Mathematica Aeterna, Vol. 1, 2011, no. 03, 137 - 148

Series that can be differentiated term-wise m times if the function is m-smooth

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Abstract

Let $f \in C^m(-\pi, \pi)$, where $m > 0$ is an integer. An algorithm is proposed for representing f as a convergent series which admits m times term-wise differentiation. This algorithm is illustrated by numerical examples. It can be used, for example, for acceleration of convergence of Fourier series. The algorithm is generalized to the case when f is piecewise-$C^m(-\pi, \pi)$ function with known positions of finitely many jump discontinuities and the sizes of the jumps and to the case when these positions and the sizes of the jumps are unknown. A jump discontinuity point s is a point at which at least one of the quantities $d_j := f^{(j)}(s-0) - f^{(j)}(s+0) \neq 0$, where $0 \leq j \leq m$.

Mathematics Subject Classification: 40A30, 42A20, 65B10

Keywords: Series; convergence rates; term-wise differentiation

1 Introduction. Formulation of the results.

Suppose that $f \in C^m(-\pi, \pi)$, where $m > 0$ is an integer. If one expands f into 2π-periodic Fourier series, then, in general, the extended 2π-periodic function has jump discontinuities at the points π and $-\pi$, and the corresponding Fourier series of this function f

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx} \tag{1}$$

has Fourier coefficients

$$f_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{2}$$

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of the order $O(n^{-1})$, and cannot be term-wise differentiated $m > 0$ times.

The goal in this paper is to find a series, representing $f \in C^m(T)$ on the interval $T := [-\pi, \pi]$ and such that it can be m times term-wise differentiated:

$$f^{(j)} = \sum_n c_n \phi_n^{(j)}(x), \qquad 0 \le j \le m. \tag{3}$$

This statement of the problem apparently is new, but it has close relation with classical problems, for example, with methods of acceleration of convergence of Fourier series ([7]), with estimating a function from a truncated Fourier series ([1], [2], [4]), with stable differentiation of piecewise-smooth functions and edge detection ([8], [9], pp. 197-217).

Therefore the problem we have stated is of interest both theoretically and in applications.

A related basic result in analysis is a theorem of A.Haar, which says that the Fourier-Haar series converges uniformly for any continuous on $[0, 1]$ function f to this function ([5]).

Our goal can be achieved in many ways. Let us propose a simple way that can be used numerically and is similar to one of the methods of acceleration of the rate of decay of the Fourier coefficients ([7]).

Step 1.

Choose a polynomial $P_{2m+1}(x)$ of degree $2m + 1$,

$$P_{2m+1}(x) = \sum_{k=0}^{2m+1} a_k x^k \tag{4}$$

such that

$$f^{(j)}(\pi) = P_{2m+1}^{(j)}(\pi), \qquad f^{(j)}(-\pi) = P_{2m+1}^{(j)}(-\pi), \qquad 0 \le j \le m. \tag{5}$$

These conditions yield a linear algebraic system for the unknown $2m + 2$ coefficients a_k, $0 \le k \le 2m + 1$. We prove in Lemma 1.2 below that this linear algebraic system has a solution and this solution is unique. Therefore the polynomial $P_{2m+1}(x)$ is uniquely determined by the above linear algebraic system.

Denote

$$g(x) := f(x) - P_{2m+1}(x). \tag{6}$$

Then

$$g^{(j)}(-\pi) = g^{(j)}(\pi) = 0, \qquad 0 \le j \le m. \tag{7}$$

Therefore, the Fourier series of the function $g(x)$ on the interval T can be m-times term-wise differentiated:

$$g^{(j)}(x) = \sum_n g_n (in)^j e^{inx}, \qquad 0 \le j \le m, \tag{8}$$

where

$$g_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx. \tag{9}$$

Step 2.

Therefore the series

$$f(x) = P_{2m+1}(x) + \sum_n g_n e^{inx} \tag{10}$$

can be m times term-wise differentiated.

This is proved in Lemma 1.3 below.

Combining these results yields the following Theorem.

Theorem 1.1 *If $f \in C^m(T)$ and $P_{2m+1}(x)$ satisfies conditions (5), then the series (10) can be m times term-wise differentiated.*

The Fourier series, obtained after the $m-$th term-wise differentiation converges, in general, in $L^2(T)$.

Lemma 1.2 *Conditions (5) determine $P_{2m+1}(x)$ uniquely.*

Proof. Since conditions (5) constitute a linear algebraic system with the $2m+2$ unknowns a_k, it is sufficient to prove that the corresponding homogeneous system has only the trivial solution. The corresponding homogeneous system says that polynomial $P_{2m+1}(x)$ of degree $2m+1$ has zeros at the points π and $-\pi$ of multiplicity $m+1$ each, so it has $2m+2$ zeros counting multiplicities. This implies that $P_{2m+1}(x) = 0$ identically. Consequently, $a_k = 0$ for $0 \le k \le 2m+1$. Lemma 1.2 is proved. □

Lemma 1.3 *If conditions (7) hold, then relation (8) holds.*

Proof. Integrating by parts m times the formula for g_n and using the conditions

$$g^{(j)}(\pi) = g^{(j)}(-\pi) = 0, \qquad 0 \le j \le m,$$

one gets

$$g_n = \frac{1}{2\pi(in)^m}\int_{-\pi}^{\pi} g^{(m)}(x)e^{-inx}dx.$$

Since $g^{(m)}(x) \in C(T)$, its Fourier coefficients are in ℓ^2. Consequently, relation (8) holds. Lemma 1.3 is proved. □

If the polynomial $P_{2m+1}(x)$ is found, then the series

$$f(x) = P_{2m+1}(x) + \sum_{n=-\infty}^{\infty} g_n e^{inx} \tag{11}$$

can be $m-$times differentiated term-wise, so we have achieved the goal.

The conclusion of the Theorem is an immediate consequence of Lemmas 1.2 and 1.3.

If one assumes that f is piecewise-smooth in T, that is, there are finitely many discontinuity points $s_p \in (-\pi, \pi)$, $1 \le p \le P$, the jumps values $h_p^{(j)} := |f^{(j)}(s_p - 0) - f^{(j)}(s_p + 0)|$, $0 \le j \le m$, and the positions of the jumps are known, that is, the numbers s_p are known, then one may use a method similar to the one that was described above. Namely, the function f is now not a $C^m(T)$ function, but piecewise- $C^m(T)$ function. Suppose for simplicity that there is only one discontinuity point s_1. Then define a polynomial $Q_{m,1}$ of degree m from the conditions similar to (5):

$$f^{(j)}(s_1 - 0) - Q_{m,1}^{(j)}(s_1) = f^{(j)}(s_1 + 0), \qquad 0 \le j \le m. \tag{12}$$

These conditions yield a linear algebraic system for the unknown $m + 1$ coefficients $q_{k,1}$, $0 \le k \le m$, of the polynomial $Q_{m,1}$.

As in Lemma 1.2, one proves that the polynomial $Q_{m,1}$ of degree m is uniquely determined by the conditions (12). The function $f_1(x) = f(x) - Q_{m,1}(x)$ in $(-\pi, s]$, $f_1(x) = f(x)$ in $[s, \pi)$, is $C^m(T)$ function, and to this function one may apply Theorem 1.1. If there are several discontinuity points, then one uses similar method and the number P of the polynomials $Q_{m,p}$ is equal to the number of discontinuity points.

Let us consider now a more difficult problem when the position of discontinuity points s_p is not known. For simplicity assume that there is just one discontinuity point $s \in T$.

The algorithm starts with finding the position of s. This can be done by using the method from [8], where the case of noisy measurements of the function f was treated. In the simpler case when the values of f are given exactly, the algorithm for locating the position of the jump s can be considerably simplified. One may use the following algorithm for locating the discontinuity point with the jump h, defined above. Denote $M := \sup_{x \in T, x \neq s} |f'(x)|$. Choose an integer N such that $M\pi/N < h/8$. Consider a partition of T by the points x_i, $x_i := -\pi + i2\pi/N$, $0 \le i \le N$. Then on any interval (x_i, x_{i+1}) which does not contain s, one has $d_i := |f_i - f_{i+1}| \le 2\pi M/N < h/4$, while on the interval, containing s, one has $d_i > 7h/8$. Thus, calculating d_i for $0 \le i \le N$ one finds the interval of length $2\pi/N$ where the jump point s is located. Increasing N one can find the position of s with any desired accuracy if f is known exactly, that is noise-free. If f is known with some noise, then the algorithm from [8] can be applied for finding the position of the discontinuity point s. In this case s cannot be located with an arbitrary desired accuracy.

Let us describe a method for representing a piecewise-$C^m(R)$ function f as a series which can be term-wise differentiated m times. This method is more general than the one described above, it does not requre finding the polynomial $P_{2m+1}(x)$.

Let s_k be a jump discontinuity point of f, $1 \le k \le K$,

$$d_{k,j} := f^{(j)}(s_k - 0) - f^{(j)}(s_k + 0), \qquad 0 \le j \le m, \qquad q := \min_{1\le k\le K} |s_k - s_{k+1}|. \tag{13}$$

Let $h(x) = 0$ for $x \le 0$, $0 \le h(x) \le 1$, $h(x) = 1$ for $x \in (0, 0.5q)$, $h(x) = 0$ for $x \ge 0.9q$, $h \in C^\infty(0, q)$, and

$$f_1(x) := f(x) + Q_K(x), \qquad Q_K(x) := \sum_{k=1}^{K}\sum_{n=0}^{m} d_{k,n}\frac{(x-s_k)^n}{n!}h(x-s_k), \tag{14}$$

where

$$d_{k,n} := f^{(n)}(s_k - 0) - f^{(n)}(s_k + 0). \tag{15}$$

Theorem 1.4 *The function f_1 is $C^m(R)$ if f is piecewise- $C^m(R)$ function with jump discontinuity points s_k and the sizes of the jumps $d_{k,n}$, $1 \le k \le K$, $0 \le n \le m$.*

Proof. By definition, the function $f_1 \in C^m(\Delta)$ if Δ does not contain discontinuity points of f. Therefore to prove Lemma 1.2 it is sufficient to check that

$$f_1^{(j)}(s_k - 0) = f_1^{(j)}(s_k + 0), \qquad 0 \le j \le m, \quad 1 \le k \le K. \tag{16}$$

Let us verify (15) for an arbitrary $k \le K$ and an arbitrary $j \le m$.

We will use the following formula:

$$[x^n h(x)]^{(j)}|_{x=+0} = 0, \quad j < n \text{ or } j > n; \quad [x^n h(x)]^{(n)}|_{x=+0} = n!; \; [x^n h(x)]^{(j)}|_{x=-0} = 0. \tag{17}$$

Let us check this. By the Leibniz formula one has:

$$[x^n h(x)]^{(j)}(x) = \sum_{i=0}^{j} C_i^j (x^n)^{(i)} h^{(j-i)}(x). \tag{18}$$

If $j < n$ then $n - i > j - i$, and $(x^n)^{(i)} = \frac{n!}{i!}x^{n-i}$, $h^{(p)}(x) = \delta^{(p-1)}(x) + \eta_p(x)$, $p \ge 0$ is an integer, $\delta(x)$ is the delta-function, $\eta_p(x) \in C_0^\infty(R)$. One has $x^\mu \delta^{(\nu-1)}(x) = 0$ if $\mu \ge \nu > 0$. Therefore $(x^n)^{(i)} h^{(j-i)}(x) = 0$ if $j < n$. If $j > n$ then the summation in (17) is up to $i = n$, because $(x^n)^{(i)} = 0$ if $i > n$. If

$j = n$, then in the sum in (18) only the term with $i = n$ does not vanish, and this term is equal to $n!$. Thus, formula (16) is verified.

From this formula the conclusion of Theorem 1.4 follows. Indeed,

$$f_1^{(j)}(s_k + 0) = f^{(j)}(s_k + 0) + d_{k,j} = f^{(j)}(s_k - 0) = f_1^{(j)}(s_k - 0). \tag{19}$$

Theorem 1.4 is proved. □

The function $h(x)$ can be constructed analytically.

2 Numerical results

2.1 Computing the coefficients of P_{2m+1}

Let

$$P_{2m+1}(x) = \sum_{i=0}^{2m+1} c_i x^i, \qquad m = 0, 1, 2, ..., . \tag{20}$$

Let us find the coefficients of P_{2m+1} from the following equations

$$P_{2m+1}^{(k)}(\pi) = f^{(k)}(\pi), \qquad P_{2m+1}^{(k)}(-\pi) = f^{(k)}(-\pi), \qquad k = 0, ..., m. \tag{21}$$

From equations (20) and (21) one gets a linear algebraic system of $2m + 2$ equations with $2m + 2$ unknowns $(c_i)_{i=0}^{2m+1}$:

$$\begin{pmatrix} M_{m+1}(\pi) \\ M_{m+1}(-\pi) \end{pmatrix} \begin{pmatrix} c_{2m+1} \\ c_{2m} \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} g(\pi) \\ g(-\pi) \end{pmatrix}, \qquad g(x) = \begin{pmatrix} f(x) \\ f^{(1)}(x) \\ \vdots \\ f^{(m)}(x) \end{pmatrix} \tag{22}$$

where

$$M_n(x) = \begin{pmatrix} x^{2n-1} & x^{2n-2} & \cdots & 1 \\ (2n-1)x^{2n-2} & (2n-2)x^{2n-3} & \cdots & 0 \\ (2n-1)(2n-2)x^{2n-3} & (2n-2)(2n-3)x^{2n-4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (2n-1)(2n-2)\cdots(n+1)x^n & (2n-2)(2n-3)\cdots nx^{n-1} & \cdots & 0 \end{pmatrix}. \tag{23}$$

The condition number of the matrix

$$H_n := \begin{pmatrix} M_n(\pi) \\ M_n(-\pi) \end{pmatrix} \tag{24}$$

increases very fast when n increases (see Table 1). Thus, it is difficult to compute c_i, $i = 1, .., 2m + 1$, with high accuracy from equation (22) if m is large.

Let us propose an alternative way to compute $(c_i)_{i=0}^{2m+1}$.
One has

$$P_{2m+1}(x) = \sum_{i=0}^{m} f^{(i)}(\pi)U_i(x) + \sum_{i=0}^{m} f^{(i)}(-\pi)L_i(x), \tag{25}$$

where L_i and U_i are polynomials of degree $2m+1$ satisfying

$$L_i^{(j)}(-\pi) = U_i^{(j)}(\pi) = \delta_{ij}, \qquad L_i^{(j)}(\pi) = U_i^{(j)}(-\pi) = 0, \qquad i, j = 0, ..., m. \tag{26}$$

Let us find L_i and U_i. From the second equation in (26), one gets

$$L_k(x) = (x-\pi)^{m+1} \sum_{i=0}^{m} \ell_{k,i}(x+\pi)^i, \qquad k = 0, ..., m. \tag{27}$$

This implies

$$\sum_{i=0}^{m} \ell_{k,i}(x+\pi)^i = \frac{L_k(x)}{(x-\pi)^{m+1}}, \qquad k = 0, ..., m. \tag{28}$$

From equation (28) one obtains

$$\ell_{k,i} = \frac{1}{i!}\frac{d^i}{dx^i}\left(\frac{L_k(x)}{(x-\pi)^{m+1}}\right)|_{x=-\pi}, \qquad k = 0, ..., m. \tag{29}$$

This, the Leibniz rule, and (26) imply

$$\ell_{k,i} = \begin{cases} \frac{C_i^k}{i!}\frac{d^{i-k}}{dx^{i-k}}\left(\frac{1}{(x-\pi)^{m+1}}\right)|_{x=-\pi} & \text{if} \quad i \geq k \\ 0 & \text{if} \quad i < k \end{cases}. \tag{30}$$

If $i \geq k$ then

$$\begin{aligned} \ell_{k,i} &= \frac{C_i^k}{i!}\frac{d^{i-k}}{dx^{i-k}}\left(\frac{1}{(x-\pi)^{m+1}}\right)|_{x=-\pi} \\ &= \frac{1}{k!(i-k)!}(-1)^{i-k}(m+1)(m+2)...(m+i-k)\frac{1}{(-2\pi)^{m+i-k+1}} \\ &= \frac{1}{k!(i-k)!}\frac{(-1)^{m+1}(m+i-k)!}{m!(2\pi)^{m+i-k+1}}. \end{aligned} \tag{31}$$

Therefore,

$$L_k(x) = (x-\pi)^{m+1} \sum_{i=k}^{m} \frac{(-1)^{m+1}(m+i-k)!}{k!(i-k)!m!(2\pi)^{m+i-k+1}}(x+\pi)^i, \qquad k = 0, 1, ..., m. \tag{32}$$

By a similar argument one obtains

$$U_k(x) = (x+\pi)^{m+1}\sum_{i=0}^{m} u_{k,i}(x-\pi)^i, \qquad k = 0, ..., m, \tag{33}$$

where

$$u_{k,i} = \begin{cases} \frac{1}{k!(i-k)!}(-1)^{i-k}(m+1)(m+2)...(m+i-k)\frac{1}{(2\pi)^{m+i-k+1}} & \text{if} \quad i \geq k \\ 0 & \text{if} \quad i < k. \end{cases} \tag{34}$$

So

$$U_k(x) = (x+\pi)^{m+1}\sum_{i=k}^{m} \frac{(-1)^{i-k}(m+i-k)!}{k!(i-k)!m!(2\pi)^{m+i-k+1}}(x-\pi)^i, \qquad k = 0, 1, ..., m. \tag{35}$$

2.2 Computing Fourier coefficients

In our experiments the coefficients of the Fourier series are computed by Filon's method, which yields an accurate results when one computes integral of oscillating functions ([3]).

According to Filon's method (see, e.g., [3], p.151-153), one uses the following formulas:

$$\begin{aligned} \int_a^b f(x)\cos(kx)dx &\approx h\Big(\alpha[f(b)\sin(kb) - f(a)\sin(ka)] + \beta C_{2n} + \gamma C_{2n-1}\Big), \\ \int_a^b f(x)\sin(kx)dx &\approx h\Big(-\alpha[f(b)\cos(kb) - f(a)\cos(ka)] + \beta S_{2n} + \gamma S_{2n-1}\Big), \end{aligned} \tag{36}$$

where

$$\begin{aligned} \alpha = \alpha(\theta) &= \frac{\theta^2 + \theta\sin\theta\cos\theta - 2\sin^2\theta}{\theta^3}, \\ \beta = \beta(\theta) &= \frac{2[\theta(1+\cos^2\theta) - 2\sin\theta\cos\theta]}{\theta^3}, \qquad \theta := hk, \qquad h = \frac{b-a}{2n}, \\ \gamma = \gamma(\theta) &= \frac{4(\sin\theta - \theta\cos\theta)}{\theta^3}, \end{aligned} \tag{37}$$

and

$$C_{2n} = \sum_{j=1}^{n-1} f(x_{2j})\cos(kx_{2j}) + \frac{1}{2}[f(a)\cos(ka) + f(b)\cos(kb)], \qquad x_j = a + jh,$$
$$C_{2n-1} = \sum_{j=1}^{n} f(x_{2j-1})\cos(kx_{2j-1}),$$
$$S_{2n} = \sum_{j=1}^{n-1} f(x_{2j})\sin(kx_{2j}) + \frac{1}{2}[f(a)\sin(ka) + f(b)\sin(kb)],$$
$$S_{2n-1} = \sum_{j=1}^{n} f(x_{2j-1})\sin(kx_{2j-1}). \tag{38}$$

2.3 Numerical experiments

Numerical experiments are done with the function $f(x) = e^x$. In our experiments, we use Filon's method to compute the Fourier coefficients of $f(x) - P_{2m+1}(x)$.

Table 1: Condition number of H_n, $n = 2, ..., 10$.

n	cond(H_n)
2	59.5
3	992.2
4	23417.8
5	725022.5
6	28011136.4
7	1302272486.5
8	70885319047.2
9	4423628332689.4
10	311370017168572.4

In all figures, we denote by S_n the n-th partial sum of the Fourier series of $\psi_m(x) := f(x) - P_{2m+1}(x)$ and by a_n and b_n the Fourier coefficients of $\psi_m(x)$. We have

$$S_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j\cos(jx) + b_j\sin(jx), \qquad n = 0, 1, ..., \tag{39}$$

where

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi_m(x)\cos(jx)dx, \qquad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi_m(x)\sin(jx)dx. \tag{40}$$

Figure 1 plots the functions $f - P_{2m+1}$ and $f - P_{2m+1} - S_{20}$ for $m = 2$. It can be seen from the plots that the maximal value of $f - P_{2m+1} - S_{20}$ is smaller than that of $f - P_{2m+1}$ by a factor 2×10^{-4}.

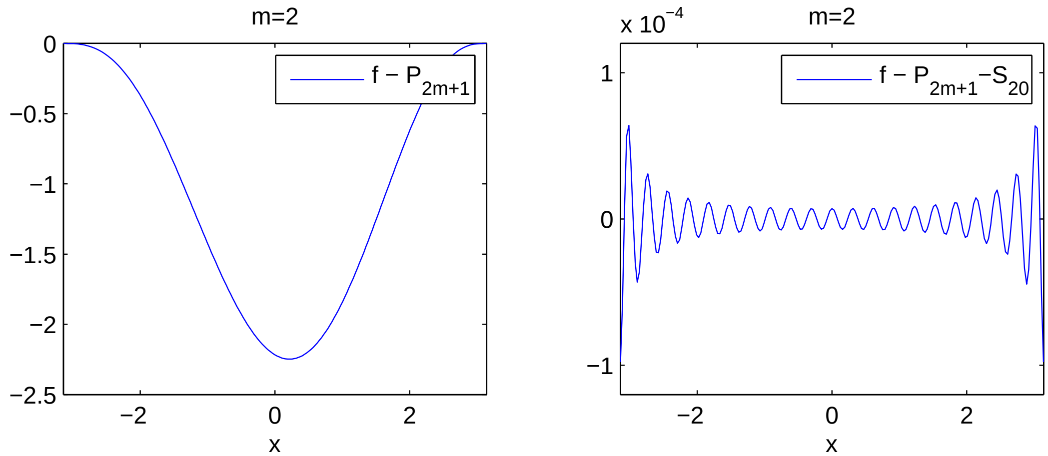

Figure 1: Plots of $f - P_{2m+1}$ and $f - P_{2m+1} - S_{20}$ for $m = 2$.

Figure 2 plots the functions $f - P_{2m+1}$ and $f - P_{2m+1} - S_{20}$ for $m = 5$. It can be seen from Figure 2 that the maximal value of $f - P_{2m+1} - S_{20}$ is smaller than that of $f - P_{2m+1}$ by a factor 2×10^{-8}. Thus, the accuracy "gain" by using $m = 5$ instead of $m = 2$ is a factor of 10^{-4}. This is a consequence of the fact that the coefficients of $f(x) - P_{2m+1}$ decrease at the rate not slower than $O(\frac{1}{n^{m+1}})$.

Figure 3 plots the function $f - P_{2m+1} - S_{10}$ for $m = 2$ and $m = 5$. From Figure 1 and Figure 3, we can see that there is no accuracy improvement by using S_{20} instead of S_{10} for approximating the function $f - P_5$. However, one can see from Figure 2 and Figure 3 that the accuracy gain by using S_{20} instead of S_{10} to approximate the function $f - P_{11}$ is a factor of 10^{-2}. Again, this is a consequence of the fact that the coefficients of $f(x) - P_{2m+1}$ decrease at the rate $O(\frac{1}{n^{m+1}})$.

Figure 4 plots the functions $\log_{10}(n^{m+1}|a_n|)$ and $\log_{10}(n^{m+1}|b_n|)$ for $n = 1, 2, ..., 100$, where a_n and b_n are the Fourier coefficients of the function $\phi_m(x) = f(x) - P_{2m+1}(x)$ (see (40)). We have used $m = 2$ and $m = 5$ in the left and right figures, respectively. It follows from Figure 4 that the Fourier coefficients a_n and b_n of $f(x) - P_{2m+1}(x)$ decreases at the rate not slower than $O(\frac{1}{n^{m+1}})$. This agrees with the result in Theorem 1.1.

ACKNOWLEDGEMENTS. The author is grateful to his student N.S.Hoang for obtaining the numerical results presented in this paper.

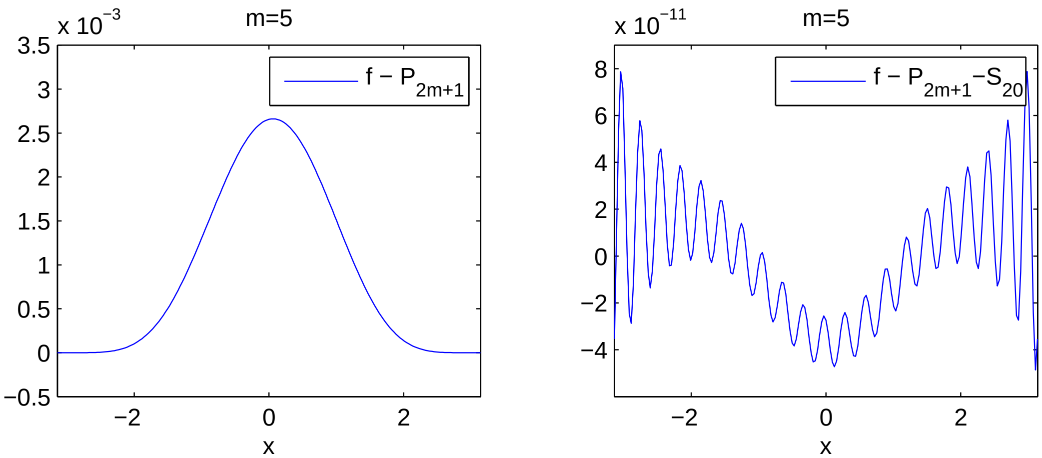

Figure 2: Plots of $f - P_{2m+1}$ and $f - P_{2m+1} - S_{20}$ for $m = 5$.

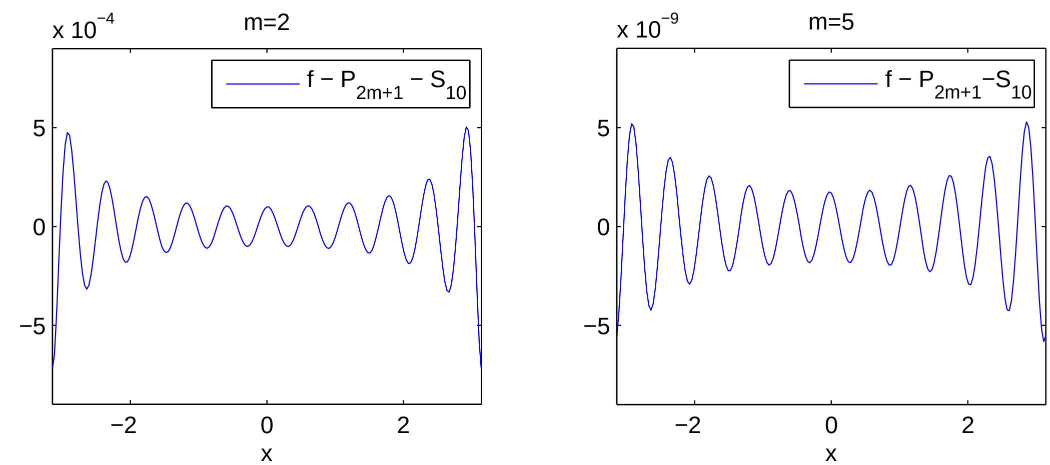

Figure 3: Plots of $f - P_{2m+1} - S_{10}$ for $m = 2$ and $m = 5$.

References

[1] C. Brezinski, Extrapolation algorithms for filtering series of functions and treating Gibbs phenomenon, Numer. Algorithms, 36, N4, (2004), 309-329.

[2] J. Boyd, Acceleration of algebraically converging Fourier series when the coefficients have series in powers of $\frac{1}{n}$, J.Comp. Phys., 228, N5, (2009), 1404-1411.

[3] P. Davis and P. Rabinowitz, *Methods of numerical integration*, Academic Press, London, 1984.

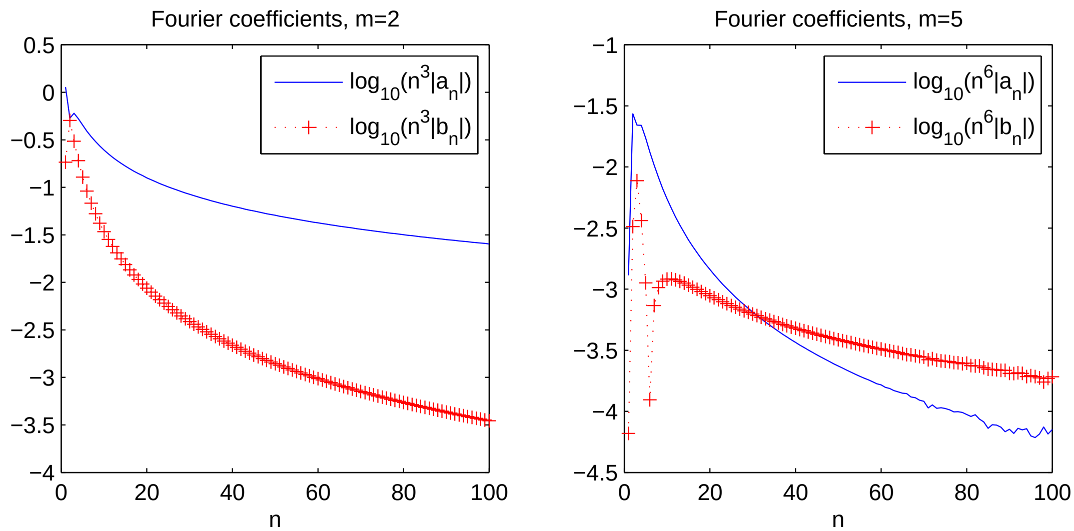

Figure 4: Plots of Fourier coefficients of $e^x - P_{2m+1}$ for $m = 2$ and $m = 5$.

[4] K. Eckhoff, Accurate reconstruction of functions of finite regularity from truncated Fourier series expansions, Math. of Comput., 64(210), (1995), 671-690.

[5] A.Haar, Zur Theorie der orthogonalen Funktionsysteme, Math. Ann., 69, (1910), 331-371.

[6] A. I. Katsevich, A. G. Ramm, Nonparametric estimation of the singularities of a signal from noisy measurements, Proc. Amer. Math. Soc., 120, N8, (1994), 1121-1134.

[7] V.I.Krylov, L. G. Kruglikova, *Handbook of numerical harmonic analysis,* Israel Progr. for Sci. Transl., Jerusalem, 1969.

[8] A. G. Ramm, Finding discontinuities of piecewise-smooth functions, JIPAM (Journ of Inequalities in Pure and Appl. Math.) 7, N2, Article 55, pp. 1-7 (2006).

[9] A. G. Ramm, *Dynamical systems method for solving operator equations,* Elsevier, Amsterdam, 2007.

Received: March, 2011