Mathematica Aeterna, Vol. 1, 2011, no. 04, 217 - 236

Generalized Ulam-Hyers stability of an AQ-functional equation in quasi-beta-normed spaces

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abstract

In this paper, we introduce and investigate the general solution of a new functional equation

$$f\left(\frac{x+y}{a}+\frac{z+w}{b}\right)+f\left(\frac{x+y}{a}-\frac{z+w}{b}\right)=\frac{1}{a^2}\left[(1+a)f(x+y)+(1-a)f(-x-y)\right]$$
$$+\frac{1}{b^2}\left[f(z+w)+f(-z-w)\right]$$

where $a,b \geq 2$ and discuss its Generalized Hyers - Ulam - Rassias stability in Quasi -β-normed spaces.

Mathematics Subject Classification: [2000]: 39B52, 39B82,39B72,47Jxx.

Keywords: Generalized Ulam-Hyers stability, Quadratic function, Additive function, Quasi-β-spaces, (β, p)-Banach spaces,Contractively subadditive, Expansively super additive

1. INTRODUCTION

In 1940, S. M. Ulam [32] , while he was giving a talk before the mathematics club of the University of wisconsin, he proposed a number of importent unsolved problems. One of the problem is the stability of functional equation. In the last five decades the problem was tackled by numerous authors [1,2,6,8,12,18,22,26]. It's solutions via various forms of functional equations like additive, quadratic, cubic and quartic and its mixed forms were discussed.

Ulam's stability problem states as follows:

Let G be a group and let H be a metric group with metric d(.,.). Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers[12] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces. He proved the following celebrated theorem.

Theorem 1.1 (25)**.** *Let E, E' be Banach spaces and let $f : E \to E'$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. Then the limit $a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $a : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$ then a is linear.

From the above property, the additive functional equation $f(x + y) = f(x) + f(y)$ has Hyers-Ulam stability on (E, E') or alternatively that it is stable in the sense of Hyers and Ulam. In 1951, T.Aoki [2] generalized the Hyers theorem and later in 1978, Th.M.Rassias [25] proved a generalization of Hyers theorem, which allows the cauchy difference to be unbounded. It states as follows:

Theorem 1.2 (25)**.** *Let E,E' be two Banach spaces and let $\theta \in [0, \infty)$ and $p \in [0, 1)$. If a function $f : E \to E'$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \left[||x||^p + ||y||^p\right]$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in E$. Moreover, it $f(tx)$ is continuous in t for each fixed $x \in E$ then T is linear.

These ideas become a powerful tool for studying the stability of several functional equations and they have been called Hyers-Ulam-Rassias stability, In 1982-84, J.M.Rassias [22] in the above Theorem [25], he replaced the sum by the product of powers of norms , which is given in the following Theorem .

Theorem 1.3 (22)**.** *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon \, \|x\|^p \, \|y\|^p \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \, \|x\|^{2p} \tag{1.2}$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. If $p > \frac{1}{2}$ the inequality (1.1) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p} - 2} \, \|x\|^{2p}$$

for all $x \in E$. If in addition $f : E \to E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}-$ linear mapping.

In 1983, Skof proved Hyers-Ulam-Rassias stability problem for quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.3}$$

for a class of functions $f : A \to B$, where A is a normed space and B is a Banach space (see [2][14]). Many results are available on various quadratic functional equations, one can see ([5][7][15][17] [19]). S.M.Jung [15] investegated the Hyers-Ulam-Rassias stability of the quadratic functional equation on pexider type

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y).$$

The generalized Hyers-Ulam-Rassias stability of a quadratic equation

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z)$$

was discussed by B.H.Bae and K.W.Kim [3]. In 2005, K.W.Jun and H.M.Kim [18] obtained the general solution of a generalized quadratic and additive type functional equation of the form

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y)$$

for any integer a with $a \neq -1, 0, 1$. J.M.Rassias [**?**, 24]erived the stability of the generalized version of the above quadratic equation

$$Q(a_1x_1 + a_2x_2) + Qf(a_2x_1 - a_1x_2) = \left(a_1^2 + a_2^2\right)[Q(x_1) + Q(x_2)]$$

which covers a wide range of quadratic functional equations in two variables. Recently , K.Ravi and R.Kodandan [29] discussed the stability of Additive and Quadratic functional equation

$$f\left(\frac{xz}{y} + \frac{yw}{x}\right) + f\left(\frac{xz}{y} - \frac{yw}{x}\right) = 2f\left(\frac{xz}{y}\right) + f\left(\frac{yw}{x}\right) + f\left(-\frac{yw}{x}\right)$$

where $x, y \neq 0$, in non-Archimedian spaces.

In this paper, we introduce and investigate the general solution of a new functional equation

$$f\left(\frac{x+y}{a}+\frac{z+w}{b}\right)+f\left(\frac{x+y}{a}-\frac{z+w}{b}\right)=\frac{1}{a^2}\left[(1+a)f(x+y)+(1-a)f(-x-y)\right] + \frac{1}{b^2}\left[f(z+w)+f(-z-w)\right] \tag{1.4}$$

and discuss its Generalized Hyers-Ulam-Rassias stability of this equation in quasi-β-Normed spaces. It may be noted that $f(x)=ax^2+bx+c$ is a solution of the functional equation (1.4)

Before giving the main results, we will present here some basic facts concerning quasi-β-Normed spaces and some prelimenary results. We fix a real number β with $0<\beta\leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$ .Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following: Let X be a linear space. A quasi-norm $\|\cdot\|$ is real - valued function on X satisfying the following:
(i) $\| x \|\geq 0$ for all $x\in X$ and $\| x \|=0$ if and only if $x=0$.
(ii) $\| \lambda x \| =| \lambda |^{\beta} . \| x \|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$.
(iii) There is a constant $K\geq 1$ such that $\| x+y \|\leq K\left(\| x \| + \| y \|\right)$ for all $x,y\in X$. The pair $(X,\|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.A quasi-β-Banach space is a complete quasi-β-normed space.
A qusi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0<p\leq 1)$ if

$$\| x+y \|^p\leq\| x \|^p + \| y \|^p$$

for all $x,y\in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space . We can refer to [3,30] for the concept of quasi-normed spaces and p-Banach space. Given a p-norm, the formula $d(x,y)=\| x+y \|^p$ gives us a translation invarient metric on X. By the Aoki-Rolewicz theorem [30] (see also [3]), each quasi-norm is equivalent to some p-norm, since it is much easier to work with p-norms than quasi-norms. henceforth we restrict our attention mainly to p-norms. In [31], J.Tabor has investigated a version of the Hyers-Rassias-Gajda theorem (see[8]) in quasi-Banach spaces. We recall that a subadditive function is a function $\phi:E_1\to E_2$, having a domain E_1 and a codomain $(E_2,\leq)$ that are both closed under additive, with the following property:

$$\phi(x+y) \leq \phi(x) + \phi(y), \forall x, y \in E_1.$$

Now we say that a function $\phi : E_1 \to E_2$ is contractively subadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \leq L\left[\phi(x) + \phi(y)\right], \forall x, y \in E_1.$$

Then ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. It follows by the contractively subadditive condition of ϕ that $\phi(\lambda x) \leq \lambda L \phi(x)$ and so $\phi(\lambda^i x) \leq (\lambda L)^i \phi(x), i \in \mathbb{N}$, for all $x \in E_1$ and all positive integer $\lambda \geq 2$. Similarly, we say that function $\phi : E_1 \to E_2$ is expansively superadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}\left[\phi(x) + \phi(y)\right], \forall x, y \in E_1.$$

Then ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi\left(\frac{x}{2^n}\right) \leq \left(\frac{L}{2}\right)^n \phi(x)$. We observe that an expansively super additive mapping ϕ satisfies the following properties $\phi(\lambda x) \geq \left(\frac{\lambda}{L}\right)\phi(x)$ and so $\phi\left(\frac{x}{\lambda^i}\right) \geq \left(\frac{L}{\lambda}\right)^i \phi(x), i \in \mathbb{N}$, for all $x \in E_1$ and all positive integer $\lambda \geq 2$.

2. **Solution of Functional Equation (1.4)**

In this Section, let E_1 and E_2 denote real vectors spaces, we will prove the following two main theorems.

Theorem 2.1. *If $f : E_1 \to E_2$ is an even function satisfying (1.4) for all $x, y, z, w \in E_1$ then f is quadratic.*

Proof. Replace (x, y, z, w) by $(0, 0, 0, 0)$ in (1.4), we obtain

$$f(0) = 0. \tag{2.1}$$

The function f is even and therefore $f(-x) = f(x)$ for all $x \in E_1$. Using evenness in (1.4) we obtain

$$f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) = \frac{2}{a^2}f(x+y) + \frac{2}{b^2}f(z+w), \tag{2.2}$$

for all $x, y, z, w \in E_1$ Replace (z, w) by $(0, 0)$ and using (2.1) in (2.2), we obtain

$$f\left(\frac{x+y}{a}\right) = \frac{1}{a^2}f(x+y), \quad \forall \quad x, y \in E_1. \tag{2.3}$$

Replacing x by 0 in (2.3), we arrive that

$$f\left(\frac{y}{a}\right) = \frac{1}{a^2} f(y), \quad \forall \quad y \in E_1. \tag{2.4}$$

Again replacing y by ax in (2.4), we obtain

$$f(ax) = a^2 f(x), \quad \forall \quad x \in E_1. \tag{2.5}$$

Replacing $[(x,y),(z,w)]$ by $[(ax,ay),(bz,bw)]$ in (2.2) and using equation (2.5) , we obtain

$$f\left[(x+y)+(z+w)\right] + f\left[(x+y)-(z+w)\right] = 2f(x+y) + 2f(z+w). \tag{2.6}$$

Replacing $(x+y, z+w)$ by (u,v) in (2.6), we obtain

$$f(u+v) + f(u-v) = 2f(u) + 2f(v). \tag{2.7}$$

Again replacing (u,v) by (x,y) in (2.7), we obtain

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \forall \quad x, y \in E_1.$$

Therefore $f : E_1 \to E_2$ is quadratic. □

Theorem 2.2. *If $f : E_1 \to E_2$ be an odd function, satisfying (1.4) for all $x, y \in E_1$. Then f is additive.*

Proof. Using oddness of f and using (2.1) in (1.4) , we obtain

$$f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) = \frac{2}{a} f(x+y). \tag{2.8}$$

Replacing (z,w) by $(0,0)$ and using (2.1) in (2.7), we obtain

$$f\left(\frac{x+y}{a}\right) = \frac{1}{a} f(x+y), \quad \forall \quad x, y \in E_1. \tag{2.9}$$

Replacing x by y in (2.8) , we obtain

$$f\left(\frac{2y}{a}\right) = \frac{1}{a} f(2y), \quad \forall \quad y \in E_1. \tag{2.10}$$

Replacing $2y$ by ax in (2.10), we arrive

$$f(ax) = a f(x), \quad \forall \quad x \in E_1. \tag{2.11}$$

Replacing $[(x,y),(z,w)]$ by $[(ax,ay),(az,aw)]$ in (2.8) and using equation (2.11), we obtain

$$f\left[(x+y)+(z+w)\right]+f\left[(x+y)-(z+w)\right]=2f(x+y), \quad \forall \quad x,y,z,w\in E_1. \tag{2.12}$$

Replacing $(x+y,z+w)$ by (u,v) in (2.11), we obtain

$$f(u+v)+f(u-v)=2f(u). \tag{2.13}$$

Interchanging u ,v and using oddness in (2.13), we obtain

$$f(u+v)-f(u-v)=2f(v). \tag{2.14}$$

Adding (2.13) and (2.14) , we get

$$f(u+v)=f(u)+f(v). \tag{2.15}$$

Replacing (u,v) by (x,y) in(2.15), we obtain

$$f(x+y)=f(x)+f(y), \quad \forall \quad x,y\in E_1.$$

Therefore the mapping $f:E_1\to E_2$ is additive. □

3. HYERS - ULAM - RASSIAS STABILITY OF EQUATION (1.4)

In this Section, we assume that E_1 is a linear space over $\mathbb{K}$ and E_2 is a (β,p) Banach space with p-norm $\|.\|_{E_2}$. Let K be the modulus of concavity of $\|.\|_{E_2}$ Now we are going to investigate the modified Ulam-Hyers Stability of the functional equation (1.4). For notational convenience, we denote for a given mapping $f:E_1\to E_2$ and a scalar $\mu\in\mathbb{K}$, the difference operator $D_\mu f: E_1\times E_1\times E_1\times E_1\to E_2$ of equation (1.4) by

$$\begin{aligned} D_\mu\ f(x,y,z,w) = f\left(\frac{\mu x+\mu y}{a}+\frac{\mu z+\mu w}{b}\right)+f\left(\frac{\mu x+\mu y}{a}-\frac{\mu z+\mu w}{b}\right)\\ -\frac{1}{a^2}\left[(a+1)\mu f(x+y)+(a-1)\mu f(-x-y)\right]\\ -\frac{1}{b^2}\left[\mu f(z+w)+\mu f(-z-w)\right] \end{aligned}$$

for all $x,y,z,w\in E_1$.

Theorem 3.1. *Assume that there exists a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ for which an odd mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \phi(x, y, z, w) \tag{3.1}$$

for all $x, y, z, w \in E_1$, and that the map ϕ is contractively subadditive with a constant L satisfying $a^{1-\beta}L < 1$. Then there exists a unique additive mapping $A : E_1 \to E_2$ which satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^{\beta} \frac{\phi(x, 0, 0, 0)}{\sqrt[p]{\left(\frac{a^{\beta-1}}{L}\right)^{p} - 1}} \tag{3.2}$$

for all $x \in E_1$.

Proof. Using oddness and (2.1) in (3.1), we obtain

$$\left\| f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) - \frac{2}{a} f(x+y) \right\|_{E_2} \leq \phi(x, y, z, w). \tag{3.3}$$

For all $x, y, z, w \in E_1$. Replace (y, z, w) by $(0, 0, 0)$ in (3.3), we obtain

$$\left\| 2f\left(\frac{x}{a}\right) - \frac{2}{a} f(x) \right\|_{E_2} \leq \phi(x, 0, 0, 0), \qquad \forall x \in E_1. \tag{3.4}$$

Again replacing x by ax in (3.4) and simplifing, we get

$$\left\| f(x) - \frac{1}{a} f(ax) \right\|_{E_2} \leq \frac{1}{2^{\beta}} \phi(ax, 0, 0, 0) \tag{3.5}$$

for all $x \in E_1$. Therefore it follows from in (3.5) that when we replace $a^i x$ in the place of x and by iterative method

$$\begin{aligned}
\left\| \frac{f(a^l x)}{a^l} - \frac{f(a^m x)}{a^m} \right\|_{E_2}^{p} &\leq \sum_{i=l}^{m-1} \frac{(aL)^p}{2^{\beta p} a^{\beta p i}} \left\| f(a^i x) - \frac{f(a^{i+1} x)}{a} \right\|_{E_2}^{p} \\
&\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{1}{a^{\beta p i}} \phi(a^i x, 0, 0, 0)^p \\
&\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{(aL)^{pi}}{a^{\beta p i}} \phi(x, 0, 0, 0)^p \\
&\leq \frac{(aL)^p}{2^{\beta p}} \phi(x, 0, 0, 0)^p \sum_{i=l}^{m-1} \left(a^{1-\beta} L\right)^{pi}.
\end{aligned} \tag{3.6}$$

for all $x \in E_1$ and for any $m > l \geq 0$. Thus it follows that a sequence $\left\{ \frac{f(a^m x)}{a^m} \right\}$ is a cauchy in E_2 and so it converges. Therefore we see that a mapping $A : E_1 \to E_2$ defined by $A(x) = \lim_{m\to\infty} \frac{f(a^m x)}{a^m}$ is well defined for all $x \in E_1$. In addition it is clear from (3.1) that the following inequality

$$\begin{aligned}\|D_1 A(x,y,z,w)\|_{E_2}^p &= \lim_{m\to\infty} \frac{\|D_1 f(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{\beta pm}} \\ &\leq \lim_{m\to\infty} \frac{\|\phi(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{\beta pm}} \\ &\leq \lim_{m\to\infty} (a^{1-\beta} L)^{\beta pm} \phi(x,y,z,w)^p = 0\end{aligned}$$

holds for all $x, y, z, w \in E_1$ and so the mapping A is additive. Taking the limit $m \to \infty$ in (3.6) with $l = 0$, we find that

$$\begin{aligned}\|f(x) - A(x)\|_{E_2}^p &\leq \left(\frac{aL}{2^\beta}\right)^p \phi(x,0,0,0)^p \sum_{i=0}^{\infty} \left(a^{1-\beta} L\right)^{pi} \\ &\leq \left(\frac{aL}{2^\beta}\right)^p \phi(x,0,0,0)^p \frac{1}{1 - (a^{1-\beta} L)^p}\end{aligned}$$

therefore, we get

$$\|f(x) - A(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^\beta \frac{\phi(x,0,0,0)}{\sqrt[p]{\left(\frac{a^{\beta-1}}{L}\right)^p - 1}}.$$

To prove uniqueness, we assume now that there is another function $A' : E_1 \to E_2$ which satisfies (1.4) and the inequality (3.2) then it follows that $A'(ax) = aA'(x)$,$A'(a^m x) = a^m A'(x)$ for all $x \in E_1$ and all $m \in N$. Thus

$$\begin{aligned}\left\| \frac{f(a^m x)}{a^m} \right\|_{E_2} &= \frac{1}{a^{\beta m}} \left\| f(a^m x) - A'(a^m x) \right\|_{E_2} \\ &\leq \frac{aL}{a^{\beta m}} \left(\frac{a}{2}\right)^\beta \frac{\phi(a^m x,0,0,0)}{\sqrt[p]{a^{\beta p} - (aL)^p}} \\ &\leq (aL) \left(\frac{a}{2}\right)^\beta \left(a^{1-\beta} L\right)^m \frac{\phi(x,0,0,0)}{\sqrt[p]{a^{\beta p} - (aL)^p}}\end{aligned}$$

for all $x \in E_1$ and all $m \in N$. Allow $m \to \infty$, we get

$$\left\| A(x) - A'(x) \right\| = 0$$

for all $x \in E_1$, which completes the proof of uniqueness. $\square$

Theorem 3.2. *Assume that there exists a mapping* $\phi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ *for which an odd mapping* $f : E_1 \to E_2$ *satisfies the inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \phi(x, y, z, w) \tag{3.7}$$

for all $x, y, z, w \in E_1$, *and that the map* ϕ *is expansively superadditive with a constant* L *satisfying* $a^{\beta-1}L < 1$. *Then there exists a unique mapping* $A : E_1 \to E_2$ *which satisfies (1.4) and the inequality*

$$\|f(x) - A(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^{\beta} \frac{\phi(x, 0, 0, 0)}{\sqrt[p]{1 - (a^{\beta-1}L)^p}} \tag{3.8}$$

for all $x \in E_1$.

Proof. From (3.4), we obtain

$$\left\| f(x) - af\left(\frac{x}{a}\right) \right\| \leq \left(\frac{a}{2}\right)^{\beta} \phi(x, 0, 0, 0) \tag{3.9}$$

it follows from (3.9) with $\frac{x}{a^i}$ in place of x and iterative method that

$$\begin{aligned} \left\| a^l f\left(\frac{x}{a^l}\right) - a^m f\left(\frac{x}{a^m}\right) \right\|_{E_2}^p &\leq \sum_{i=l}^{m-1} a^{\beta p i} \left\| f\left(\frac{x}{a^i}\right) - af\left(\frac{x}{a^{i+1}}\right) \right\|_{E_2}^p \\ &\leq \left(\frac{a}{2}\right)^{\beta p} \sum_{i=l}^{m-1} a^{\beta p i} \phi\left(\frac{x}{a^i}, 0, 0, 0\right)^p \\ &\leq \left(\frac{a}{2}\right)^{\beta p} \phi(x, 0, 0, 0) \sum_{i=l}^{m-1} \left(a^{\beta-1}L\right)^{pi} \end{aligned} \tag{3.10}$$

for all $x \in E_1$ and for any $m > l \geq 0$. Therefore we see that a mapping $A : E_1 \to E_2$ defined by

$$A(x) = \lim_{m\to\infty} a^m f\left(\frac{x}{a^m}\right)$$

is well defined for all $x \in E_1$. Taking the limit $m \to \infty$ in (3.10) with $l = 0$, we find that the mapping A satisfying the inequality (3.8) near the approximate mapping $f : E_1 \to E_2$ of (1.4). The remaining proof is similar to that of Theorem 3.1. $\square$

Theorem 3.3. *Assume that an odd mapping $f : E_1 \to E_2$ satisfies*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \phi(x,y,z,w)$$

for all $x,y,z,w \in E_1$. If a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \to [0,\infty)$ satisfies

$$\Phi(x,0,0,0) = \sum_{i=0}^{\infty} \frac{K^i \phi(a^{i+1}x,0,0,0)}{a^{\beta i}} < \infty \text{ and } \lim_{m\to\infty} \frac{K^m \phi(a^m x,0,0,0)}{a^{\beta m}} = 0$$

for all $x,y,z,w \in E_1$. Then there exists a unique additive mapping $A : E_1 \to E_2$ such that A satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq \frac{K}{2^\beta}\Phi(x,0,0,0), \forall x \in E_1.$$

Proof. It follows from (3.5) with $a^i x$ in place of x and iterative method that

$$\left\|f(x) - f\left(\frac{a^m x}{a^m}\right)\right\|_{E_2} \leq \frac{K}{2^\beta}\sum_{i=0}^{m-2} \frac{K^i \phi(a^{i+1}x,0,0,0)}{a^{\beta i}} + \frac{1}{2^\beta}\frac{K^{m-1}\phi(a^m x,0,0,0)}{a^{\beta(m-1)}} \tag{3.11}$$

or all $x \in E_1$ and for any $m > 1$, which is considered to be (3.5) for $m = 1$. In fact, we see by computation

$$\begin{aligned}\left\|f(x) - \frac{f(a^{m+1}x)}{a^{m+1}}\right\|_{E_2} &\leq K\left\|f(x) - \frac{f(ax)}{a}\right\|_{E_2} + \frac{K}{a^\beta}\left\|f(x) - \frac{f(a^{m+1}x)}{a^m}\right\|_{E_2} \\ &\leq \frac{K}{2^\beta}\phi(ax,0,0,0) + \frac{K^2}{(2a)^\beta}\sum_{i=0}^{m-2}\frac{K^i\phi(a^{i+2}x,0,0,0)}{a^{\beta i}} \\ &\qquad + \frac{K^m}{(2a)^\beta}\frac{\phi(a^{m+1}x,0,0,0)}{a^{\beta(m-1)}} \\ &\leq \frac{K}{2^\beta}\sum_{j=0}^{m-1}\frac{K^j\phi(a^{j+1}x,0,0,0)}{a^{\beta j}} + \frac{K^m}{2^\beta}\frac{\phi(a^{m+1}x,0,0,0)}{a^{\beta m}},\end{aligned}$$

for all $x \in E_1$,which proves the inequality (3.11) for $m+1$ by induction.

Thus follows that a sequence $\left\{\frac{f(a^m x)}{a^m}\right\}$ is cauchy in E_2 and it converges.Therefore we see that a mapping $A : E_1 \to E_2$ defined by $A(x) = \lim_{m\to\infty}\frac{f(a^m)}{a^m}$ is well defined for all $x \in E_1$. The remaining proof is similar to that of Theorem 3.1. □

Theorem 3.4. *Assume that an odd mapping $f : E_1 \to E_2$ satisfies*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \phi(x,y,z,w)$$

for all $x, y, z, w \in E_1$. If a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ satisfies

$$\Phi(x,0,0,0) = \sum_{i=0}^{\infty} \left(a^{\beta}K\right)^{i} \phi\left(\frac{x}{a^{i}},0,0,0\right) < \infty \text{ and } \lim_{m\to\infty} \left(a^{\beta}K\right)^{m} \phi\left(\frac{x}{a^{m}},0,0,0\right) = 0.$$

for all $x, y, z, w \in E_1$. Then there exists a unique additive mapping $A : E_1 \to E_2$ such that A satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq K \left(\frac{a}{2}\right)^{\beta} \Phi(x,0,0,0),$$

for all $x \in E_1$.

Proof. It follows from(3.9) with $\frac{x}{a^i}$ and the similar method to (3.11) that

$$\begin{aligned} \left\|f(x) - a^{m} f\left(\frac{x}{a^{m}}\right)\right\|_{E_2} \leq K \left(\frac{a}{2}\right)^{\beta} \sum_{i=0}^{m-2} \left(a^{m}K\right)^{i} \phi\left(\frac{x}{a^{i}},0,0,0\right) \\ + \left(\frac{a}{2}\right)^{\beta} \left(a^{\beta}K\right)^{m-1} \phi\left(\frac{x}{a^{m-1}},0,0,0\right) \end{aligned}$$

for all $x \in E_1$ and for any $m > 1$. Therefore we see that a mapping $A : E_1 \to E_2$ defined by $A(x) = \lim_{m\to\infty} a^m f\left(\frac{x}{a^m}\right)$ is well defined for all $x \in E_1$. The remaining proof is similar to that of Theorem 3.3. □

Corollary 3.5. *Let E_1 be a quasi-α-normed linear space with quasi-α-norm $\| \,.\, \|$. if there exists a fixed real number $r \in R$ such that an odd mapping $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \theta \left(\|x\|^{r} + \|y\|^{r} + \|z\|^{r} + \|w\|^{r}\right)$$

for all $x, y, z, w \in E_1\, (E_1 \setminus \{0\}\ if\ r \leq 0)$, then there exists a unique additive mapping $A : E_1 \to E_2$ which satisfies Eq.(1.4) and the inequality

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{K\theta}{2^{\beta}} \frac{a^{\alpha r}}{1 - K a^{\alpha r - \beta}} & if\ K a^{\alpha r} < a^{\beta}, \\ \left(\frac{K\theta a^{\beta}}{2^{\beta}}\right) \frac{1}{1 - K a^{\beta - \alpha r}} & if\ K a^{\beta} < a^{\alpha r}, \end{cases}$$

for all $x \in E_1\, (E_1 \setminus \{0\}\ if\ r \leq 0)$.

Proof. By replacing $\phi(x,y,z,w)$ by $(\|x\|^{r} + \|y\|^{r} + \|z\|^{r} + \|w\|^{r})$ in Theorem 3.1 and Theorem 3.2, we obtain above result. □

Theorem 3.6. *Assume that there exists a mapping $\varphi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ for which an even mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \varphi(x, y, z, w) \tag{3.12}$$

for all $x, y, z \in E_1$ and that the map ϕ is contractively subadditive with a constant L satisfying $a^{1-2\beta}L < 1$. Then there exists a unique quadratic mapping $Q : E_1 \to E_2$ which satisfies (1.4) and the inequality

$$\|f(x) - Q(x)\|_{E_2} \leq \left(\frac{a^2}{2}\right)^{\beta} \frac{\varphi(x, 0, 0, 0)}{\sqrt[p]{\left(\frac{a^{2\beta-1}}{L}\right)^p - 1}}, \tag{3.13}$$

for all $x \in E_1$.

Proof. Using evenness in (3.12), we obtain

$$\left\| f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) - \frac{2}{a^2} f(x+y) - \frac{2}{a^2} f(z+w) \right\|_{E_2} \leq \varphi(x, y, z, w). \tag{3.14}$$

For all $x, y, z, w \in E_1$. Replace (y, z, w) by $(0, 0, 0)$ in (3.14), we obtain

$$\left\| 2f\left(\frac{x}{a}\right) - \frac{2}{a^2} f(x) \right\|_{E_2} \leq \varphi(x, 0, 0, 0), \qquad \forall x \in E_1. \tag{3.15}$$

Again replacing x by ax in (3.15) and simplifing, we get

$$\left\| f(x) - \frac{1}{a^2} f(ax) \right\|_{E_2} \leq \frac{1}{2^\beta} \varphi(ax, 0, 0, 0) \tag{3.16}$$

for all $x \in E_1$.Therefore it follows from in (3.16) that when we replace $a^i x$ in the place of x and by iterative method

$$\begin{aligned}
\left\| \frac{f(a^l x)}{a^{2l}} - \frac{f(a^m x)}{a^{2m}} \right\|_{E_2}^p &\leq \sum_{i=l}^{m-1} \frac{(aL)^p}{2^{\beta p} a^{2\beta p i}} \left\| f(a^i x) - \frac{f(a^{i+1} x)}{a^2} \right\|_{E_2}^p \\
&\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{1}{a^{\beta p i}} \varphi(a^i x, 0, 0, 0)^p \\
&\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{(aL)^{pi}}{a^{2\beta p i}} \varphi(x, 0, 0, 0)^p \\
&\leq \frac{(aL)^p}{2^{\beta p}} \varphi(x, 0, 0, 0)^p \sum_{i=l}^{m-1} \left(a^{1-2\beta} L\right)^{pi}. \qquad (3.17)
\end{aligned}$$

for all $x \in E_1$ and for any $m > l \geq 0$. Thus it follows that a sequence $\left\{\frac{f(a^m x)}{a^{2m}}\right\}$ is a cauchy sequence in E_2 and so it converges. Therefore we see that a mapping $A : E_1 \to E_2$ defined by $Q(x) = \lim_{m\to\infty} \frac{f(a^m x)}{a^{2m}}$ is well defined for all $x \in E_1$. In addition it is clear from (3.12) that the following inequality

$$\begin{aligned}\|D_1 Q(x,y,z,w)\|_{E_2}^p &= \lim_{m\to\infty} \frac{\|D_1 f(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{2\beta pm}} \\ &\leq \lim_{m\to\infty} \frac{\|\varphi(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{2\beta pm}} \\ &\leq \lim_{m\to\infty} (a^{1-2\beta}L)^{\beta pm} \varphi(x,y,z,w)^p = 0\end{aligned}$$

holds for all $x, y, z, w \in E_1$ and so the mapping Q is quadratic. Taking the limit $m \to \infty$ in (3.17) with $l = 0$, we find that

$$\begin{aligned}\|f(x) - Q(x)\|_{E_2}^p &\leq \left(\frac{aL}{2^\beta}\right)^p \varphi(x,0,0,0)^p \sum_{i=0}^{\infty} \left(a^{1-2\beta}L\right)^{pi} \\ &\leq \left(\frac{aL}{2^\beta}\right)^p \varphi(x,0,0,0)^p \frac{1}{1-(a^{1-2\beta}L)^p}\end{aligned}$$

therefore, we get

$$\|f(x) - Q(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^\beta \frac{\varphi(x,0,0,0)}{\sqrt[p]{\left(\frac{a^{2\beta-1}}{L}\right)^p - 1}}.$$

To prove uniqueness, we assume now that there is another function $Q' : E_1 \to E_2$ which satisfies (1.4) and the inequality (3.13) then it follows that $Q'(ax) = aQ'(x)$,$Q'(a^m x) = a^m Q'(x)$ for all $x \in E_1$ and all $m \in N$. Thus

$$\begin{aligned}\left\|\frac{f(a^m x)}{a^m}\right\|_{E_2} &= \frac{1}{a^{2\beta m}} \left\|f(a^m x) - Q'(a^m x)\right\|_{E_2} \\ &\leq \left(\frac{aL}{a^{2\beta m}}\right)^m \left(\frac{a^2}{2}\right)^\beta \frac{\varphi(a^m x,0,0,0)}{\sqrt[p]{a^{\beta p} - (aL)^p}} \\ &\leq (aL)\left(\frac{a}{2}\right)^\beta \left(a^{1-2\beta}L\right)^m \frac{\varphi(x,0,0,0)}{\sqrt[p]{a^{2\beta p} - (aL)^p}}\end{aligned}$$

for all $x \in E_1$ and all $m \in N$. Allow $m \to \infty$, we get

$$\left\|Q(x) - Q'(x)\right\| = 0$$

for all $x \in E_1$, which completes the proof of uniqueness. □

Theorem 3.7. *Assume that there exists a mapping $\varphi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ for which an even mapping $f : E_1 \to E_2$ satisfies the inequality (3.12) and that the map φ is expansively superadditive with a constant L satisfying $a^{2\beta-1}L < 1$. Then there exists a unique quadratic mapping $Q : E_1 \to E_2$ which satisfies (1.4) and the inequality*

$$\|f(x) - Q(x)\|_{E_2} \leq \left(\frac{a^2}{2}\right)^{\beta} \frac{\varphi(x,0,0,0)}{\sqrt[p]{(1-a^{2\beta-1})^p}} \tag{3.18}$$

for all $x \in E_1$.

Proof. From (3.15), we obtain

$$\left\|f(x) - a^2 f\left(\frac{x}{a}\right)\right\| \leq \left(\frac{a^2}{2}\right)^{\beta} \varphi(x,0,0,0) \tag{3.19}$$

it follows from (3.19) with $\frac{x}{a^i}$ in place of x and iterative method that

$$\begin{aligned} \left\|a^{2l} f\left(\frac{x}{a^l}\right) - a^{2m} f\left(\frac{x}{a^m}\right)\right\|_{E_2}^p &\leq \sum_{i=l}^{m-1} a^{2\beta p i} \left\|f\left(\frac{x}{a^i}\right) - a^2 f\left(\frac{x}{a^{i+1}}\right)\right\|_{E_2}^p \\ &\leq \left(\frac{a^2}{2}\right)^{\beta p} \sum_{i=l}^{m-1} a^{2\beta p i} \varphi\left(\frac{x}{a^i}, 0,0,0\right)^p \\ &\leq \left(\frac{a^2}{2}\right)^{\beta p} \varphi(x,0,0,0) \sum_{i=l}^{m-1} \left(a^{2\beta-1}L\right)^{pi} \end{aligned} \tag{3.20}$$

for all $x \in E_1$ and for any $m > l \geq 0$. Therefore we see that a mapping $Q : E_1 \to E_2$ defined by

$$Q(x) = \lim_{m\to\infty} a^{2m} f\left(\frac{x}{a^m}\right)$$

is well defined for all $x \in E_1$. Taking the limit $m \to \infty$ in (3.20) with $l = 0$, we find that the mapping Q satisfing the inequality (3.18) near the approximate mapping $f : E_1 \to E_2$ of (1.4). The remaining proof is similar to that of Theorem 3.6. □

Theorem 3.8. *Assume that an even mapping $f : E_1 \to E_2$ satisfies*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \varphi(x,y,z,w)$$

for all $x, y, z, w \in E_1$. If a mapping $\varphi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ satisfies

$$\Psi(x,0,0,0) = \sum_{i=0}^{\infty} \frac{K^i \varphi\left(a^{i+1}x,0,0,0\right)}{a^{2\beta i}} < \infty \text{ and } \lim_{m\to\infty} \frac{K^m \varphi\left(a^m x,0,0,0\right)}{a^{2\beta m}} = 0$$

for all $x, y, z, w \in E_1$. Then there exists a unique quadratic mapping $Q : E_1 \to E_2$ such that Q satisfies (1.4) and the inequality

$$\|f(x) - Q(x)\|_{E_2} \leq \frac{K}{2^\beta}\Psi(x,0,0,0), \forall x \in E_1.$$

Proof. It follows from (3.16) with $a^i x$ in place of x and iterative method that

$$\left\| f(x) - f\left(\frac{a^m x}{a^{2m}}\right) \right\|_{E_2} \leq \frac{K}{2^\beta} \sum_{i=0}^{m-2} \frac{K^i \phi(a^{i+1}x,0,0,0)}{a^{\beta i}} + \frac{1}{2^\beta} \frac{K^{m-1}\phi(a^m x,0,0,0)}{a^{\beta(m-1)}} \tag{3.21}$$

for all $x \in E_1$ and for any $m > 1$, which is considered to be (3.9) for $m = 1$. In fact, we see by computation

$$\begin{aligned}
\left\| f(x) - \frac{f(a^{m+1}x)}{a^{2(m+1)}} \right\|_{E_2} &\leq K \left\| f(x) - \frac{f(ax)}{a^2} \right\|_{E_2} + \frac{K}{a^{2\beta}} \left\| f(ax) - \frac{f(a^{2(m+1)}x)}{a^m} \right\|_{E_2} \\
&\leq \frac{K}{2^\beta}\varphi(ax,0,0,0) + \frac{K^2}{(2a^2)^\beta} \sum_{i=0}^{m-2} \frac{K^i \varphi(a^{i+2}x,0,0,0)}{a^{2\beta i}} \\
&\quad + \frac{K^m}{(2a)^\beta} \frac{\varphi(a^{m+1}x,0,0,0)}{a^{2\beta(m-1)}} \\
&\leq \frac{K}{2^\beta} \sum_{j=0}^{m-1} \frac{K^j \varphi(a^{j+1}x,0,0,0)}{a^{2\beta j}} + \frac{K^m}{2^\beta} \frac{\varphi(a^{m+1}x,0,0,0)}{a^{2\beta m}},
\end{aligned}$$

for all $x \in E_1$, which proves the inequality (3.21) for $m+1$ by induction.

Thus follows that a sequence $\left\{\frac{f(a^m x)}{a^m}\right\}$ is cauchy in E_2 and it converges.Therefore we see that a mapping $A : E_1 \to E_2$ defined by $A(x) = \lim_{m\to\infty} \frac{f(a^m)}{a^m}$ is well defined for all $x \in E_1$. The remaining proof is similar to that of theorem 3.6. $\square$

Theorem 3.9. *Assume that an even mapping $f : E_1 \to E_2$ satisfies*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \phi(x,y,z,w)$$

for all $x, y, z, w \in E_1$. If a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \to [0, \infty)$ satisfies

$$\Psi(x,0,0,0) = \sum_{i=0}^{\infty} \left(a^{2\beta}K\right)^i \phi\left(\frac{x}{a^i},0,0,0\right) < \infty \text{ and } \lim_{m\to\infty} \left(a^{2\beta}K\right)^m \phi\left(\frac{x}{a^m},0,0,0\right) = 0.$$

For all $x, y, z, w \in E_1$. Then there exists a unique quadratic mapping $Q : E_1 \to E_2$ such that Q satisfies (1.4) and the inequality

$$\|f(x) - Q(x)\|_{E_2} \leq K\left(\frac{a^2}{2}\right)^{\beta} \Psi(x,0,0,0).$$

For all $x \in E_1$

Proof. It follows from(3.19) with $\frac{x}{a^i}$ and the similar method to (3.21) that

$$\begin{aligned}\left\|f(x) - a^{2m} f\left(\frac{x}{a^m}\right)\right\|_{E_2} \leq K\left(\frac{a^2}{2}\right)^{\beta} \sum_{i=0}^{m-2} \left(a^{2m}K\right)^i \varphi\left(\frac{x}{a^i},0,0,0\right) \\ + \left(\frac{a^2}{2}\right)^{\beta} \left(a^{2\beta}K\right)^{m-1} \varphi\left(\frac{x}{a^{m-1}},0,0,0\right)\end{aligned}$$

for all $x \in E_1$ and for any $m > 1$. Therefore we see that a mapping $Q : E_1 \to E_2$ defined by $Q(x) = \lim_{m\to\infty} a^m f\left(\frac{x}{a^m}\right)$ is well defined for all $x \in E_1$. The remaining proof is similar to that of Theorem 3.8. □

Corollary 3.10. *Let E_1 be a quasi-α-normed linear space with quasi-α-norm $\| \,.\, \|$. if there exists a fixed $r \in R$ such that an even mapping $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|D_1 f(x,y,z,w)\|_{E_2} \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right)$$

for all $x, y, z, w \in E_1$ ($E_1 \setminus \{0\}$ if $r \leq 0$), then there exists a unique quadratic mapping $Q : E_1 \to E_2$ which satisfies Eq.(1.4) and the inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{K\theta}{2^\beta} \frac{a^{2\alpha r}}{1-Ka^{\alpha r - 2\beta}} & if\ Ka^{\alpha r} < a^{2\beta}, \\ \left(\frac{K\theta a^{2\beta}}{2^\beta}\right) \frac{1}{1-Ka^{2\beta-\alpha r}} & if\ Ka^{2\beta} < a^{\alpha r}, \end{cases}$$

for all $x \in E_1$ ($E_1 \setminus \{0\}$ if $r \leq 0$),

Proof. Replacing $\varphi(x,y,z,w)$ by $\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right)$ in Theorem 3.6 and Theorem 3.7, we obtain above result. □

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