution (b) VIM solution (c) Multivariate Padé approximation of VIM solution

Figure 1: Example 1 solutions for $\alpha = 2.0$.

Table 1: Numerical values when $\alpha = 2.0$ for Example 1.

x	t	u_{VIM}	u_{MPA}	u_{Exact}
0.01	0.01	$0.9999985714 \times 10^{-12}$	$0.9999985717 \times 10^{-12}$	0.1×10^{-11}
0.02	0.02	$0.6399963428 \times 10^{-10}$	$0.6399963427 \times 10^{-10}$	0.64×10^{-10}
0.03	0.03	$0.7289906264 \times 10^{-9}$	$0.7289906262 \times 10^{-9}$	0.729×10^{-9}
0.04	0.04	$0.4095906365 \times 10^{-8}$	$0.4095906364 \times 10^{-8}$	0.4096×10^{-8}
0.05	0.05	$0.156244184 \times 10^{-7}$	$0.156244184 \times 10^{-7}$	0.15625×10^{-7}
0.06	0.06	$0.4665359983 \times 10^{-7}$	$0.4665359983 \times 10^{-7}$	0.46656×10^{-7}
0.07	0.07	$0.1176407612 \times 10^{-6}$	$0.1176407612 \times 10^{-6}$	0.117649×10^{-6}
0.08	0.08	$0.2621200197 \times 10^{-6}$	$0.2621200198 \times 10^{-6}$	0.262144×10^{-6}
0.09	0.09	$0.5313794632 \times 10^{-6}$	$0.5313794631 \times 10^{-6}$	0.531441×10^{-6}
0.1	0.1	$0.9998570239 \times 10^{-6}$	$0.9998570234 \times 10^{-6}$	0.1×10^{-5}



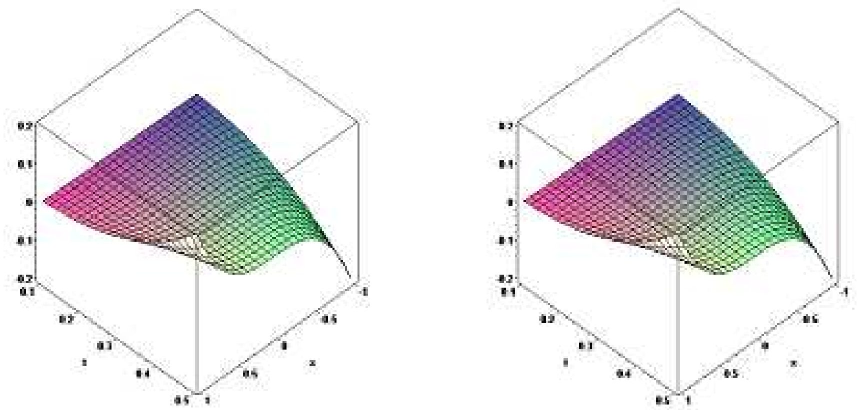
(a) VIM solution

(b) Multivariate Padé approximation of VIM solution

Figure 2: Example 1 solutions for $\alpha = 1.5$.

Table 2: Numerical values when $\alpha = 1.5$ for Example 1.

x	t	u_{VIM}	u_{MPA}
0.01	0.01	$0.4867606662 \times 10^{-10}$	$0.4867625260 \times 10^{-10}$
0.02	0.02	$0.1705755572 \times 10^{-8}$	$0.1705792441 \times 10^{-8}$
0.03	0.03	$0.1381609983 \times 10^{-7}$	$0.1381692273 \times 10^{-7}$
0.04	0.04	$0.6128672929 \times 10^{-7}$	$0.6129422227 \times 10^{-7}$
0.05	0.05	$0.1952751831 \times 10^{-6}$	$0.1953168894 \times 10^{-6}$
0.06	0.06	$0.5044516999 \times 10^{-6}$	$0.5046216523 \times 10^{-6}$
0.07	0.07	$0.1127182674 \times 10^{-5}$	$0.1127740978 \times 10^{-5}$
0.08	0.08	$0.2264536447 \times 10^{-5}$	$0.2266102644 \times 10^{-5}$
0.09	0.09	$0.4194040677 \times 10^{-5}$	$0.4197934702 \times 10^{-5}$
0.1	0.1	$0.7284137720 \times 10^{-5}$	$0.7292939395 \times 10^{-5}$



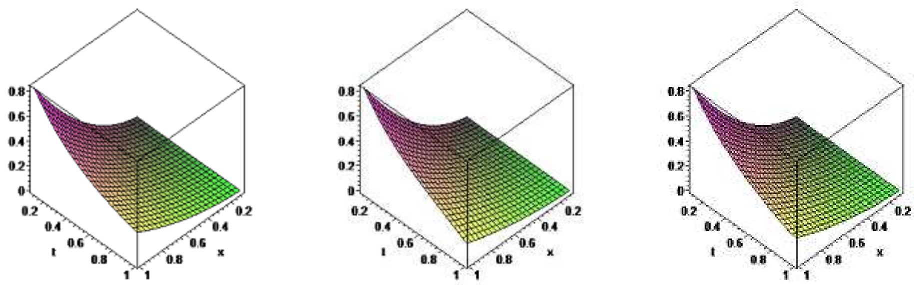
(a) VIM solution (b) Multivariate Padé approximation of VIM solution

Figure 3: Example 1 solutions for $\alpha = 1.75$.

Table 3: Numerical values when $\alpha = 1.75$ for Example 1.

x	t	u_{VIM}	u_{MPA}
0.01	0.01	$0.6713514104 \times 10^{-11}$	$0.6713514105 \times 10^{-11}$
0.02	0.02	$0.3281759859 \times 10^{-9}$	$0.3281759859 \times 10^{-9}$
0.03	0.03	$0.3204055164 \times 10^{-8}$	$0.3204055165 \times 10^{-8}$
0.04	0.04	$0.1616230193 \times 10^{-7}$	$0.1616230193 \times 10^{-7}$
0.05	0.05	$0.5675792133 \times 10^{-7}$	$0.5675792134 \times 10^{-7}$
0.06	0.06	$0.1584901621 \times 10^{-6}$	$0.1584901621 \times 10^{-6}$
0.07	0.07	$0.3777857545 \times 10^{-6}$	$0.3777857544 \times 10^{-6}$
0.08	0.08	$0.8019718443 \times 10^{-6}$	$0.8019718443 \times 10^{-6}$
0.09	0.09	$0.1558181826 \times 10^{-5}$	$0.1558181826 \times 10^{-5}$
0.1	0.1	$0.2823149542 \times 10^{-5}$	$0.2823149543 \times 10^{-5}$

As it is presented above in Example 2. we obtained multivariate Padé approximations of variational iteration solution of Eq. (64) for values of $\alpha = 2.0$, $\alpha = 1.5$, and $\alpha = 1.75$. Tables 4 - 6, and Figures 4 - 6 show the approximate solutions for Eq. (64) obtained for the three different values of α using the variational iteration method (VIM) and the multivariate Padé approximation (MPA). The values of $\alpha = 2.0$ is the only case for which we know the exact solution $u(x,t) = (x/t+1)^2$ and the results of multivariate Padé approximation (MPA) are in excellent agreement with the exact solution and those obtained by the variational iteration method (VIM).

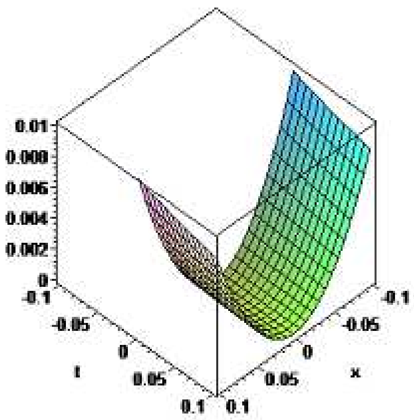


(a) Exact solution (b) VIM solution (c) Multivariate Padé approximation of VIM solution

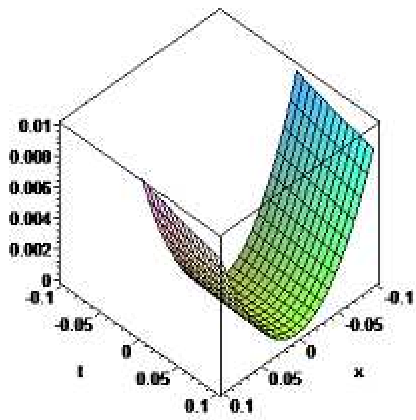
Figure 4: Example 2 solutions for $\alpha = 2.0$.

Table 4: Numerical values when $\alpha = 2.0$ for Example 2.

x	t	u_{VIM}	u_{MPA}	u_{Exact}
0.01	0.01	0.00009802960494	0.00009802960495	0.00009802960494
0.02	0.02	0.0003844675124	0.0003844675125	0.0003844675125
0.03	0.03	0.0008483363175	0.0008483363176	0.0008483363182
0.04	0.04	0.001479289934	0.001479289934	0.001479289941
0.05	0.05	0.002267573656	0.002267573655	0.002267573696
0.06	0.06	0.003203987014	0.003203987014	0.003203987184
0.07	0.07	0.004279849200	0.004279849200	0.004279849769
0.08	0.08	0.005486966839	0.005486966839	0.005486968450
0.09	0.09	0.006817603925	0.006817603924	0.006817607945
0.1	0.1	0.008264453717	0.008264453716	0.008264462810



(a) VIM solution

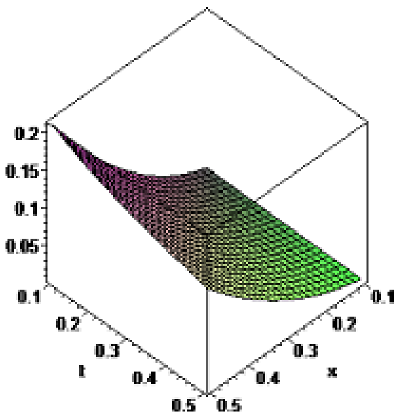


(b) Multivariate Padé approximation of VIM solution

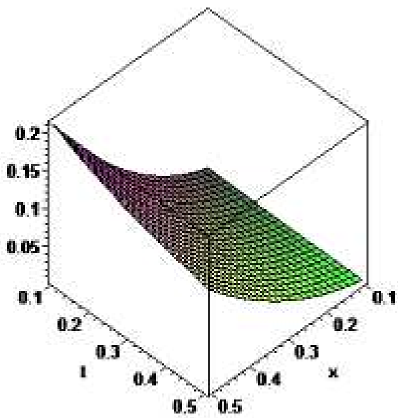
Figure 5: Example 2 solutions for $\alpha = 1.5$.

Table 5: Numerical values when $\alpha = 1.5$ for Example 2.

x	t	u_{VIM}	u_{MPA}
0.01	0.01	0.00009805742207	0.00009805742208
0.02	0.02	0.0003848949142	0.0003848949143
0.03	0.03	0.0008504258525	0.0008504258524
0.04	0.04	0.001485685664	0.001485685663
0.05	0.05	0.002282722748	0.002282722747
0.06	0.06	0.003234501119	0.003234501118
0.07	0.07	0.004334811900	0.004334811899
0.08	0.08	0.005578192194	0.005578192193
0.09	0.09	0.006959850295	0.006959850295
0.1	0.1	0.008475596550	0.008475596548



(a) VIM solution



(b) Multivariate Padé approximation of VIM solution

Figure 6: Example 2 solutions for $\alpha = 1.75$.

Table 6: Numerical values when $\alpha = 1.75$ for Example 2.

x	t	u_{VIM}	u_{MPA}
0.01	0.01	0.00009805185529	0.00009805185529
0.02	0.02	0.0003847966767	0.0003847966767
0.03	0.03	0.0008499060377	0.0008499060378
0.04	0.04	0.001484004096	0.001484004096
0.05	0.05	0.002278566886	0.002278566886
0.06	0.06	0.003225836632	0.003225836632
0.07	0.07	0.004318746188	0.004318746188
0.08	0.08	0.005550851279	0.005550851280
0.09	0.09	0.006916269160	0.006916269160
0.1	0.1	0.008409622712	0.008409622712

6. Conclusion

We know and it can be seen from the references that variational iteration method (VIM) has been applied to fractional differential equations. By comparison with variational iteration method (VIM), the fundamental goal of this work has been to construct an approximate solution for nonlinear and linear partial differential equations of fractional order by using multivariate Padé approximation. The goal has been achieved by using the multivariate Padé approximation (MPA) and the variational iteration method (VIM). The present work shows the validity and great potential of the multivariate Padé approximation for solving nonlinear partial differential equations of fractional order from the numerical results. For the values of

$\alpha = 2.0$ in example 1 and for the values of $\alpha = 2.0$ in example 2, numerical results obtained using the multivariate Padé approximation (MPA) and the variational iteration method (VIM) are in excellent agreement with exact solutions and each other. For the values of $\alpha = 1.5$, $\alpha = 1.75$, in example 1 and for the values of $\alpha = 1.5$, $\alpha = 1.75$ in example 2, numerical results show that the results of multivariate Padé approximation are in excellent agreement with those results obtained by the variational iteration method (VIM). The basic idea described in this paper is expected to be further employed to solve other similar problems in fractional calculus.

References

[1] J. Abouir, A. Cuyt, P. Gonzalez-Vera, and R. Orive. On the Convergence of General Order Multivariate Padé-Type Approximants. *Journal of Approximation Theory*, 86:216–228, 1996.

[2] G. Adomian. A review of the decomposition method in applied mathematics. *Fractional Calculus and Applied Analysis*, 135:501–544, 1988.

[3] G. Adomian. *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer Academic Publishers, Boston, 1994.

[4] M. Caputo. Linear models of dissipation whose Q is almost frequency independent. part II. *Journal of the Royal Astronomical Society*, 13:529–539, 1967.

[5] A. Cuyt. Multivariate Padé-approximant. *Journal of Mathematical Analysis and Applications*, 96:283–293, 1983.

[6] A. Cuyt. A review of multivariate Padé approximation theory. *Journal of Computational and Applied Mathematics*, 12:221–232, 1985.

[7] A. Cuyt. How well can the concept of Padé approximant be generalized to the multivariate case? *Journal of Computational and Applied Mathematics*, 105:25–50, 1985.

[8] A. Cuyt and L. Wuytack. *Nonlinear Methods in Numerical Analysis*. Elsevier Science Publishers B.V, Amsterdam, 1987.

[9] A. Cuyt, L. Wuytack, and H. Werner. On the continuity of the multivariate Padé operator. *Journal of Computational and Applied Mathematics*, 11:95–102, 1984.

[10] L. Debnath and D. Bhatta. Solutions to few linear fractional inhomogeneous partial differential equations in fluid mechanics. *Fractional Calculus and Applied Analysis*, 7:153–192, 2004.

[11] R. Gorenflo. Afterthoughts on interpretation of fractional derivatives and integrals. In P. Rusev, I. Di-movski, and V. Kiryakovai, editors, *Transform Methods and Special Functions*, pages 589–591, Sofia, 1998. Bulgarian Academy of Sciences, Institute of Mathematics ands Informatics.

[12] Ph. Guillaume and A. Huard. Multivariate Padé Approximants. *Journal of Computational and Applied Mathematics*, 121:197–219, 2000.

[13] Ph. Guillaume, A. Huard, and V. Robin. Generalized Multivariate Padé Approximants. *Journal of Approximation Theory*, 95:203–214, 1998.

[14] J.H. He. Semi-inverse method of establishing generalized principlies for fluid mechanics with emphasis on turbomachinery aerodynamics. *International Journal of Turbo and Jet Engines*, 14(1):23–28, 1997.

[15] J.H. He. Variational iteration method for delay differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 2(4):235–236, 1997.

[16] J.H. He. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Computer Methods in Applied Mechanics and Engineering*, 167:57–68, 1998.

[17] J.H. He. Approximate solution of nonlinear differential equations with convolution product nonlinearities. *Computer Methods in Applied Mechanics and Engineering*, 167:69–73, 1998.

[18] J.H. He. Nonlinear oscillation with fractional derivative and its applications. *International Conference on Vibrating Engineering '98*, pages 288–291, 1998.

[19] J.H. He. Some applications of nonlinear fractional differential equations and their approximations. *Bulletin of Science and Technology*, 15(2):86–90, 1999.

[20] J.H. He. Variational iteration method – a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics*, 34:699–708, 1999.

[21] J.H. He. Variational iteration method for autonomous ordinary differential systems. *Applied Mathematics and Computation*, 114:115–123, 2000.

[22] J.H. He. Variational theory for linear magneto-electro-elasticity. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2(4):309–316, 2001.

[23] J.H. He. Variational principle for Nano thin film lubrication. *International Journal of Nonlinear Sciences and Numerical Simulation*, 4(3):313–314, 2003.

[24] J.H. He. Variational principle for some nonlinear partial differential equations with variable coefficients. *Chaos Solitons and Fractals*, 19(4):847–851, 2004.

[25] J.H. He. Variational iteration method: Some recent results and new interpretations. *Journal of Computational and Applied Mathematics*, 207(1):3–17, 2007.

[26] J.H. He and X.H. Wu. Variational iteration method: New development and applications. *Computers and Mathematics with Applications*, 54(7-8):881–894, 2007.

[27] M. Inokuti, H. Sekine, and T. Mura. General use of the lagrange multiplier in non-linear mathematical physics. In S. Nemat-Nasser, editor, *Variational Method in the Mechanics of Solids*, pages 156–162, Oxford, 1978. Pergamon Press.

[28] A. Luchko and R. Groneflo. *The initial value problem for some fractional differential equations with the Caputo derivative*. Preprint series A0–98, Fachbreich Mathematik und Informatik, Freic Universitat Berlin, 1997.

[29] F. Mainardi. *Fractional calculus: Some basic problems in continuum and statistical mechanics*. Springer-Verlag, New York, 1997.

[30] K.S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley and Sons, Inc, New York, 1993.

[31] S. Momani. An explicit and numerical solutions of the fractional KdV equation. *Mathematics and Computers in Simulation*, 70(2):110–118, 2005.

[32] S. Momani. Non-perturbative analytical solutions of the space- and time-fractional Burgers equations. *Chaos, Solitons and Fractals*, 28(4):930–937, 2006.

[33] S. Momani and S. Abuasad. Application of He's variational iteration method to Helmholtz equation. *Chaos Solitons and Fractals*, 27(5):1119–1123, 2006.

[34] S. Momani and Z. Odibat. Analytical approach to linear fractional partial differential equations arising in fluild mechanics. *Physics Letters A*, 355:271–279, 2006.

[35] S. Momani and Z. Odibat. Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method. *Applied Mathematics and Computation*, 177:488–494, 2006.

[36] S. Momani and Z. Odibat. Approximate solutions for boundary value problems of time-fractional wave equation. *Applied Mathematics and Computation*, 181:767–774, 2006.

[37] S. Momani and Z. Odibati. Numerical comparison of methods for solving linear differential equations of fractional order. *Chaos Solitons and Fractals*, 31:1248–1255, 2007.

[38] S. Momani and R. Qaralleh. Numerical approximations and Padé approximants for a fractional population growth model. *Applied Mathematical Modelling*, 31:1907–1914, 2007.

[39] S. Momani and N. Shawagfeh. Decomposition method for solving fractional Riccatti differential equations. *Applied Mathematics and Computation*, 182:1083–1092, 2006.

[40] Z. Odibat and S. Momani. An explicit and numerical solutions of the fractional KdV equation. *International Journal of Nonlinear Sciences and Numerical Simulation*, 7(1):15–27, 2006.

[41] Z. Odibat and S. Momani. Numerical methods for nonlinear differential equations of fractional order. *Applied Mathematical Modelling*, 32:28–39, 2008.

[42] Z. Odibat and S. Momani. The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics. *Computers and Mathematics with Applications*, 58:2199–2208, 2009.

[43] K.B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, New York, 1974.

[44] I. Podlubny. *Fractional Differential Equations*. Academic Press, New York, 1999.

[45] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fractional Calculus and Applied Analysis*, 5:367–386, 2002.

[46] A. Rèpaci. Nonlinear dynamical systems: On the accuracy of Adomian's decomposition method. *Applied Mathematics Letters*, 3(3):35–39, 1990.

[47] V. Turut, E. Celik, and M. Yigider. Multivariate Padé approximation for solving partial differential equations (PDE). *International Journal For Numerical Methods In Fluids*, 66(9):1159–1173, 2011.

[48] A. Wazwaz. A new algorithm for calculating Adomian polynomials for nonlinear operators. *Applied Mathematics and Computation*, 111:53–69, 2000.

[49] A. Wazwaz and S. El-Sayed. A new modification of the Adomian decomposition method for linear and nonlinear operators. *Applied Mathematics and Computation*, 122:393–405, 2001.

[50] P. Zhou. Explicit construction of multivariate Padé approximants. *Journal of Computational and Applied Mathematics*, 79:1–17, 1997.

[51] P. Zhou. Multivariate Padé Approximants Associated with Functional Relations. *Journal of Approximation Theory*, 93:201–230, 1998.