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Order regularity for Birkhoff interpolation with lacunary polynomials

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Abstract

In this short paper we present sufficient conditions for the order regularity problem in Birkhoff interpolation with lacunary polynomials. These conditions are a generalization of the Atkinson-Sharma theorem.

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1 Introduction

An *algebraic Birkhoff interpolation problem* [7] is defined by a triplet (X, E, C) where $E = (e_{ij})_{i=1,j=0}^{m,\ \ q}$ is an interpolation matrix with $e_{ij} \in \{0, 1\}$ and exactly n ones, $X = (x_1, \ldots, x_m)$ is an m-tuple of distinct real points (nodes) and $C = (c_{ij} : e_{ij} = 1)$ is a system of n real values. The objective of the algebraic Birkhoff interpolation problem (X, E, C) is to find a polynomial $p(x)$ of degree less than n that satisfies

$$D^{(j)}p(x_i) = c_{ij}, \quad e_{ij} = 1. \tag{1}$$

A *degree sequence* is an n-tuple of integers $K = (k_1, \ldots, k_n)$ such that $0 \leq k_1 < \cdots < k_n$. A *K-algebraic Birkhoff interpolation problem* [4] is defined

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by a quartet (X, E, K, C); in this more general problem, the goal is to find a lacunary polynomial $p(x) = \sum_{i=1}^{n} a_i \frac{x^{k_i}}{k_i!}$ that satisfies (1).

The interpolation conditions determine a linear system. The *generalized Vandermonde matrix*

$$V(X, E, K) = \left(\frac{x_i^{k_1-j}}{(k_1-j)!}, \ldots, \frac{x_i^{k_n-j}}{(k_n-j)!};\ e_{ij} = 1 \right)$$

is the coefficient matrix of the system, where we have adopted a certain order of the elements $e_{ij} = 1$ and we agree on $1/k! = 0$ if $k < 0$.

Obviously, the interpolation problem (X, E, K, C) has a unique solution if and only if the matrix $V(X, E, K)$ is regular. An interpolation matrix E is said to be *conditionally K-regular* if there exists a node system X such that $V(X, E, K)$ is regular. When $V(X, E, K)$ is regular for every node system $X = (x_1, \ldots, x_m)$ such that $a \leq x_1 < \cdots < x_m \leq b$, E is said to be *order K-regular* on $[a, b]$; otherwise, E is *order K-singular* on $[a, b]$.

An interpolation matrix is said to be *normal* if it has as many ones as columns. For a normal matrix $E = (e_{ij})_{i=1,j=0}^{m,\, n-1}$, the *Pólya condition* is

$$\sum_{k=0}^{j} \sum_{i=1}^{m} e_{ik} \geq j + 1, \quad j = 0, 1, \ldots, n-1.$$

In *algebraic interpolation*, the degree system is $K = (0, 1, \ldots, n-1)$. In this case the conditional regularity of an interpolation matrix can be characterized by the Pólya condition [2], and the Atkinson-Sharma theorem [1] provides us with very general sufficient conditions to decide on order regularity. Moreover, the regularity of the generalized Vandermonde matrix is not affected by affine transformations of the node system, hence the property of order regularity does not depend on the choice of the interval.

In the K-algebraic case, the situation is quite different. In fact, if we do not impose any additional condition on the interval and the degree sequence, even Lagrange interpolation turns out to be order K-singular.

If no constraints on the interval are imposed, that is, in the case $-\infty \leq x_1 < \cdots < x_m \leq +\infty$, it is possible to characterize order K-regularity of Lagrange and Hermite-Sylvester interpolation [6], [3]. Nevertheless, it seems unlikely that more general results can be obtained in this direction. In contrast, if we prevent the nodes from changing sign, then a full generalization of the Atkinson-Sharma theorem to the K-algebraic problem can be made. A preliminary study restricted to the case of two-node problems can be found in [5].

2 Definitions and previous results

Next we present several definitions and results stated in [4] that are relevant to the present work.

The *derivative sequence* of an interpolation matrix E is the nondecreasing sequence $Q(E) = (q_1, \dots, q_n)$ whose elements are the derivative orders specified in E. For instance, in the interpolation matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{2}$$

the derivative sequence is $Q(E) = (0, 0, 1, 5, 6, 7)$. An interpolation matrix E with n ones satisfies the *Pólya K-condition* with respect to a degree sequence $K = (k_1, \dots, k_n)$ if $q_i \le k_i$, $i = 1, \dots, n$; in this case, we write $Q(E) \le K$.

Theorem 2.1 *The interpolation matrix E is conditionally K-regular if and only if E satisfies the* Pólya K-condition.

Lemma 2.2 *Let $K = (k_1, \dots, k_n)$ and $Q = (q_1, \dots, q_n)$ be nondecreasing sequences and let K', Q' represent the nondecreasing sequences obtained from K and Q after inserting, in the right place, a new element r. If $Q \le K$, then $Q' \le K'$.*

We will also need the concept of supported sequence and the Atkinson-Sharma theorem. Roughly speaking, a *sequence* in an interpolation matrix E is a maximal block of ones placed in a row of E. More precisely, a sequence $S_{ij}^r(E)$ is a set of r elements in E that $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+r-1} = 1$ and such that the elements $e_{i,j-1}$ and $e_{i,j+r}$, if they exist, are both zero. The sequence $S_{ij}^r(E)$ is *odd* if r is odd. The sequence $S_{ij}^r(E)$ is *upper supported* if there exists an element $e_{i_1,j_1} = 1$ such that $i_1 < i$ and $j_1 < j$; in this case, we say that the element $e_{i_1,j_1} = 1$ is an *upper support* of E. When there exists an element $e_{i_2,j_2} = 1$ such that $i < i_2$ and $j_2 < j$, the sequence $S_{ij}^r(E)$ is *lower supported* and we call e_{i_2,j_2} a *lower support*. The sequence $S_{ij}^r(E)$ is said to be *supported* if it has upper and lower supports.

Theorem 2.3 (Atkinson-Sharma) *Let E be a normal interpolation matrix. If E satisfies the Pólya condition and contains no odd supported sequences, then E is order regular for algebraic interpolation.*

3 Main result: a generalization of the Atkinson-Sharma theorem

Let E be an interpolation matrix with n ones, $K = (k_1, \dots, k_n)$ an arbitrary degree sequence and $K^* = (k_1^*, \dots, k_r^*)$ the degree sequence made up of the

$r = k_n - n + 1$ elements of $K_{k_n} = (0, 1, \ldots, k_n)$ which are not in K. We say that a sequence $S_{ij}^r(E)$ is *upper K-supported* when it has a lower support and, in addition, it has either an upper support or $k_1^* < j$. Let $Q_1(E) = \left(q_1^1, \ldots, q_{n_1}^1\right)$ denote the derivative sequence of the first row of E. We say that the interpolation matrix E is *upper K-inclusive* if the elements q_i^1 are all in K. For example, the interpolation matrix

$$E = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

has no supported sequences. If we consider the degree sequence $K = (0, 2, 3, 5)$, we get $K^* = (1, 4)$ and $k_1^* = 1$. Therefore, the sequence in the first row $S_{12}^2(E)$ is upper K-supported. We also note that $Q_1(E) = (2, 3)$; hence E satisfies the upper K-inclusive condition. We are now in a position to state and prove the main results.

Theorem 3.1 *Let E be an interpolation matrix that satisfies the Pólya K-condition and the upper K-inclusive property. If E contains no odd upper K-supported sequences, then E is order K-regular on $[0, +\infty)$.*

Proof. If $k_n = n - 1$, we are in the algebraic case. In this case the Pólya K-condition is equivalent to the classic Pólya condition [4] and K^* has no elements, so E has no odd supported sequences. From Theorem 2.3, it follows that E is order regular on $\mathbb{R}$. In the proper K-algebraic case, it is $k_n > n-1$ and K^* has $r = k_n - n + 1 > 0$ elements. Let $X = (x_1, \ldots, x_m)$ be a node system. We first consider the case $0 < x_1 < \cdots < x_m$. We form an interpolation matrix

$$E^* = \left(\frac{F_1}{\bar{E}} \right)$$

with $m + 1$ rows and $k_n + 1$ columns where F_1 is a row which has ones in the places specified by K^* and $\bar{E}$ is an $m \times (k_n + 1)$ matrix that has ones in the same places as E. We take the node system $y = (0, x_1, \ldots, x_m)$ and the following basis for the space $\mathcal{P}_{k_n}$ of polynomials of degree $\leq k_n$

$$\frac{x^{k_1^*}}{k_1^*!}, \ldots, \frac{x^{k_r^*}}{k_r^*!}, \frac{x^{k_1}}{k_1!}, \ldots, \frac{x^{k_n}}{k_n!}. \tag{3}$$

We also arrange the elements $e_{ij}^* = 1$ of E^* according to the lexicographic order of the pairs (i, j) with prevalence of the first index. Then, the triplet (Y, E^*, K_{k_n}) defines an algebraic Birkhoff problem whose generalized Vandermonde matrix has the following structure

$$V\left(Y, E^*, K_{k_n}\right) = \left(\begin{array}{c|c} I_r & O_{r \times n} \\ \hline M & V\left(X, E, K\right) \end{array} \right) \tag{4}$$

where I_r is the identity matrix of order r, $O_{r\times n}$ is a zero matrix, M is an $n\times r$ matrix and $V\left(X,E,K\right)$ is the generalized Vandermonde matrix of the triplet $\left(X,E,K\right)$. If we apply r times Lemma 2.2, it results that E^* satisfies the Pólya condition. Now, let us suppose that E^* has an odd supported sequence $S^l_{ij}(E^*)$. If $S^l_{ij}(E^*)$ has an upper support in row 1, then we get a sequence $S^l_{i-1,j}(E)$ which is upper K-supported; if no upper support exists for $S^l_{ij}(E^*)$ in row 1, then $S^l_{i-1,j}(E)$ is an upper supported sequence and, in consequence, it is also K-supported. To complete this case, we observe that E has no odd K-supported sequences; therefore, E^* has no odd supported sequences either, and by Theorem 2.3, the generalized Vandermonde matrix $V\left(Y,E^*,K_{k_n}\right)$ is regular. From the structure of $V\left(Y,E^*,K_{k_n}\right)$, it follows that $V\left(X,E,K\right)$ is also regular.

Now, we consider the case $x_1=0$. The node system is $X=(0,x_2,\dots,x_m)$, $0<x_2<\cdots<x_m$. For this case we construct an interpolation matrix $E^*=\left(e^*_{ij}\right)$ with m rows and k_n+1 columns that has ones in the same places as E and, in addition, $e^*_{1,k^*_j}=1$, $j=1,\dots,r$. Let E_1 denote the first row of E. As E is upper K-inclusive, E_1 has no common elements with K^* and E^* has exactly k_n+1 ones. We also note that the derivative system $Q(E^*)$ can be obtained from $Q(E)$ by adding the r new elements $k^*_1,\dots,k^*_n$. We arrange the basis of $\mathcal{P}_{k_n}$ as in (3) and set out the elements $e^*_{ij}=1$ placing first $e_{1,k^*_1},\dots,e_{1,k^*_r}$ and then the rest in a given order. With this arrangement, the generalized Vandermonde matrix $V\left(X,E^*,K_{k_n}\right)$ has the block structure shown in (4). As in the previous case, the triplet $\left(X,E^*,K_{k_n}\right)$ defines an algebraic problem that satisfies the Pólya condition. If we suppose that there exists an odd supported sequence $S^l_{ij}(E^*)$, then we get an odd upper K-supported sequence $S^l_{ij}(E)$. In consequence, no odd supported sequences exist in E^* and $V\left(X,E,K\right)$ is regular. $\square$

An analogous result can be established for order K-regularity on $(-\infty,0]$. To this end, we say that the sequence $S^r_{ij}(E)$ is *lower K-supported* when it has an upper support and it has either a lower support or $k^*_1<j$. We also represent by $Q_m(E)=\left(q^m_1,\dots,q^m_{n_m}\right)$ the derivative sequence of the last row of E, and say that the interpolation matrix E is *lower K-inclusive* if all the elements q^m_i are in K.

Theorem 3.2 *Let E be a lower K-inclusive matrix that satisfies the Pólya K-condition. If E contains no odd lower K-supported sequences, then E is order K-regular on* $(-\infty,0]$.

Proof. If we proceed as in the proof of Theorem 3.1, only a few obvious changes must be made. In the case $x_1<\cdots<x_n<0$, we build the matrix

$$E^*=\left(\frac{\bar{E}}{F_{m+1}}\right)$$

where F_{m+1} is a row which has ones in places $k_1^*, \ldots, k_r^*$. We also take the basis of $\mathcal{P}_{k_n}$

$$\frac{x^{k_1}}{k_1!}, \ldots, \frac{x^{k_n}}{k_n!}, \frac{x^{k_1^*}}{k_1^*!}, \ldots, \frac{x^{k_r^*}}{k_r^*!}, \tag{5}$$

the node system $Y = (x_1, \ldots, x_n, 0)$ and arrange the elements $e_{ij}^* = 1$ according to the lexicographic order of the pairs (i, j) with prevalence of the first index. We get the generalized Vandermonde matrix

$$V(Y, E^*, K_{k_n}) = \left(\begin{array}{c|c} V(X, E, K) & O_{n\times r} \\ \hline M & I_r \end{array}\right) \tag{6}$$

The interpolation matrix E^* is normal and satisfies the Pólya condition; moreover, as E has no odd lower K-supported sequences, E^* has no odd supported sequences either, and we can conclude that $V(X, E, K)$ is regular. In the second case $x_1 < \cdots < x_n = 0$, the node system is $X = (x_1, \ldots, x_{n-1}, 0)$. We now form the $m \times (k_n + 1)$ extended matrix E^* by adding r ones to the last row of E in positions $k_1^*, \ldots, k_n^*$; the lower K-inclusive property guarantees the correctness of the process. We take the basis of $\mathcal{P}_{k_n}$ as in (5) and choose an arrangement of the elements $e_{ij}^* = 1$ such that the elements $e_{m,k_1^*}^*, \ldots,\ e_{m,k_r^*}^*$ are placed at the end. The generalized Vandermonde matrix $V(X, E^*, K_{k_n})$ thus obtained has the block structure shown in (6). As in the previous case, we know that E satisfies the Pólya K-condition and has no odd lower K-supported sequences; hence, it results that E^* satisfies the conditions of Theorem 2.3 and we can conclude that $V(X, E, K)$ is regular. $\square$

Finally, it is worth noting that the K-inclusive properties have not been used in the cases corresponding to nonzero nodes. Hence, the following results hold.

Theorem 3.3 *Let E be an interpolation matrix that satisfies the Pólya K-condition. If E contains no odd upper K-supported sequences, then E is order K-regular on* $(0, +\infty)$.

Theorem 3.4 *Let E be an interpolation matrix that satisfies the Pólya K-condition. If E contains no odd lower K-supported sequences, then E is order K-regular on* $(-\infty, 0)$.

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