Mathematica Aeterna, Vol. 2, 2012, no. 3, 203 - 213

Bipolar Valued Fuzzy Sets in Γ-Semigroups

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Abstract

In this paper the notions of bipolar valued fuzzy subsemigroup, bipolar valued fuzzy bi-ideal, bipolar valued fuzzy (1,2)- ideal and bipolar valued fuzzy ideal in a Γ-semigroup have been introduced. Some important characterizations are also obtained.

Mathematics Subject Classification: 20M12, 03F55, 03E72

Keywords: Γ-Semigroup, Regular Γ-semigroup, Simple Γ-semigroup, Bipolar valued fuzzy set, Bipolar valued fuzzy subsemigroup, Bipolar valued fuzzy bi-ideal, Bipolar valued fuzzy (1,2)-ideal, Bipolar valued fuzzy left(right) ideal.

1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[8]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by Lofti Zadeh[20] in his classic paper in 1965. The concept of bipolar-valued fuzzy sets was introduced by Lee[12, 13]. In traditional fuzzy sets the membership degree range is $[0, 1]$. The membership degree is the degree of belonging ness of an element to a set. The membership degree 1 indicates that an element completely belongs to its corresponding set, the membership degree 0 indicates that an element does not belong to the corresponding set and the membership degree on the interval $(0, 1)$ indicate the partial membership to the corresponding set. Sometimes, membership degree also means the satisfaction degree of elements to some property corresponding to a set and its counter property. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is increased from the interval $[0, 1]$ to the interval $[-1, 1]$.

In a bipolar-valued fuzzy set the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property and the membership degrees on $[-1, 0]$ indicate that elements somewhat satisfy the implicit counter-property. Jun et al.[9] introduced the notion of bipolar fuzzy subalgebra and bipolar fuzzy ideal in BCH-algebras. Lee[11] introduced the notion of bipolar fuzzy subalgebra and bipolar fuzzy ideal in BCK/BCI-algebras. The concept of bipolar valued fuzzy translation and bipolar valued fuzzy S-extension of a bipolar valued fuzzy subalgebra in BCK/BCI-algebra was introduced by Y.B. Jun, H.S. Kim and K.J. Lee[10].

In 1981 M.K. Sen[15] introduced the notion of Γ-semigroup as a generalization of semigroup and ternary semigroup. We call this Γ-semigroup a both sided Γ-semigroup. In 1986 M.K. Sen and N.K. Saha[17] modified the definition of Sen's Γ-semigroup. This newly defined Γ-semigroup is known as one sided Γ-semigroup. Γ-semigroups have been analyzed by a lot of mathematicians, for instance by Chattopadhyay[1], Dutta and Adhikari[3, 4, 5], Hila[6, 7], Chinram[2], Sen et al.[16, 18]. T.K. Dutta and N.C. Adhikari[3, 4, 5] mostly worked on both sided Γ-semigroups. They defined operator semigroups of such type of Γ-semigroups and established many results and obtained many correspondences between a Γ-semigroup and its operator semigroups. In this paper we have considered both sided Γ-semigroups. The purpose of this paper is as stated in the abstract.

2 Preliminaries

In this section we discuss some elementary definitions that we use in the sequel.

Definition 2.1 [*4*] *Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.*

Example 1 [4] Let S be the set of all integers of the form $4n + 1$ and Γ be the set of all integers of the form $4n + 3$ where n is an integer. If $a\alpha b$ is $a + \alpha + b$ and $\alpha a\beta$ is $\alpha + a + \beta$(usual sum of integers) for all $a, b \in S$ and for all $\alpha, \beta \in \Gamma$, then S is a Γ-semigroup.

Definition 2.2 *A subsemigroup of a Γ-semigroup S is a non-empty subset I of S such that $I\Gamma I \subseteq I$.*

Definition 2.3 *A subsemigroup I of a Γ-semigroup S is called a bi-ideal of S if $I\Gamma S\Gamma I \subseteq I$.*

Definition 2.4 *A subsemigroup I of a Γ-semigroup S is called a (1,2)-ideal of S if $I\Gamma S\Gamma I\Gamma I \subseteq I$.*

Definition 2.5 [4] *A left ideal(right ideal) of a Γ-semigroup S is a non-empty subset I of S such that $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). If I is both a left ideal and a right ideal of S, then we say that I is an ideal of S.*

Definition 2.6 [10] *Let S be the universe of discourse. A bipolar valued fuzzy set φ in S is an object having the form $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ where the negative membership degree $\varphi^- : S \to [-1, 0]$ is a mapping denotes the satisfaction degree of an element x to some implicit counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ and the positive membership degree $\varphi^+ : S \to [0, 1]$ is a mapping denotes the satisfaction degree of an element x to the property corresponding to $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.*

Remark 1: [10] (1) If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

(2) If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, then it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ but some what satisfies the counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

(3) If $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$, then the membership function of the property overlaps that of its counter property over the some portion of the domain.

Remark 2: For the sake of simplicity we shall use the symbol $\varphi = (S; \varphi^-, \varphi^+)$ for the bipolar valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

Definition 2.7 *For a bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ and $(\alpha, \beta) \in [-1, 0) \times (0, 1]$, the sets $N(\varphi; \alpha) := \{x \in S : \varphi^-(x) \leq \alpha\}$ and $P(\varphi; \beta) := \{x \in S : \varphi^+(x) \geq \beta\}$ are called the negative α-cut and positive β-cut of $\varphi = (S; \varphi^-, \varphi^+)$ respectively. The set $C(\varphi; (\alpha, \beta)) := N(\varphi; \alpha) \cap P(\varphi; \beta)$ is called the bipolar (α, β)-cut of $\varphi = (S; \varphi^-, \varphi^+)$.*

3 Main Results

Definition 3.1 *A non-empty bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy subsemigroup of S if*

(1) $\varphi^-(x\gamma y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$,

(2) $\varphi^+(x\gamma y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$.

Definition 3.2 *A bipolar valued fuzzy subsemigroup $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy bi-ideal of S if*

(1) $\varphi^-(x\alpha y\beta z) \leq \max\{\varphi^-(x), \varphi^-(z)\}$ $\forall x, y, z \in S$ *and* $\forall \alpha, \beta \in \Gamma$,

(2) $\varphi^+(x\alpha y\beta z) \geq \min\{\varphi^+(x), \varphi^+(z)\}$ $\forall x, y, z \in S$ *and* $\forall \alpha, \beta \in \Gamma$.

Definition 3.3 *A bipolar valued fuzzy subsemigroup $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy (1,2)-ideal of S if*

(1) $\varphi^-(x\alpha\omega\beta(y\gamma z)) \leq \max\{\varphi^-(x), \varphi^-(y), \varphi^-(z)\}$ $\forall x, \omega, y, z \in S$ *and* $\forall \alpha, \beta, \gamma \in \Gamma$,

(2) $\varphi^+(x\alpha\omega\beta(y\gamma z)) \geq \min\{\varphi^+(x), \varphi^+(y), \varphi^+(z)\}$ $\forall x, \omega, y, z \in S$ *and* $\forall \alpha, \beta, \gamma \in \Gamma$.

Example 2 Let S be the set of all non positive integers and Γ be the set of all non positive even integers. Then S is a Γ-semigroup if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy subset of S, defined as follows

$$\varphi^-(x) = \begin{cases} -1, & \text{if } x = 0 \\ -0.1, & \text{if } x = -1, -2 \\ -0.2, & \text{if } x < -2 \end{cases} \quad \text{and} \quad \varphi^+(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.1, & \text{if } x = -1, -2 \\ 0.2, & \text{if } x < -2 \end{cases}.$$

Then the bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of S is a bipolar valued fuzzy subsemigroup of S.

Theorem 3.4 *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy set of a Γ-semigroup S and for $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, $C(\varphi; (\alpha, \beta))$ be a non-empty bipolar (α, β)-cut of $\varphi = (S; \varphi^-, \varphi^+)$. Then $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup(bi-ideal, $(1, 2)$-ideal) of S if and only if $C(\varphi; (\alpha, \beta))$ is a subsemigroup(resp. bi-ideal, $(1, 2)$-ideal) of S.*

Proof: Let $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup of S and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(\varphi; (\alpha, \beta)) \neq \phi$. Let $x, y \in C(\varphi; (\alpha, \beta))$ and $\gamma \in \Gamma$. Then $\varphi^-(x) \leq \alpha, \varphi^-(y) \leq \alpha, \varphi^+(x) \geq \beta, \varphi^+(y) \geq \beta$. Since $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup of S, so $\varphi^-(x\gamma y) \leq \max\{\varphi^-(x), \varphi^-(y)\} \leq \alpha$ and $\varphi^+(x\gamma y) \geq \min\{\varphi^+(x), \varphi^+(y)\} \geq \beta$. So $x\gamma y \in C(\varphi; (\alpha, \beta))$. Hence $C(\varphi; (\alpha, \beta))$ is a subsemigroup of S.

In order to prove the converse, we have to show that $\varphi = (S; \varphi^-, \varphi^+)$ satisfies the conditions of Definition 3.1. If the conditions of Definition 3.1 is false, then there exist $x, y \in S$ and $\gamma \in \Gamma$ with the following cases:

Case (1) : $\varphi^-(x\gamma y) > \max\{\varphi^-(x), \varphi^-(y)\}, \varphi^+(x\gamma y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$,

Case (2) : $\varphi^-(x\gamma y) > \max\{\varphi^-(x), \varphi^-(y)\}, \varphi^+(x\gamma y) < \min\{\varphi^+(x), \varphi^+(y)\}$,

Case (3) : $\varphi^-(x\gamma y) \leq \max\{\varphi^-(x), \varphi^-(y)\}, \varphi^+(x\gamma y) < \min\{\varphi^+(x), \varphi^+(y)\}$.

For Case (1), let $\alpha = \max\{\varphi^-(x), \varphi^-(y)\}$ and $\beta = \min\{\varphi^+(x), \varphi^+(y)\}$. Then $x, y \in C(\varphi; (\alpha, \beta))$ but $x\gamma y \notin C(\varphi; (\alpha, \beta))$. This is a contradiction. In Case (2) and (3) we also get contradiction. So the conditions of Definition 3.1 are true. Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup of S. Similarly we can prove the other cases also.

Definition 3.5 *A non-empty bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy left ideal of S if*

(1) $\varphi^-(x\gamma y) \leq \varphi^-(y)$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$,

(2) $\varphi^+(x\gamma y) \geq \varphi^+(y)$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$.

Definition 3.6 *A non-empty bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy right ideal of S if*

(1) $\varphi^-(x\gamma y) \leq \varphi^-(x)$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$,

(2) $\varphi^+(x\gamma y) \geq \varphi^+(x)$ $\forall x, y \in S$ *and* $\forall \gamma \in \Gamma$.

Definition 3.7 *A non-empty bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of a Γ-semigroup S is called a bipolar valued fuzzy ideal of S if it is a bipolar valued fuzzy left ideal and bipolar valued fuzzy right ideal of S.*

Example 3 Let $S = \{e, a, b\}$ and $\Gamma = \{\gamma\}$, where γ is defined on S with the following cayley table:

γ	e	a	b
e	e	e	e
a	e	a	e
b	e	e	b

Then S is a Γ-semigroup. Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy subset of S, defined as follows:

S	e	a	b
φ^-	-1	-0.1	-0.2
φ^+	1	0.2	0.4

Then the bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of S is a bipolar valued fuzzy ideal of S.

Theorem 3.8 *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy set of a Γ-semigroup S and for $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, $C(\varphi; (\alpha, \beta))$ be a non-empty bipolar (α, β)-cut of $\varphi = (S; \varphi^-, \varphi^+)$. Then $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy left(fuzzy right ideal, ideal) of S if and only if $C(\varphi; (\alpha, \beta))$ is a left ideal(resp. right ideal, ideal) of S.*

Proof: Let $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy left ideal of S and $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ be such that $C(\varphi; (\alpha, \beta)) \neq \phi$. Let $x \in S$ and $y \in C(\varphi; (\alpha, \beta))$ and $\gamma \in \Gamma$. Then $\varphi^-(y) \leq \alpha, \varphi^+(y) \geq \beta$. Since $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy left ideal of S, so $\varphi^-(x\gamma y) \leq \varphi^-(y) \leq \alpha$ and $\varphi^+(x\gamma y) \geq \varphi^+(y) \geq \beta$. So $x\gamma y \in C(\varphi; (\alpha, \beta))$. Hence $C(\varphi; (\alpha, \beta))$ is a left ideal of S.

In order to prove the converse, we have to show that $\varphi = (S; \varphi^-, \varphi^+)$ satisfies the conditions of Definition 3.5 If the conditions of Definition 3.5 is false, then there exist $x, y \in S$ and $\gamma \in \Gamma$ with the following cases:

Case (1) : $\varphi^-(x\gamma y) > \varphi^-(y), \varphi^+(x\gamma y) \geq \varphi^+(y)$,

Case (2) : $\varphi^-(x\gamma y) > \varphi^-(y), \varphi^+(x\gamma y) < \varphi^+(y)$,

Case (3) : $\varphi^-(x\gamma y) \leq \varphi^-(y), \varphi^+(x\gamma y) < \varphi^+(y)$.

For Case (1), let $\alpha = \frac{\varphi^-(x\gamma y)+\varphi^-(y)}{2}$. Then $\varphi^-(x\gamma y) > \alpha > \varphi^-(y)$, consequently, $y \in C(\varphi; (\alpha, \varphi^+(y)))$ but $x\gamma y \notin C(\varphi; (\alpha, \varphi^+(y)))$. This is a contradiction.

For Case (3), let $\beta = \frac{\varphi^+(x\gamma y)+\varphi^+(y)}{2}$. Then $\varphi^+(x\gamma y) < \beta < \varphi^+(y)$, consequently, $y \in C(\varphi; (\varphi^-(y), \beta))$ but $x\gamma y \notin C(\varphi; (\varphi^-(y), \beta))$. This is a contradiction.

For Case (2), let $\alpha = \frac{\varphi^-(x\gamma y)+\varphi^-(y)}{2}$ and $\beta = \frac{\varphi^+(x\gamma y)+\varphi^+(y)}{2}$. Then $\varphi^-(x\gamma y) > \alpha \geq \varphi^-(y)$ and $\varphi^+(x\gamma y) < \beta \leq \varphi^+(y)$, consequently, for some $(\alpha, \beta) \in [-1, 0) \times (0, 1], y \in C(\varphi; (\alpha, \beta))$ but $x\gamma y \notin C(\varphi; (\alpha, \beta))$. This is a contradiction. So the conditions of Definition 3.5 are true. Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy left ideal of S. Similarly we can prove the other cases also.

Definition 3.9 *A Γ-semigroup S is called left(right) duo if every left(resp. right) ideal of S is a two sided ideal of S.*

Definition 3.10 *A Γ-semigroup S is called duo if it is left and right duo.*

Definition 3.11 *A Γ-semigroup S is called bipolar fuzzy left(right) duo if every bipolar valued fuzzy left(resp. right) ideal of S is a bipolar valued fuzzy ideal of S.*

Theorem 3.12 *In a regular left duo(right duo, duo) Γ-semigroup S, following are equivalent: (1) $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy bi-ideal of S, (2) $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy $(1,2)$-ideal of S.*

Proof: $(1) \Rightarrow (2)$: Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy bi-ideal of S and $x, w, y, z \in S, \alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned}\varphi^-(x\alpha w\beta(y\gamma z)) &= \varphi^-((x\alpha w\beta y)\gamma z) \leq \max\{\varphi^-(x\alpha w\beta y), \varphi^-(z)\} \\ &\leq \max[\max\{\varphi^-(x), \varphi^-(y)\}, \varphi^-(z)] \\ &= \max\{\varphi^-(x), \varphi^-(y), \varphi^-(z)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x\alpha w\beta(y\gamma z)) &= \varphi^+((x\alpha w\beta y)\gamma z) \geq \min\{\varphi^+(x\alpha w\beta y), \varphi^+(z)\} \\ &\geq \min[\min\{\varphi^+(x), \varphi^+(y)\}, \varphi^+(z)] \\ &= \min\{\varphi^+(x), \varphi^+(y), \varphi^+(z)\}.\end{aligned}$$

Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy $(1,2)$-ideal of S.

$(2) \Rightarrow (1)$: Let S be a regular and left duo Γ-semigroup and $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy $(1,2)$-ideal of S. Let $x, w, y \in S, \alpha, \delta \in \Gamma$. Since S is regular and left duo, we have $x\alpha w \in (x\Gamma S\Gamma x)\Gamma S \subseteq x\Gamma S\Gamma x$, which implies $x\alpha w = x\beta s\gamma x$ for some $s \in S$ and $\beta, \gamma \in \Gamma$. Then

$$\begin{aligned}\varphi^-(x\alpha w\delta y) &= \varphi^-((x\beta s\gamma x)\delta y) = \varphi^-(x\beta s\gamma(x\delta y)) \\ &\leq \max\{\varphi^-(x), \varphi^-(x), \varphi^-(y)\} \\ &= \max\{\varphi^-(x), \varphi^-(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x\alpha w\delta y) &= \varphi^+((x\beta s\gamma x)\delta y) = \varphi^+(x\beta s\gamma(x\delta y)) \\ &\geq \min\{\varphi^+(x), \varphi^+(x), \varphi^+(y)\} \\ &= \min\{\varphi^+(x), \varphi^+(y)\}.\end{aligned}$$

Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy bi-ideal of S.

Theorem 3.13 *In a regular left duo(right duo, duo) Γ-semigroup S, following are equivalent: (1) $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy right ideal(resp. fuzzy left ideal, fuzzy ideal) of S, (2) $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy bi-ideal of S.*

Proof: $(1) \Rightarrow (2)$: Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy right ideal of S and let $x, y, z \in S, \alpha, \beta \in \Gamma$. Then

$$\varphi^-(x\alpha y\beta z) = \varphi^-(x\alpha(y\beta z)) \leq \varphi^-(x) \leq \max\{\varphi^-(x), \varphi^-(z)\}$$

and

$$\varphi^+(x\alpha y\beta z) = \varphi^+(x\alpha(y\beta z)) \geq \varphi^+(x) \geq \min\{\varphi^+(x), \varphi^+(z)\}.$$

Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy bi-ideal of S. Similarly we can prove the other cases.

(2) $\Rightarrow$ (1) : Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy bi-ideal of S and $x, y \in S, \gamma \in \Gamma$. Then $x\gamma y \in S$. Since S is regular and left duo in view of the fact that $S\Gamma x$ is a left ideal we obtain, $x\gamma y \in (x\Gamma S\Gamma x)\Gamma S \subseteq x\Gamma S\Gamma x$. This implies that there exist elements $z \in S, \alpha, \beta \in \Gamma$ such that $x\gamma y = x\alpha z\beta x$. Then

$$\varphi^-(x\gamma y) = \varphi^-(x\alpha z\beta x) \leq \max\{\varphi^-(x), \varphi^-(x)\} = \varphi^-(x)$$

and

$$\varphi^+(x\gamma y) = \varphi^+(x\alpha z\beta x) \geq \min\{\varphi^+(x), \varphi^+(x)\} = \varphi^+(x).$$

Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy right ideal of S. Similarly we can prove the other cases also.

Theorem 3.14 *Let S be a regular Γ-semigroup. If S is left(right) duo, then S is bipolar fuzzy left (right) duo.*

Proof: Let us assume that S is a left duo Γ-semigroup. Let $\varphi = (S; \varphi^-, \varphi^+)$ be any bipolar valued fuzzy left ideal of S and $a, b \in S, \gamma \in \Gamma$. Then, since the left ideal $S\Gamma a$ is a two-sided ideal of S, and since S is regular, we have

$$a\gamma b \in (a\Gamma S\Gamma a)\Gamma b \subseteq (S\Gamma a)\Gamma S \subseteq S\Gamma a.$$

This implies that there exist elements $x \in S, \alpha \in \Gamma$ such that $a\gamma b = x\alpha a$. Then, since $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy left ideal of S, so $\varphi^-(x\alpha a) \leq \varphi^-(a), \varphi^+(x\alpha a) \geq \varphi^+(a)$ and then $\varphi^-(a\gamma b) \leq \varphi^-(a), \varphi^+(a\gamma b) \geq \varphi^+(a)$. Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar fuzzy ideal of S. Hence we deduce that S is bipolar fuzzy left duo. Similarly we can prove the other case also.

Definition 3.15 [*19*] *A Γ-semigroup S is called left-zero(right zero) if $x\gamma y = x$(resp. $x\gamma y = y$) $\forall x, y \in S, \forall \gamma \in \Gamma$.*

Theorem 3.16 *In a left zero(right zero) Γ-semigroup S every bipolar valued fuzzy left(resp. right) ideal is constant function.*

Proof: Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy left ideal of S and $x \in S$. Since S is left zero so there exist $y \in S$ and $\gamma \in \Gamma$ such that $x = x\gamma y$. Then $\varphi^-(x) = \varphi^-(x\gamma y) \leq \varphi^-(y)$ and $\varphi^+(x) = \varphi^+(x\gamma y) \geq \varphi^+(y)$. Again $\varphi^-(y) = \varphi^-(y\gamma x) \leq \varphi^-(x)$ and $\varphi^+(y) = \varphi^+(y\gamma x) \geq \varphi^+(x)$. So, $\varphi^-(x) = \varphi^-(y)$ and $\varphi^+(x) = \varphi^+(y) \forall x, y \in S$. Hence $\varphi = (S; \varphi^-, \varphi^+)$ constant function. Similarly we can prove the other case also.

Definition 3.17 *A Γ-semigroup S is said to be left(right) simple if S has no proper left(resp. right) ideals.*

Definition 3.18 *If a Γ-semigroup S has no proper ideals, then we say that S is simple.*

Definition 3.19 *A Γ-semigroup S is said to be bipolar fuzzy left(bipolar fuzzy right) simple if every bipolar valued fuzzy left(resp. bipolar valued fuzzy right) ideal of S is a constant function.*

Definition 3.20 *A Γ-semigroup S is said to be bipolar fuzzy simple if every bipolar valued fuzzy ideal of S is a constant function.*

Theorem 3.21 *Let S be Γ-semigroup. If S is left simple(right simple, simple), then S is bipolar fuzzy left simple(resp. bipolar fuzzy right simple, bipolar fuzzy simple).*

Proof: Let us assume that S is left simple Γ-semigroup. Let $\varphi = (S; \varphi^-, \varphi^+)$ be any bipolar valued fuzzy left ideal of S and $a, b \in S$. Then there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $b = x\alpha a$ and $a = y\beta b$ and so we obtain

$$\varphi^-(a) = \varphi^-(y\beta b) \leq \varphi^-(b) = \varphi^-(x\alpha a) \leq \varphi^-(a)$$

and

$$\varphi^+(a) = \varphi^+(y\beta b) \geq \varphi^+(b) = \varphi^+(x\alpha a) \geq \varphi^+(a).$$

Consequently, $\varphi^-(a) = \varphi^-(b)$ and $\varphi^+(a) = \varphi^+(b)$. Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a constant function. Consequently, S is bipolar fuzzy left simple. Similarly we can prove the other case also.

Theorem 3.22 *Let S be a left(right) simple Γ-semigroup. Then every bipolar valued fuzzy bi-ideal of S is a bipolar valued fuzzy right ideal(resp. bipolar valued fuzzy left ideal) of S.*

Proof: Let $\varphi = (S; \varphi^-, \varphi^+)$ be any bipolar valued fuzzy bi-ideal of S and $a, b \in S$. Since S is left simple, so there exist $x \in S, \gamma \in \Gamma$ such that $b = x\gamma a$. Then

$$\begin{aligned}\varphi^-(a\alpha b) &= \varphi^-(a\alpha x\gamma a) \leq \max\{\varphi^-(a), \varphi^-(a)\} \\ &= \varphi^-(a) \forall \alpha \in \Gamma\end{aligned}$$

and

$$\begin{aligned}\varphi^+(a\alpha b) &= \varphi^+(a\alpha x\gamma a) \geq \min\{\varphi^+(a), \varphi^+(a)\} \\ &= \varphi^+(a) \forall \alpha \in \Gamma.\end{aligned}$$

Hence $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy right ideal of S. Similarly we can prove the other case also.

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Received: January, 2012