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Length spectra and systole for a set of closed hyperbolic surface in genus 2

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Abstract

In this paper we determine the length of the two shortest closed geodesics for closed hyperbolic surface of genus 2 without twist in the Fenchel-Nielsen coordinates (in a specific pants decomposition). We also give systole in other cases.

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Keywords: geodesic, hyperbolic surface, length spectrum

1 Introduction

In 1966, M. Kac asked: "Can one hear the shape of a drum ?" [6] which can be translated as "Can the Laplacian spectra of a compact riemaniann manifold determine its isometry class ?". See [2] for an introduction on Laplacian spectra. We must keep in mind that in the case of closed hyperbolic surfaces the Laplacian spectra and the length spectra (the well ordered sequence of lengths of all closed geodesics with multiplicities) determine each other.

J. Milnor swiftly gave a negative answer, finding non isometric but isospectral 16- dimensional flat torus. In 1980, M-F. Vignéras first proved that closed hyperbolic surfaces are not rigid, constructing two arithmetic surfaces isospectral but non isometric [9] before T. Sunada gave general criterion to construct isospectral manifolds [8]. P. Buser then used this result to construct explicitly, for any genus $g \geq 4$ counter examples for the Kac's conjecture [3]. Sunada's technique seems not to be used in genus 2 and 3 where the question still remains open. However, according to the rigidity result obtained by P. Buser and K-D. Semmler on hyperbolic one-holed torus [4] we may expect positive answer in genus $g = 2$. This article is written

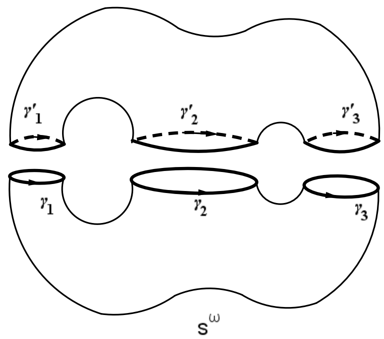

Figure 1: A model for Teichmüller in genus 2

following this point of view, knowing that all rigidity results need explicit explanation of the bottom of the spectra. Bear in mind that spectra determine topology ([3], chap VII): isospectral twins then live in the same Teichmüller space.

2 Main Results

To describe the bottom of the length spectra, we use a geometric realization of the Teichmüller space in genus 2, initially seen as the equivalence classes of hyperbolic surfaces marked by a diffeomorphism (we shall consult [3]).
So let us choose lengths $l_1, l_2, l_3 \in \mathbf{R}_+^*$ and angles (called *twists*) $\theta_1, \theta_2, \theta_3 \in \mathbf{R}$. We then construct a closed surface of genus 2 pasting two isometric pants with boundaries γ_i, γ_i' having same length l_i, according to the pasting condition $\gamma_i(t) \sim \gamma_i'(\theta_i - t)$ (we use 1-periodic parametrization for boundaries).
Any marked surface contains, in its equivalence class, a unique surface S^ω. See [3] for a proof. The l_i and θ_i are the *Fenchel-Nielsen coordinates* of S associated to this pants decomposition.

With this presentation, we describe now the *systole* (the shortest closed geodesic) as well as the second value of the length spectra for a specific family of surfaces. For S with coordinates ω, we order the lengths l_i and note $x \leq y \leq z$ the corresponding values. We then obtain the following

Theorem 1:
Let S be a closed hyperbolic surface with all $\theta_i = 0$ in Fenchel-Nielsen coordinates.

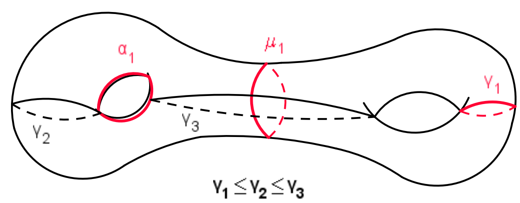

Figure 2: The candidates for systole

With the above notation, the systole of S is given by at least one of the following:

$$\cosh(\frac{\text{syst}}{2}) = \cosh x \quad ; \quad \cosh(\frac{\text{syst}}{2}) = \frac{\cosh x + \cosh y \cosh z}{\sinh y \sinh z}$$
$$\cosh(\frac{\text{syst}}{2}) = \frac{2(\cosh x)^2 + 2(\cosh y)^2 + (\cosh z)^2 + 4\cosh x \cosh y \cosh z + 1}{\sinh z}$$

The corresponding curves will be geometrically identified and we will also give an algorithm for the first two lengths following the coordinates. We will finally establish generalizations, where $z_0 = z_0(x, y)$ is determined by the first two lengths in the pants decomposition (cf. Annex).

Corollary 1:
Let S be a closed hyperbolic surface with $z \geq z_0$ and the corresponding twist vanished in Fenchel-Nielsen coordinates. The systole of S is

$$\cosh(\frac{\text{syst}}{2}) = \frac{2(\cosh x)^2 + 2(\cosh y)^2 + (\cosh z)^2 + 4\cosh x \cosh y \cosh z + 1}{\sinh z}$$

Corollary 2:
Let S be a closed hyperbolic surface with $\cosh z \leq 2$ in Fenchel-Nielsen coordinates. The systole of S is $2x$.

No explicit formula for the systole of closed hyperbolic surfaces is known. However we will find in [1] a numerical approach which gives an approximate value of the systole in any genus g. The technique presented in this paper to describe the bottom of the spectra differs from the one used in [4],[7] or [5] and deals with the combinatorial aspect of billiard trajectories in right angled hexagon, where the closed geodesics of S "really" lives.

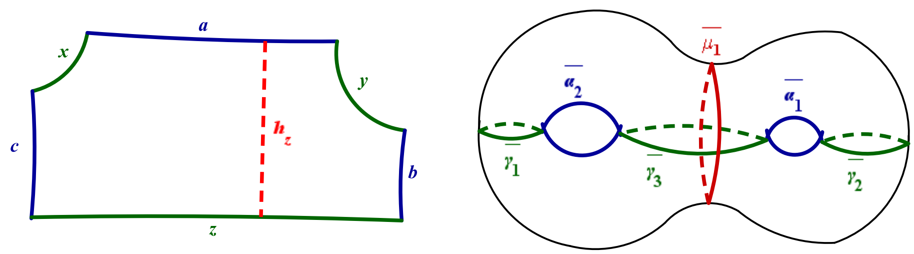

Figure 3: The system Ω

3 Notation

Each surface S^ω is identified with four copies of the same right angled hexagon $\mathcal{H}$. We use the following notations

Notation	Length	Geodesic in S^ω (untwisted)
x	$\inf\{l_i/2\}$	$\bar{\gamma}_1$
z	$\sup\{l_i/2\}$	$\bar{\gamma}_3$
y		$\bar{\gamma}_2$
b	opposed side of x in $\mathcal{H}$	$\bar{\alpha}_1$
c	opposed side of y in $\mathcal{H}$	$\bar{\alpha}_2$
a	opposed side of z in $\mathcal{H}$	$\bar{\alpha}_3$
h_z	commun perpendicular between a and z	$\bar{\mu}_1$
h_y	commun perpendicular between c and y	$\bar{\mu}_2$
h_x	commun perpendicular between b and x	$\bar{\mu}_3$

Note that lengths of geodesics are twice (or four times for $\bar{\mu}_i$) as long as lengths given in the grid. Without ambiguity we write β for the geodesic or the length of the geodesic.
The right column gives a *fundamental system* of closed geodesics on S^ω called Ω. By analogy we will write $\Omega^{(n)}$ for the set of n-fold geodesics of $\Omega = \Omega^{(1)}$.

4 Proofs

We first assume that all twists take the value zero.
Let $\beta \neq \gamma_i$ be a closed geodesic of S decomposed into N β-*pieces* (its successive intersections with the pants of the pants decomposition): N is 1 or is even. We lift the geodesic into a piece geodesic curve in $\mathcal{H}$ and note X the number of γ-*arc* in the lift (this is a geodesic segment joining two sides of a, b, c in the hexagon).
Note that if $N = 1$, β is contained in a pant so is longer than γ_i.

Lemma 4.1. *If $N \geq 4$ or $N \geq 2, X \geq 1$, there are at least two curves in $\Omega \cup \Omega^{(2)}$ shorter than β.*

Proof:
Each β-piece is longer than $\bar{\alpha}_1/2$ or $\bar{\mu}_1/2$. If $X \geq 1$ we use the continuity and the closure of the lift to show that the length of β is greater than $\bar{\gamma}_1$. ✠

Lemma 4.2. *If $N = 2, X = 0$ and $\beta \notin \Omega$, there are at least two curves in Ω shorter than β.*

Proof:
We examine each case with a combinatorial elementary study and use, if necessary, the triangular inequality.✠

Finally we have to order the lengths of the fundamental system to find the two shortest geodesics. As $b \leq c \leq a$ and $h_z \leq h_y \leq h_x$, one expects that these curves are $\bar{\alpha}_1, \bar{\gamma}_1$ or $\bar{\mu}_1$. After an elementary study we obtain the results given in annex where can be found a complete description of the situation.
We have in particular the following result (description of the systole if z increases):

Value of x	**Value of y**	**Systole**
$\cosh x \leq \sqrt{2}$	$\cosh y \leq \mathbf{y_2}$	systole is $\bar{\gamma}_1$, then $\bar{\mu}_1$
	$\mathbf{y}_2 < \cosh y \leq \mathbf{y}_3$	systole is $\bar{\gamma}_1$, then $\bar{\alpha}_1$, then $\bar{\mu}_1$
	$\cosh y > \mathbf{y}_3$	systole is $\bar{\alpha}_1$, then $\bar{\mu}_1$
$\sqrt{2} < \cosh x \leq 2$	$\cosh y \leq \mathbf{y}_3$	systole is $\bar{\gamma}_1$, then $\bar{\alpha}_1$, then $\bar{\mu}_1$
	$\cosh y > \mathbf{y}_3$	systole is $\bar{\alpha}_1$, then $\bar{\mu}_1$
$\cosh x > 2$		systole is $\bar{\alpha}_1$, then $\bar{\mu}_1$

Note that sometimes the bottom of the spectra is constituted of multiples of the systole. The following picture gives the evolution when z increases.

We will now examine the case where the twists need not to vanish: we can always use that, if $N \geq 2$, the length of β is always greater than at least one of the values $2b, 4h_z$.

Corollary 4.3. *Let S be a closed hyperbolic surface with Fenchel-Nielsen coordinates $z \geq \sup\{z_4, z_5\}$ (values given in annex) and the corresponding twist $\theta_z = 0$. Then the systole of S is*

$$\cosh(\frac{\text{syst}}{2}) = \frac{2(\cosh x)^2 + 2(\cosh y)^2 + (\cosh z)^2 + 4\cosh x \cosh y \cosh z + 1}{\sinh z}$$

Corollary 4.4. *Let S be a closed hyperbolic surface with Fenchel-Nielsen coordinates $\cosh z \leq 2$. Then the systole of S is $2x$.*

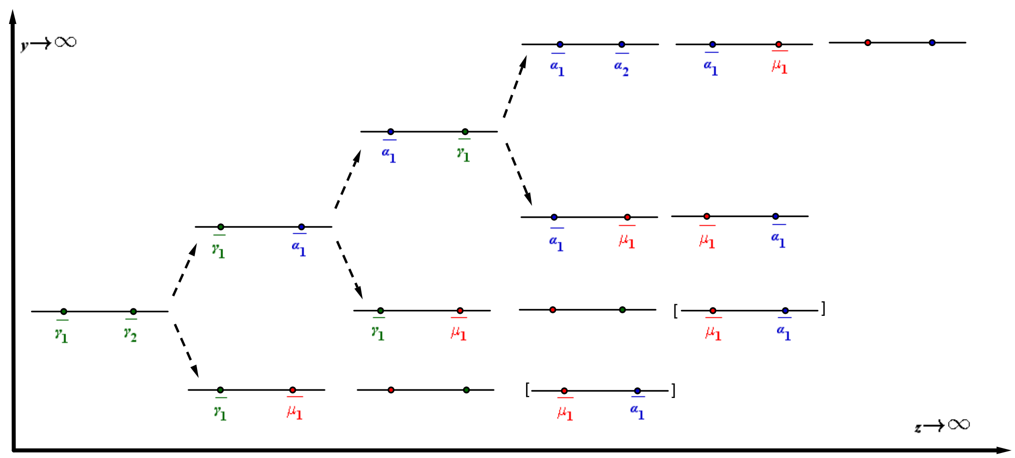

Figure 4: Evolution when z increases

Note (and this is a general observation) that the action of twists make systole longer: the untwisted surface minimizes the length in its orbit. In particular, if $\bar{\gamma}_1$ is the systole when the twists vanish, it still the case. The last corollary is a special case of this situation.
Now we can use this work as a first step toward the general case and we can expect to understand more explicitly the geometric information given by the spectra and then find where (in the Teichmüller space) there can exists isospectral twins in genus 2, if such objects exist.

5 Annex

The following grid gives explicitly (as an algorithm would) the two shortest (which are not multiples) lengths of the spectra. Let's not forget that it may happen that some multiples of the systole occur before the second value.
Note that we use strict inequalities for comprehensive presentation. The limit cases correspond to lengths equalities, so the exact multiplicity of the length studied can be determined easily. We also use notations $\mathbf{x} = \cosh x$ and the same for $\mathbf{y}, \mathbf{z}, \mathbf{x_i}, \mathbf{y_i}, \mathbf{z_i}$, where these values are defined in a final grid.

Value of $\mathbf{x}$	**Value of** $\mathbf{y}$	**Value of** $\mathbf{z}$	**Bottom of the spectra**
$\mathbf{x} < \sqrt{2}$	$\mathbf{y} < \sqrt{2}$	$\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
$\mathbf{x} < \mathbf{x}_1 < \sqrt{2}$	$\sqrt{2} < \mathbf{y} < \mathbf{y}_1$	$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
	$\mathbf{y}_1 < \mathbf{y} < \mathbf{y}_2$	$\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
	$\mathbf{y}_2 < \mathbf{y} < \mathbf{y}_3$	$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_3 < \mathbf{y}$	$\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$

Value of $\mathbf{x}$	**Value of** $\mathbf{y}$	**Value of** $\mathbf{z}$	**Bottom of the spectra**
$\mathbf{x}_1 < \mathbf{x} < \sqrt{2}$	$\sqrt{2} < \mathbf{y} < \mathbf{y}_2$	$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_1$ and $\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_1$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
	$\mathbf{y}_2 < \mathbf{y} < \mathbf{y}_1$	$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2$ and $\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2$ and $\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2 > \mathbf{z}_1$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2 > \mathbf{z}_1$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_2 > \mathbf{z}_1$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_1 < \mathbf{y} < \mathbf{y}_3$	$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_2$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_2$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y} > \mathbf{y}_3$	$\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$

Value of x	**Value of y**	**Value of z**	**Bottom of the spectra**
$\sqrt{2} < \mathbf{x} < \mathbf{x}_2$	$\mathbf{y} < \mathbf{y}_1$	$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_1 < \mathbf{y} < \mathbf{y}_3$	$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
		$\mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_3 < \mathbf{y}$	$\mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z} < \mathbf{z}_3$	

Value of $\mathbf{x}$	Value of $\mathbf{y}$	Value of $\mathbf{z}$	Bottom of the spectra
$\sqrt{2} < \mathbf{x} < \mathbf{x}_2$	$\mathbf{y}_3 < \mathbf{y}$	$\mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
$\mathbf{x}_2 < \mathbf{x} < 2$	$\mathbf{y} < \mathbf{y}_1$	$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_6 < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\gamma}_2$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}_6 < \mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}_6 < \mathbf{z}_5 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_1 < \mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_1 < \mathbf{y} < \mathbf{y}_3$	$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\alpha}_1$
		$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\gamma}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_2 < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\gamma}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_2 < \mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
		$\mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_2 < \mathbf{z}_3 < \mathbf{z}$	

Value of x	**Value of y**	**Value of z**	**Bottom of the spectra**
$\mathbf{x}_2 < \mathbf{x} < 2$	$\mathbf{y}_3 < \mathbf{y} < \mathbf{y}_4$	$\mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} < \mathbf{z}_4$ and $\mathbf{z} < \mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\gamma}_1$
		$\mathbf{z} < \mathbf{z}_3$ and $\mathbf{z}$ between $\mathbf{z}_4$ and $\mathbf{z}_5$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z} < \mathbf{z}_3$ and $\mathbf{z} > \mathbf{z}_4$ and $\mathbf{z} > \mathbf{z}_5$	$\bar{\mu}_1 < \bar{\alpha}_1$
		$< \mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_3 < \mathbf{z}$ and $\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
	$\mathbf{y}_4 < \mathbf{y}$	$\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$
$2 < \mathbf{x}$		$\mathbf{z} < \mathbf{z}_7 < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\alpha}_2$
		$\mathbf{z}_7 < \mathbf{z} < \mathbf{z}_4$	$\bar{\alpha}_1 < \bar{\mu}_1$
		$\mathbf{z}_7 < \mathbf{z}_4 < \mathbf{z}$	$\bar{\mu}_1 < \bar{\alpha}_1$

We introduce the function $\varphi(t) := \dfrac{t}{\sqrt{t^2-1}}$ and recall the following elementary trigonometric formulae in the right hexagon $\mathbf{b} = \dfrac{\mathbf{x}+\mathbf{yz}}{\sqrt{(\mathbf{y}^2-1)(\mathbf{z}^2-1)}}$ (the $\mathbf{c}$ value given by analogy) and $\mathbf{h_z} = 2(\mathbf{c}^2-1)(\mathbf{x}^2-1)-1$. The constants used are uniquely defined by the following equations:

Value	**Equation**	**Value**	**Equation**
x_1	$1+\varphi(\mathbf{x})+(\varphi(\mathbf{x}))^2-(\varphi(\mathbf{x}))^3=0$	x_2	$1+\mathbf{x}+\mathbf{x}^2-\mathbf{x}^3=0$
y_1	$\mathbf{x}+\mathbf{y}+\mathbf{y}^2-\mathbf{y}^3=0$	y_2	$(\mathbf{x}^2-1)\mathbf{y}^2-\mathbf{x}^2=0$ soit $\mathbf{y}=\varphi(\mathbf{x})$
y_3	$(\mathbf{x}-1)\mathbf{y}^2-2\mathbf{x}=0$	y_4	$(1+\mathbf{x}+\mathbf{x}^2-\mathbf{x}^3)\mathbf{y}^2+\mathbf{x}^2(\mathbf{x}-1)=0$
z_1	$\mathbf{b}=\mathbf{y}$	z_2	$\mathbf{b}=\mathbf{x}$
z_3	$\mathbf{c}=\mathbf{x}$	z_4	$\mathbf{h_z}=\mathbf{b}$
z_5	$\mathbf{h_z}=\mathbf{x}$	z_6	$\mathbf{h_z}=\mathbf{y}$
z_7	$\mathbf{h_z}=\mathbf{c}$		

References

[1] H. Akrout, *Un processus effectif de détermination des systoles pour les surfaces hyperboliques*, Geom. Dedicata **121**, 2006, 1-8.

[2] M. Berger, P. Gauduchon, E. Mazet, *Le spectre d'une variété riemanienne*, Lectures Notes in Math. **194**, 1971.

[3] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics **106**, 1992.

[4] P. Buser, K.-D. Semmler, *The geometry and spectrum of the one holed torus*, Comment. Math. Helv. **63**, 1988, 259-274.

[5] A. Haas, *Length spectra as moduli for hyperbolics surfaces*, Duke Math. J. **5**, 1985, 922-935.

[6] M. Kac, *Can One Hear the Shape of a Drum?*, The American Mathematical Monthly **73**, 1966, 1-23.

[7] E. Philippe. *Le spectre des longueurs des surface hyperboliques: un exemple de rigidité*, Sémin. Théor. Spectr. Géom. Grenoble, 2009-2010, 109-120.

[8] T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121**, 1985, 169-186.

[9] M-F. Vignéras, *Variétés riemaniennes isospectrales et non isométriques*, Ann. of Math. **112**, 1980,21-32.

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