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Pre-Service Middle School Mathematics Teachers' Ways of Thinking, Ways of Understanding and Pedagogical Approaches in Problem-Solving Process

Ortaokul Matematik Öğretmeni Adaylarının Problem Çözme Sürecindeki Düşünme Yolları, Anlama Yolları ve Pedagojik Yaklaşımları

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Özet

Bu çalışmada ortaokul matematik öğretmeni adaylarının problem çözme bağlamındaki düşünme yolları, anlama yolları ve pedagojik açıklamaları ile bunlar arasındaki ilişkilerin DNR çerçevesi kapsamında araştırılması amaçlanmaktadır. Dört ortaokul matematik öğretmeni adayından nitel araştırma yöntemlerinden klinik görüşme yoluyla toplanan veriler açık ve eksensel kodlama yaklaşımı ile analiz edilmiştir. Analiz sonuçları matematik öğretmeni adaylarının problem çözme bağlamındaki düşünme yollarının iki kategoriye ayrıldığını göstermiştir. Ayrıca bu çalışma problem çözme bağlamındaki düşünme yollarının ve özellikle kanıt şemalarının ilköğretim matematik öğretmen adaylarının pedagojik açıklamalarında etkili bir rol oynadığını açığa çıkarmıştır.

Anahtar Kelimeler: Problem çözme, düşünme yolları, anlama yolları, ilköğretim matematik öğretmen adayları.

Abstract

The aim of this study is to investigate pre-service middle school mathematics teachers' *ways of thinking (WoT), ways of understanding (WoU) and pedagogical approaches as well as the relationships among them in the context of problem-solving within the DNR framework. In this qualitatively designed study, the data was collected through clinical interviews with four pre-service middle school mathematics teachers and analyzed through open and axial coding approach. The results of the analysis indicated that pre-service mathematics teachers' WoTs in the context of problem-solving were fell into two categories. This study also revealed that WoTs and particularly proof schemes in the context of problem-solving might play effective role in pre-service middle school mathematics teachers' pedagogical approaches.*

Keywords: Problem solving, ways of thinking, ways of understanding, pre-service mathematics teachers.

1. Introduction

Problem-solving is seen as the focus of school mathematics and the basic objectives of mathematics teaching. Polya (1945) indicates that the first duty of mathematics teachers is developing students' problem-solving abilities. Schoenfeld (1992) emphasized that teaching students Polya's heuristics does not guarantee that students solve the problems successfully because of the heuristics are restrictive but not prescriptive. On the other hand, it is asserted that knowing problem-solving strategies provides systematical steps, which increase the probability of success (Ramnarain, 2014). In addition to Polya's heuristics, Schoenfeld (1992) suggested that teaching students more problem-solving strategies particular to certain type of problems and teaching metacognitive strategies that help to know when these strategies are used improves student's beliefs. Schoenfeld (1992) developed a new model but it is pointed out that model is not implementable for explaining problem-solving process (English, Lesh and Fennewald, 2008).

By investigating the development of problem-solving research in the last 50 years, English et al. (2008) discuss effectiveness of suggestions proposed in the studies. They claim that the models accepted in mathematics teaching community isolate the problem-solving and there is not a considerable evolution in this area. English et al. (2008) also point out the failure in the problem-solving area and stress that it is time to try out alternative approaches. They assert that problem-solving is related to mathematical content, thinking and reasoning process, beliefs and contextual factors rather than learning the problem-solving strategies. Schoenfeld's (1992) beliefs component and problem-solving strategies are collected under ways of thinking (WoT) which is a component of Harel's (2008) triad of determinants in DNR framework. According to Harel (2008), the prominent thing is WoTs in problem-solving and problem-solving approaches present mathematical thinking processes of individuals. Lesh and Harel (2003) stated that problem solvers' WoTs about givens in a problem should be tested and considered repeatedly. Even students obtain the same solution; they can have different WoTs in problem-solving process (Viholainen, 2011). Harel (2008) states that since the indicator of success is regarded as answering the problem correctly in school mathematics, the WoTs associated with problem-solving are not paid enough attention. When looked the studies related with WoTs, it has seen that WoTs are collected under the habits of mind (HoM). Lim and Selden (2009) suggest that there are several terms (e.g., WoTs, cognitive dispositions) in mathematics education, which are similar or support each other and they all should be considered as HoM. Mathematical HoM refers to "think about mathematics the way mathematicians do" (Cuoco, Goldenberg and Mark, 1996, p.377). Lim and Selden (2009) state that Harel's WoTs concept underscores the thinking aspect of the mathematical HoM and Harel regards HoM as internalized WoTs. HoM as WoTs also represent individual's disposition to act (Lim, Morera and Tchoshanov, 2010). Harel (2008) views problem-solving as a mental act (MA) such as generalizing, interpreting and anticipation. In the triad of determinants, when problem-solving is centered as MA, ways of understanding (WoUs) and WoTs components are explaining problem-solving act. In this case, WoUs as products of the MA are solutions of the problems and WoTs as the characteristic of MA are problem-solving approaches (Harel, 2008). WoTs associated with problem-solving are determined based on problem-solving approaches, mathematical beliefs and proof schemes in DNR (Harel, 2008). In that case characteristic of problem-solving should involve not only problem-solving approaches but also mathematical beliefs and proof schemes.

Teachers play a significant role in developing students' problem-solving ability. Some studies (e.g., Harel and Lim, 2004) found that mathematics teachers ignore students' WoUs and WoTs. Moreover, it has seen that mathematics teachers try to impose their own WoTs to students and they are not open to alternative WoTs (Harel and Lim, 2004). Considering together the need for new approaches as suggested by recent studies on problem-solving and the role of teachers in developing students' problem-solving ability, the importance of mathematics teachers' WoTs becomes more salient. In this study middle school mathematics pre-service teachers' WoTs and WoUs associated with problem-solving and the relationship between WoTs and their pedagogical approaches are investigated in DNR framework.

In accordance with this purpose, two main research questions were addressed in our study:

1. What are the pre-service mathematics teachers' WoUs, WoTs and pedagogical approaches associated with problem-solving?

2. What are the interactions between pre-service mathematics teachers' WoTs and pedagogical approaches?

1.1. Theoretical framework (The DNR System)

The DNR system, introduced by Harel (2008), is a conceptual framework that deals with teaching and learning of mathematics. In this framework, Harel (2008) introduces the triad of determinants, MA-WoUs-WoTs, to analyze students' acts of a particular mental act. Harel and Sowder (1998), as a result of their study on the mental act of proving, achieved a triad involving the concepts of proving, proof and proof schemes. In this triad, the cognitive product of the proving act is named as proof, and the cognitive characteristics that the proving act is named as a proof scheme. This triad was then generalized into MA, WoUs and WoTs to be used for different MA. One advantage of the triad of determinants is that analysis is extremely fine grained (Lim, 2006). According to Lim (2006) The MA-WoU-WoT triad can be used for two purposes as for research and for learning and teaching. For research, this triad is a tool for determining what students understand of a particular topic based on their actions and statements, and analyzing their WoUs and WoTs (Lim, 2006). Via this triad for analyzing, researchers can divide students' thinking into manageable components and may be manage the complexity in students' thinking schemes and beliefs. The components of the triad of determinants in DNR framework are illustrated in Figure 1.

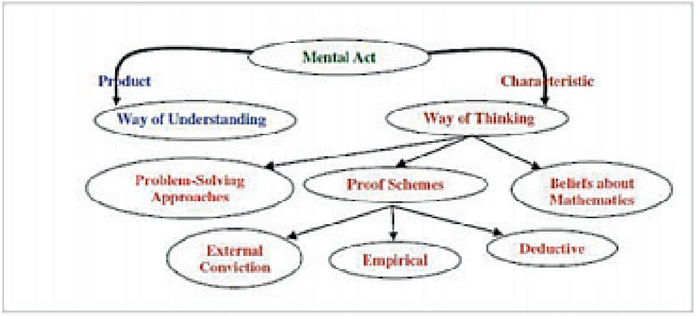


Figure 1. The triad of determinants (Harel, 2008, p. 493)

Harel and Sowder (1998) classified proof schemes into three: external , empirical and deductive. They stated that students with external proof scheme ground their justifications on similar proofs acquired previously, what teachers say, statements in the textbook, or pointless and unquestioned use of symbols. He stated that students with empirical proof scheme defend their justifications by generalizing limited number of examples or sample drawings, students with deductive proof scheme tend to use deductive reasoning and process based thinking in their justifications, and make use of definitions, theorems and axioms. Harel (2008) deals with problem-solving approaches as problem-solving strategies and he argues that problem-solving approaches refer to the cognitive characteristics of problem solving act and reflect the perspective of the problem solver. In aspect of beliefs about mathematics, Harel (2008) characterizes it as somebody's perspectives of what mathematics is, how it is created, and its intellectual or practical benefits.

2. Method

In this qualitatively designed study, the data were collected through clinical task-based interviews. The clinical interview is useful in that it provides in-depth information about individuals' WoUs and WoTs in the process of solving a mathematical problem (Koichu and Harel, 2007). The interviewer prompted the participants to read each problem aloud and explain their thinking when solving problems. This interview approach included open-ended questioning for the purpose of producing insights into the participants' problem-solving processes.

2.1. Participants

This study was carried out within the scope of an elective course (The Language of Mathematics) in Middle School Mathematics Teaching Program. The participants were selected based on purposive sampling method (Fraenkel and Wallen, 1996) from amongst the 22 students taking this course. Students were divided into four categories by their grade point average, and one volunteered pre-service teacher was selected randomly from each group.

2.2. Instrument

Problems which are used in this study are selected based on special function concept and the solution strategies required for solving them. The first problem is based on piecewise function, more specifically greatest integer function. This problem designed by the researchers was included in the study as it clearly shows the process of generalization by looking for a pattern. The second problem was adapted from the problem used by Harel and Lim (2004). Although this problem can be seen as a routine problem falling in the category of "mixture problems", it is difficult to perceive the functional relationship on which the problem is based, as supported by their results. As distinct from the first problem, the second problem entails proportional reasoning ability rather than pattern seeking. Finally, the third problem is a more challenging one based on absolute value function. Realizing the functional relationship in this problem requires geometric reasoning. In order to test the clarity of the instrument, a pilot interview with one of the pre-service middle-school teac-her was carried out. Problems used to collect in-depth data in clinical interviews are as follows:

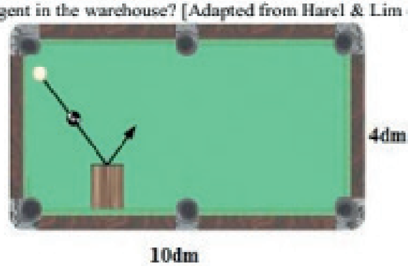
Problems	The modeling problems related to functions
Problem 1	In a parking lot, the vehicles are charged only the entrance fee of TRY 2 for parks less than one hour. For parks over one hour, the vehicles are charged double the extra period of parking per hour (the parking fee is not calculated per minute, but per hour). For parks over 12 hours, the additional parking fee is calculated per 12 hours. How would you define the relationship between the period of parking and the fee charged?
Problem 2	A chemical company received an order of 30 liters of liquid detergent. The liquid detergent is required to involve 5% bleach. However, in the warehouse of the company, tank A has liquid detergent with 20% bleach and tank B has liquid detergent with 2% bleach. a) Is it possible that this chemical company produces a solution that fulfils the customer instructions? Why? b) If the rate of bleach in the ordered detergent is increased or reduced, how would be the relationship between the liquid detergent produced and the use of detergents in tanks A and B? c) If different amounts of liquid detergent with 5% bleach are ordered, how would be the mathematical expression showing the relationship between the amount of order and the liquid detergent in the warehouse? [Adapted from Harel & Lim (2004)]
Problem 3	 You are trying to drive the black ball into the middle pocket on a miniature billiards table. The black ball is located at the point (1, 2). Your aim is to drive the ball into the pocket at the point (5, 4). Given that the coordinate of the point at which the ball hits before entering the pocket is (2, 1), write the equation of path of the ball. Is it possible that the ball enters the middle pocket? (The distances between pockets are equal.)

Figure 2. The problems used in clinical interviews

2.3. Data Analysis

The data were analyzed using an open and axial coding approach (Strauss and Corbin, 1998) that involved examining the interactions between the middle school pre-service teachers' WoUs-WoTs and their pedagogical approaches. WoUs-WoTs associated with problem-solving were characterized in terms of the framework of DNR (Harel, 2008). In this study the beliefs component of WoTs refer to the beliefs concerning problem-solving and concepts on which the problem is based. In the same vein, proof schemes are analyzed as the ways of justification that pre-service teachers use in the problem-solving process. Bell (1978) states that mathematical proof is concerned "not simply with the formal presentation of arguments, but with the student's own activity of arriving at conviction, of making verification, and of communicating convictions about results to others" (p. 48). The conceptual analysis of the data is used in order to describe WoUs-WoTs and pedagogical approaches that lead to verbal and written products.

3. Results

The results of this study reveal that the participants have two different WoTs. The participants with the first WoT (P1 and P2) tended to solve problems by the strategy of looking for patterns with special cases or by trial and error. The participants with the second WoT (P3 and P4) tended to sort out the problem through the mathematical knowledge related to mathematical concepts that the problem case evoked in their mind. It was seen that these WoTs and particularly proof schemes play an effective role in participants' pedagogical approaches. Although the participants who seemed to have weak function scheme had negative attitudes to this concept, they all had positive attitudes towards teaching these function problems in their professional lives. Furthermore, even they had difficulty in the problem-solving process; their positive attitude did not seem to change.

3.1. Pre-service teachers with the first WoT and their pedagogical approaches

The research findings suggest that two participants (P1 and P2) tended to obtain generalizations through special cases in the problem-solving process. They intended to generalize by specifying the functional relationships in the problem case. However, one of them resort to quantitative examples while the other used scenarios in order to specify the problem situation. For instance whereas one of the participants maintained her tendency to understand and justify through quantitative examples, the other participant tried to understand through different scenarios and to make a justification through associations with daily life. That is why the latter did not engage in the process of generalization until she was convinced about the problem situation.

Furthermore, in the first problem these participants were able to structure the concept with the generalization of relationships in problem-solving process. These participants, who tended to solve problems through generalizations, needed to get convinced about the problem case and preferred trial and error method for justification. They opted for convincing themselves by trying to justify the results without being prompted. Another characteristic that distinguishes these participants from the other two is that they formed a visual model, where appropriate, at the beginning of the stage of understanding the problem situation. They gave explanations that were parallel with their problem-solving strategies and proof schemes, when they were asked how they would guide students in the problem-solving process.

First pre-service teacher (P1): P1 was inclined to generalize within the problem scenario and to constitute relevant problem scenarios when new examples were required. Moreover, she needs to ensure that the problem situation was meaningful to her. For instance, P1 generalized the special cases for the first problem with the help of a diagram; however, tried to generalize by taking period as a discrete variable. In the problem case, P1 found charging per hour illogical and she resisted generalizing the results obtained by assigning rational values to the variable of period with the guidance of the interviewer. The participant obtained an inaccurate algebraic generalization by taking the period as a discrete variable, as shown below.

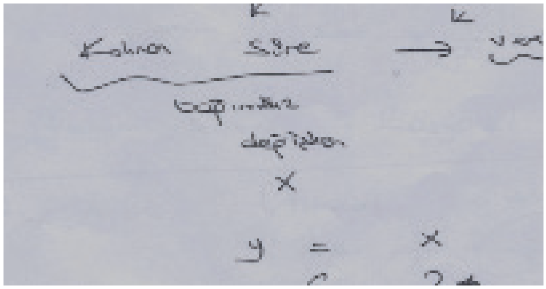


Figure 3. P1's first solution (WoUs) for the first problem

Some of the many statements showing that P1 was not able to make sense of the problem case are quoted below.

I: Extra period of parking. What does that mean?

P1: It means no additional charge is requested for 50 min. Then I will leave a the 59th minute. Does it make sense?

On the other hand, this participant was able to discern the variables and the relationships among them easily within the context of problem scenario and demonstrate them clearly with her expressions. Going far beyond, she was able to recall the concept of function based on the relationships in the problem, and grasped that the algebraic expression she obtained was a function.

I: What does the expression you wrote here mean to you?

P1: It recalls me the concept of function.

Although P1 was able to recall the concept of function in the problem easily, it was thought that she had weak function conception. P1 could not perform a function analysis based on algebraic expressions, but indicated that she comprehended the functional relationship in the problem scenario. Some of the statements that led the researchers to this conclusion were quoted below.

I: Why did you say this is a function?

P1: This is a function because there is a relationship between x and y. They vary dependently on each other.

I: For instance, is this a function (writes $x=y^2$ *on the paper)?*

P1: Yes, it is. You've given a value to y. But looking at this, I am not sure which one is dependent and which one is independent. You know, it could be seen easily in the problem case. There is a relationship between fee and time. But I couldn't understand it here. I can see the function clearly in the problem. But, to be honest, when given such an expression (shows the algebraic expression on the paper), I can't interpret it exactly.

The interviewer prompted P1 so that she could make sense of the charging procedure in the problem case. After getting convinced, she tried to rewrite the generalization; and although she failed to recall the greatest integer function, she was able to form the algebraic expression by generalizing the special cases. As seen below, P1 denoted the decimal part of number by "n" and subtracted it from the number.

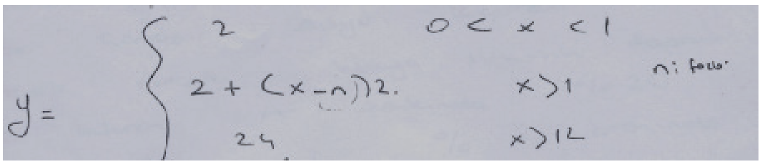


Figure 4. P1's second solution (WoUs) for the first problem

For the second question, P1 tried to understand the problem by drawing a visual model. P1 thought that the problem statement was incomplete and the scenario of the problem was nonsense also in this problem.

P1: 20% bleach in A, 2% bleach in B. (Participant tries to understand the problem by drawing a visual model.) Can this chemical company produce blach that fulfills the instructions? The question is incomplete…

I: Why do you think it is incomplete?

P1: I would add water into the mixture to reduce the rate from 20% to 5%.

I: You won't add water. This is a mixture of detergent and bleach..

P1: Then... At the moment no... This sounds nonsense to me...

After being prompted by interviewer to make sense of the problem case, the participant –like others – recalled the solution method, which she probably used in her previous experience, associated with mixture problems. When she was asked to think about the variables in the problem, she grasped the functional relationship and made proportional reasoning.

P1: For instance, suppose that this is not 5%, but increased to 100%.

I: You want it to be 100%? Then they ask for pure bleach.

P1: Yes. To see it more clearly. Then proportion of mixture will be 1, right? I don't reduce the weight of mixture here. So the proportion of the bleach increases…So only the amounts taken from A changes ... Then A should increase. Because the proportion of bleach in tank A is more….

I: What are the variables?

P1: The amounts taken from A and B are the independent variables. Then the amount of detergent formed should be the dependent variable.

P1 started solving the problem with the solution methods she recalled from similar problems and then she noticed the functional relationship in the problem through subsequent questioning. Even she said that the function was not explicit in the problem, she mentioned the function that she grasped in the problem case after she used the expression of "mixture problems".

I: On which concept or concepts is this problem based?
P1: As a concept, mixture problems. Not as clear as it is in the first question, but there is a function here in my opinion. To be honest, when I started solving the problem, I didn't think very reasonably. I directly wrote the formula given formerly. Writing this, I just applied the formula. I didn't think logically at that moment. Thinking it was a mixture problem, I did this. Actually I didn't reasoning.

For the final problem, P1 was the only student that grasped the functional relationship defined by path of the ball and recalled the absolute value function without any prompt.

I: It also asks whether the ball will enter the hole. Do you think it will enter?
P1: What now comes to my mind is similarity. Will it enter? Yes, it will. Let me first try to write. The ball proceeds both horizontally and vertically. Then it will proceed depending on y and on x. Assume that the path is t. I think I will write this as t = (x, y)... But here... the equation of path of the ball…Is this an absolute value?

In the problem case, P1 grasped that the graph of the functional relationship determined by the path of ball was a translation of the graph of absolute value function. Subsequently, in order to obtain the equations of lines in this graph, she tried to estimate the algebraic expression by taking some examples related to the translation process through the method of trial and error.

P1: Now, I'm going to translate it. I shift it one unit upwards. And two units to the right, yes. Let me see how it looks now? Now, 2 to 1. The ball is here now and the pocket is there at (5, 4). Yes, (5, 4). Because we shifted it two units to the right. If I translate two units from 3, it comes to 5. This will be at 4. The pocket is here. The ball hits here, but we should also consider the initial location of the ball, right? It was at (1, 2). The ball is here, then it will hit here, and enter there.

The justification way of this participant is parallel to the strategies she employed. Therefore, it was thought that her proof scheme was empirical proof scheme.

P1: There is a relationship between the two, and I said this is a function because I defined y to be dependent on x. For each value I assign to x, y takes a different value. That is why it is a function. For instance, suppose x is 3. For instance, when x is 1, it makes 2 liras. If x is 2, it makes 4 liras. Nothing changes here. When it is 2, it doesn't have any other value. I realized that. That is why I called it a function.

P1's pedagogical approaches were compatible with the strategies she used. She said that she would prompt students to develop scenarios and make students use visual models in the problem-solving process. In the second problem, P1 implied that she would manipulate the scenario by reflecting the relationship in the problem and using concrete materials. The excerpts below also illustrate that P1's pedagogical approach reflects her WoTs.

I: How would you prompt your students to solve this problem?

P1: Maybe I can exemplify it by something else…I mean... Let's assume there's a mixture of cotton and iron here… Iron balls are in the cotton (creates a visual models). There're three iron ball in A, there's one iron ball in B. How many iron balls should I put in A so that there're two in B. How many iron balls should be added in A for making a mixture that has the same proportion of components?

Second pre-service teacher (P2): P2 mostly used strategies such as using special examples and looking for patterns to understand and generalize problems. For instance, just like P1, this participant tried to generalize the first problem with the help of a diagram showing the special cases, but could obtain an incorrect algebraic generalization expression. The reason of obtaining an incorrect expression was her tendency to take the period as a discrete variable. The interview with P2 led the researchers to think that her learning of function concept was weak. She recalled the concept of function with the algebraic expression in the form of $y = f(x)$.

I: What's a function?

P2: Suppose the function f(x) = y. The function of y can take different values according to the x variable. I can define function as a system.

I: OK... In all cases where the dependent variable changes according to the independent variable, is there a function?

P2: Is there cases where it is not a function? It seems like a function to me now. But is it always? I'm not sure at the moment, but it may be. x is a dependent variable, there is an independent variable, then yes.

Based on special cases, P2 looked for a pattern, but she obtained an incorrect generalization. After this mistake – made by three out of four participants-, P2 drew a graph when she failed to reach the algebraic representation in the first problem. Moreover, she obtained the graph representation of the functional relationship in the problem case by determining the locus of special cases she took, as seen below.

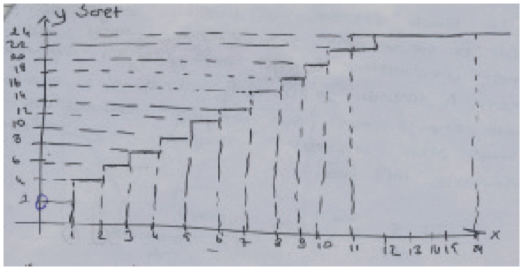


Figure 5. P2's solution (WoUs) for the first problem

In the second problem P2 went through the generalization process rapidly and obtained the algebraic expression by assigning a variable. When prompted to make reasoning, she used a visual model and tried to justify through the method of trial and error by choosing special cases. Thus, it is thought that P2's proof scheme is empirical. P2 had difficulty in understanding the third problem. In this case, she tended strategies such as using the variable as a label, using the formulas she recalled and using symbolic manipulations. In the problem statement, she initially interpreted the expression of "path of the ball" as the length of the path and accordingly calculated the distances and summed them up. Reminded by the interviewer that the equation of path of the ball was asked in the problem, P2 obtained the equation in two parts. When she was asked to express it with only a single-equation, she suggested summing up the linear equations algebraically.

P2: ... Linear equations, equations of the route taken by lines… I mean as the ball takes these two routes, I need to sum up the equations on each side, I guess. For, there are two equations and two pats.

The ways of justification used by this participant reveal that she has empirical proof scheme through such methods as giving examples, replacement and crosschecking. She tended to show the accuracy of her thoughts or operations through special cases, using expressions such as "for instance, suppose that, replacing, crosschecking".

P2's pedagogical approaches, like P1's, were compatible with the strategies she used. For example, in the first problem, she said that she would make students use strategies like looking for patterns or drawing diagram. P2 also reflected empirical proof scheme strongly in her pedagogical approaches.

I: OK, how would you prompt your student to solve this problem?

P2: I would ask her to read the question and whether she understood it. What did she understand? Let's suppose she didn't understand this part, for instance "over 1 hour". Extra charge for cars parking over 12 hours. She has to understand the charge remains same. If she doesn't understand it, I may ask her to reread the question. When guiding her, "yes, you found 2x=y" I'd say. How does this change? If it is the case for parks until 12 hours, does it yield *the same result later? Then 13th hour would be 26, but not the same charge that applies to 24 hours…*

I: What would you do if the student did not obtain the expression 2x=y?

P2: I think I'd try many other ways. I'd ask her to draw a table. I don't know, I'd ask her to write park hours from 1 to 12, and then make a calculation.

3.2. Pre-service teachers with the second WoT and their pedagogical approaches

The participants with this WoT (P3 and P4) tended to solve the problems by recalling the mathematical concepts on which the problem was based or the way of solution they had already known. It was observed that these participants needed to associate the problem situation with mathematical concepts, and used algebraic strategies such as writing equations, assigning variables and using formulas. It was seen that the participants' proof schemes differed according to their conceptual level on which the problem was based. For instance, the participant with this WoT who have strong conceptual level was thought to have the deductive proof scheme, while the one who have weak conceptual level scheme was thought to have the external proof scheme. It was thought that their pedagogical approaches were different from both the participants' with the first WoT and each other based on their proof schemes and the conceptual level of the function concept.

Third pre-service teacher (P3): P3 was tended to solve all problems by using the mathematical knowledge related to the mathematical concepts that the problem case evoked in her mind, and dominantly used to algebraic expressions. To exemplify, in the first problem, she assigned variables in the process of understanding the problem, and tried to write the generalization in the form of piecewise function. P3 failed to obtain a correct algebraic generalization although she was the only participant who was aware that the period was a continuous variable. P3 recalled the concepts of function and piecewise function in a short time, but could not recall the greatest integer function. That is why she probably failed to reach an algebraic generalization. P3 also demonstrated that she had a strong conception of function. In her justifications, P3 frequently revealed that she had deductive proof scheme.

I: Is every relationship a function?

P3: No, every relationship is not a function. In order it to be a function (drawing in the air a correspondence between two sets with her hands), there needs to be a certain domain and a range. I mean it should be defined in a set. For instance, each element in the domain should correspond to exactly one element in the range.

I: OK, well, when deciding whether there is a function here, did you check what you said right now?

P3: I didn't think of that detail. I just thought there's a relationship. Now looking from that side, one element in the domain should correspond to only *one element in the range, there shouldn't be two images. Here, when t is greater than 12, its image would be 24. Namely, when the period is smaller than 1, the fee would be 2 liras. Then it is equal to 2 liras. …Then it fulfills all criteria. We can say it's a function.*

Also P3 perceived the second problem as a routine problem, like the other participants who all handled it as a mixture problem, and tried to use a formula. When the formula she chose did not work, she decided to use the proportion. When the interviewer prompted her to make reasoning about the problem, P3 made proportional reasoning.

P3: The amount of bleach in A is greater. Then, if a higher proportion is asked, the amount of A would increase and B would reduce accordingly. To reduce the rate, we have to take less from tank A and take more from tank B. When we reduce the rate, it is needed to take more from tank B.

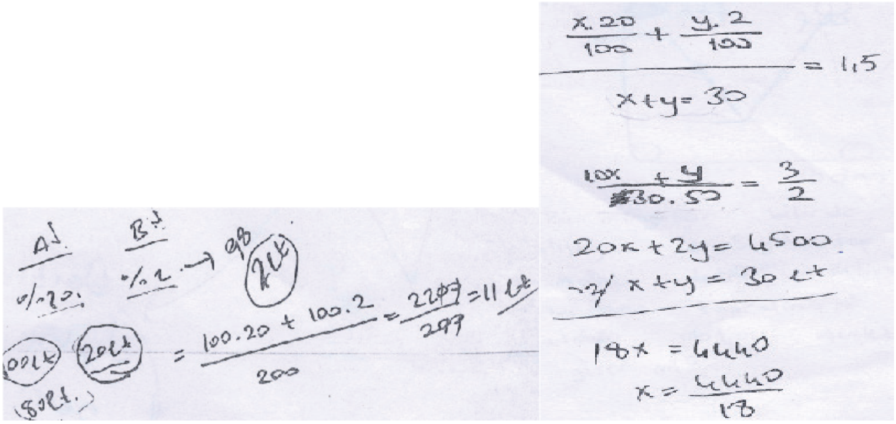


Figure 6. P3's solution (WoUs) for the second problem

In the case of the third problem, P3 failed to make sense of the problem because she interpreted the path of the ball as distance. Despite the prompt, she did not tend to try understanding the problem and make any progress to solving the problem.

P3 suggested many effective teaching strategies parallel with her strong function conception and deductive proof scheme. For example in the first problem, P3 was able to recall easily the concept of function and activities related to this concept.

I: How would you prompt your students to solve this problem?
P3: Well, I'd first ask them to read and explain the problem and note whatever they understood. I may also divide them into groups and ask them to discuss the problem. Then, I'd ask one member from each group what they obtained. What is asked in the question? Tell me what you understood. Ideas are shared and discussed in the classroom, and the problem is understood

thoroughly. If students don't think correctly, I'd ask questions to guide them. Then I may try to draw their attention to the relationship between period of parking and fee.

I: Suppose that the student doesn't see that relationship.

P3: An activity… Can't see the relationship… I can maybe arrange an activity about functions… In any set we chose… I can think of an activity now… Suppose there's a box… When you add numbers, there appears something else… Or an activity board may be developed… The changing in two quantities in function…

Fourth pre-service teacher (P4): P4 was the participant who failed to solve the problems unless she recalled the mathematical concepts underlying the problem. However, she was able to use symbolic manipulations effectively when she recalled the underlying concepts. For example, she resisted to generalize the variable of period as rational values in the first problem, despite the prompt. She obtained an incorrect algebraic expression and then expressed it as a piecewise function. When she recalled the absolute value function from this expression, she tried to write the generalization by using the absolute value.

P4: …Would it be nonsense if I write as an absolute value? No, it wouldn't.

P4, throughout the interview, used the expression of special functions and mentioned the absolute value function.

I: You frequently mention special functions. What are they?
P4: Now an absolute value function comes to my mind. What else?...
I: What does absolute value function evoke in your mind?
P4: The first problem is like that, I guess.
I: What comes to your mind in general?
P4: not $y= f(x)$, but such as $f(y) = |x|$.

P4 revealed her WoUs and WoTs more clearly in the second and third problems. For instance, when she was prompted to make reasoning, she used symbolic manipulations by assigning many variables such as x, y, n, k, A, B, but she was not obtained the generalization. The third problem is based on the concept of absolute value function that P4 recalled during the interview. Only P4 was able to solve this problem completely and obtained the equation. It was thought that P4 solved this problem successfully because she was able to recall the concept underlying the problem besides using symbolic manipulations effectively.

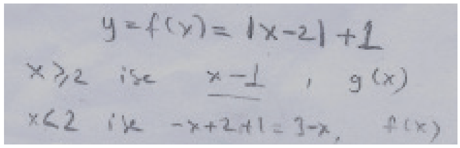


Figure 7. P4's solution (WoUs) for the third problem

P4 used the knowledge she remembered from courses or other external resources. Moreover, she tried to arrive a judgment by finding a clue from intervie-wer's questions. As a result of her responses depending on an authority, it was thought that she has the external proof scheme.

I: How can we write it more clearly?
P4: I don't remember. I remember this expression, but only this one…
I: You wanted to sum up the two lines. In the set of lines, is there addition?
P4: I get it. No, I know only vector sum.
I: Then, would it be correct to name it as the total route?
P4: I get it. No, it's not correct.
I: The route in the problem consists of the combination of two routes.
P4: I have to combine the routes, I think. Should I write a composite function?

P4's pedagogical approaches also were compatible with the strategies she used i.e. symbolic manipulation and external proof scheme. P4 failed to offer teaching strategies specific to the problem. For example in the second problem she mentioned that she could not make students solve the problem without using algebraic expressions. It is thought that the most important reason of this might be her external proof scheme.

I: Would you have your students solve this problem?
P4: It seems I have to. (Laughs)
I: Why do you think you have to?
P4: So that they learn the topic.
I: What do you think is the topic?
P4: Percent and mixture problems. It will be a hard task being a teacher. Now I realize that. (Laughs) I don't want to support rote learning, but by doing so we incite them to rote learning. They don't understand the rationale.
I: Why do you think so?
P4: I can explain it directly based on the algebraic expression. I can't tell it to a student who doesn't know algebraic expression.
I: How would you prompt them to understand?
P4: Unfortunately, I'm so attached to algebraic expressions that nothing else comes to my mind. We were able to solve these problems before knowing algebraic expressions, but I don't remember what we used to do (Laughs) I am so dependent on them that I want to solve all through algebraic expressions.

4. Summary and Discussion

This study concentrated on pre-service middle school mathematics teachers' WoUs and WoTs in problem-solving process and relationship between WoTs and pedagogical approaches. To consider problem-solving approaches as WoTs associated with problem-solving might be limited. As Schoenfeld (1992) proposed, problem-solving is a complicated process. Triad of determinants provides us a fine-grained analysis for determining WoTs and managing the parts of WoTs (Lim, 2006).

The study has revealed that the pre-service teachers present various WoUs. Although participants with the same WoTs generated different WoUs, participants with the first WoTs presented more WoUs than the others. Lim (2006) revealed that students' ways of understanding and thinking are related to each other, and that students with desirable WoTs have versatile and comprehensive WoUs regarding equations and inequalities. Lim also found that students that do not have desirable WoTs have limited WoUs.

The study has revealed that the pre-service teachers adopt two distinct WoTs. This result is parallel with the study conducted by Lim et al. (2010). They also classified pre-service teachers' problem-solving dispositions into two categories as impulsive and analytic disposition. Whereas the studies conducted in DNR framework for determining WoTs distinguish desirable WoTs from undesirable WoTs (e.g., Lim, 2006), in this study there is not a distinction between WoTs in the sense of desirable/undesirable.

It was thought that the characteristics of these WoTs, distinguished in this study, were similar to the characteristics of formal and informal thinking ways defined in the literature. The first WoT in this study is consistent with the informal thinking, and the second one is similar to the formal thinking. To exemplify, Viholainen (2011) found that informal thinkers tried to solve the problem with different strategies when the strategy they selected did not work; however, formal thinkers were inclined to use only conceptual knowledge such as definitions, theorems and rules and failed to develop non-formal strategies. In the present study as well, participants with the second WoT tried to solve the problems by using algebraic strategies and were not able to progress in problem-solving when they failed to recall the concepts on which the problems were based. On the other hand, participants with the first WoT were able to use different strategies and even constructed the concept underlying the problem in some cases. Nevertheless, because there are no generally accepted views with regard to the presence and forms of various mental structures in the literature, these WoTs were not named as formal and informal in this study. There is need for further studies to investigate the relationship among different types of thinking, mental structures and conceptual learning process.

It was seen that proof schemes have a critical role for determining WoTs. The participants with the first WoT seemed to have empirical proof scheme while the other participants were thought to have deductive or external proof scheme. Şengül and Güner (2013) found that all proof scheme types (external, analytical,empirical) are used by pre-service mathematics teachers. Also in a study in the context of function concept and proof schemes, pre-service mathematics teachers did not vary in terms of used proof schemes, some of them did not use empirical schemes any time (İskenderoğlu, Baki and İskenderoğlu, 2010). While there are not explicit differences between the participants with the first WoT (e.g., problem-solving strategies, proof schemes, pedagogical approaches), there are critical differences between the participants with the second WoT (e.g., level of conceptual schemes, proof schemes, pedagogical approaches). Therefore, it was thought that the participants had different proof schemes due to their different conceptual levels. In order to elaborate the relationship between concept formation and proof schemes, there is a need to know participants' concept schemes in a detailed way. Consequently, as underlined by English et al (2008), in the literature there is a need to handle and investigate together the problem-solving process and the conceptual development process. The present study did not aim to investigate the participants' conceptual level, but the concepts were questioned in the problem-solving process parallel with the purposes of the study. The relevant findings support the need emphasized by English et al (2008).

The interpretation of the results with regard to pedagogical approaches suggests that there is a relationship between pedagogical approaches and WoTs. The pre-service teachers were inclined to reflect the strategies they used in their pedagogical approaches. The findings of Harel and Lim (2004) also reveal that in-service teacher's WoTs and pedagogical approaches are interrelated. In this vein, Harel and Lim mentioned that pre-service teachers' WoTs should be developed and that all components of this necessity should be integrated into teacher-training system. Another factor that affects the pedagogical approaches of pre-service teachers is concept schemes. For instance P3 seemed to have a stronger concept scheme. She implied that, when providing guidance to students, she might use diverse and more effective strategies than the ones she had used in the problem-solving process. Among the strategies she suggested were different representations such as function machine. This result is parallel with Watson and Harel (2013) results. Watson and Harel (2013) concluded that mathematics teachers' knowledge of function had significant effects on their teaching at primary and middle school level. In their study, one of the teachers with good knowledge of function had resort to multiple representations, designing an activity that involved a function machine. A further study based on the findings of this study may focus on the development of WoTs used in problem-solving in learning process of a concept. A study conducted with a larger sample may present significant pedagogical implications for the development of students' problem-solving skills.

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