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Fixed Points Theorem on Three Metric Spaces

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Abstract

The Purpose of this paper is to establish a fixed point theorem for set valued mappings on three complete metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

The following fixed-point theorem was proved in [1].

Theorem 1.1: Let *(X, d)* and *(Z, σ)* be complete metric spaces. If *S* is a continuous mapping of *X* into *Z*, and *R* is a continuous mapping of *Z* into *X* satisfying the inequalities:

$$d(RSx, RSx') \leq c \max\{d(x,x'), d(x,RSx), d(x',RSx'), \sigma(Sx,Sx')\}$$

$$\sigma(SRz, SRz') \leq c \max\{\sigma(z,z'), \sigma(z,SRz), \sigma(z',SRz'), d(Rz,Rz')\}$$

for all *x, x'* in *X*, and *z, z'* in *Z*, where $0 \leq c < 1$, then *RS* has a unique fixed point *u* in *X* and *RS* has a unique fixed point *w* in *Z*. Further *Su* = *w* and *Rw* = *u*.

The next theorem was proved in [2].

Theorem 1.2: Let *(X, d), (Y, ρ)* and *(Z, σ)* be complete metric spaces and suppose
T is a continuous mapping of *X* into *Y, S* is a continuous mapping of *Y* into *Z* and *R* is a continuous mapping of *Z* into *X* satisfying the inequalities:

$$d(RSTx, RSy) \le c\max\{d(x, RSy), d(x, RSTx), \rho(y, Tx), \sigma(Sy, STx)\},$$
$$\rho(TRSy, TRz) \le c\max\{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy)\},$$
$$\sigma(STRz, STx) \le c\max\{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. Then *RST* has a unique fixed point u in X, *TRS* has a unique fixed point v in Y and *STR* has a unique fixed point w in Z. Further $Tu = v$, $Sv = w$ and $Rw = u$.
The next theorem was proved in [3].

Theorem 1.3: Let *(X, d), (Y, ρ)* and *(Z, σ)* be complete metric spaces and suppose *T* is a continuous mapping of *X* into *Y, S* is a continuous mapping of *Y* into *Z* and *R* is a continuous mapping of *Z* into *X* satisfying the inequalities:

$$d(RSTx, RSTx') \le c\,max\ \{d(x, x'), d(x, RSTx), d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}$$
$$\rho(TRSy, TRSy') \le c\,max\ \{\rho(y, y'), \rho(y, TRSy), \rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}$$
$$\sigma(STRz, STRz') \le c\,max\ \{\sigma(z, z'), \sigma(z, STRz), \sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}$$

for all x, x' in X, y, y' in Y and z, z' in Z where $0 \le c < 1$. Then *RST* has a unique fixed point u in X, *TRS* has a unique fixed point v in Y and *STR* has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ *and* $Rw = u$.

As a generalization of result in [3], following related fixed point theorem for set valued mappings was obtained in [4].

Theorem 1.4: Let *(X, d), (Y, ρ)* and *(Z, σ)* be complete metric spaces and suppose *T* is a continuous mapping of *X* into *Y, S* is a continuous mapping of *Y* into *Z* and *R* is a continuous mapping of *Z* into *X* satisfying the inequalities:

$$\delta_1(RSTx, RSTx') \le c\max\ \{d(x, x'), \delta_1(x, RSTx), \delta_1(x', RSTx'), \delta_2(Tx, Tx'), \delta_3(STx, STx')\}$$
$$\delta_2(TRSy, TRSy') \le c\max\ \{\rho(y, y'), \delta_2(y, TRSy), \delta_2(y', TRSy'), \delta_3(Sy, Sy'), \delta_1(RSy, RSy')\}$$
$$\delta_3(STRz, STRz') \le c\max\ \{\sigma(z, z'), \delta_3(z, STRz), \delta_3(z', STRz'), \delta_1(Rz, Rz'), \delta_2(TRz, TRz')\}$$

for all x, x' in X, y, y' in Y and z, z' in Z where $0 \le c < 1$. Then *RST* has a unique fixed point u in X, *TRS* has a unique fixed point v in Y and *STR* has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ *and* $Rw = u$.
Here the function δ is to be defined as follows:
Let *(X, d)* be a complete metric space and Let $B(X)$ be the set of all nonempty subset of *X*. The function $\delta(A, B)$ with *A* and *B* in $B(X)$ is defined by
$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$.
If *A* consists a single point a, we write $\delta(A, B) = \delta(a, B)$ and if *B* also consists of a single point b, we write $\delta(A, B) = \delta(a, b) = d(a, b)$. It follows immediately from

the definition that
$\delta(A,B)=\delta(B,A)\geq 0$,
$\delta(A,B)\leq\delta(A,C)+\delta(C,B)$, for all *A,B,C* in *B*(*X*).
For some preliminary definitions such as the convergence of a sequence of sets in $B(X)$ and the continuity if a mapping *T* of *X* into *B*(*X*), we refer (cf[4]).
The following related fixed point theorem was proved in [5].

Theorem 1.5 : Let *(X, d), (Y, ρ)* and *(Z, σ)* be complete metric spaces and suppose *T* is a continuous mapping of *X* into *Y*, *S* is a continuous mapping of *Y* into *Z* and *R* is a continuous mapping of *Z* into *X* satisfying the inequalities:

$$d^2(RSTx,RSTx')\leq c\max\{d(x',RSTx')\rho(y,y'),d(x,RSTx)d(RSTx,RSTx'),$$
$$d(x',RSTx')\sigma(z,z'),d(x,x')d(x',RSTx)\},$$
$$\rho^2(TRSy,TRSy')\leq c\max\{\rho(y',TRSy')\sigma(z,z'),\rho(y,TRSy)\rho(TRSy,TRSy'),$$
$$d(x,x')\rho(y',TRSy'),\rho(y',TRSy)\rho(y,y')\},$$
$$\sigma^2(STRz,STRz')\leq c\max\{\sigma(z',STRz')d(x,x'),\sigma(z',STRz')\sigma(z,STRz),$$
$$\sigma(z',STRz')\rho(y,y'),\sigma(z',STRz)\sigma(z,z')\}.$$

for all *x, x′* in *X*, *y, y′* in *Y* and *z, z′* in *Z* where 0≤ c < 1. Then *RST* has a unique fixed point *u* in *X*, *TRS* has a unique fixed point *v* in *Y* and *STR* has a unique fixed point *w* in *Z*. Further, *Tu = v, Sv = w* and *Rw = u*.

2. MAIN RESULT

We now prove the following related fixed point theorem for set valued mappings, which generalizes the result in [5].

Theorem 1.6 : Let *(X, d_1), (Y, d_2)* and *(Z, d_3)* be complete metric spaces and suppose *T* is a continuous mapping of *X* into *B*(*Y*), *S* is a continuous mapping of *Y* into *B*(*Z*) and *R* is a continuous mapping of *Z* into *B*(*X*) satisfying the inequalities:

$$\delta_1^2(RSTx,RSTx')\leq c\max\{\delta_1(x',RSTx')\delta_2(Tx,Tx'),\delta_1(x,RSTx)\delta_1(RSTx,RSTx'),$$
$$\delta_1(x',RSTx')\delta_3(STx,STx'),d_1(x,x')\delta_1(x',RSTx')\} \quad \text{...............}(1)$$
$$\delta_2^2(TRSy,TRSy')\leq c\max\{\delta_2(y',TRSy')\delta_3(Sy,Sy'),\delta_2(y,TRSy)\delta_2(TRSy,TRSy'),$$
$$\delta_1(RSy,RSy')\delta_2(y',TRSy'),\delta_2(y',TRSy')d_2(y,y')\} \quad \text{..............}(2)$$
$$\delta_3^2(STRz,STRz')\leq c\max\{\delta_3(z',STRz')\delta_1(Rz,Rz'),\delta_3(z',STRz')\delta_3(z,STRz),$$
$$\delta_3(z',STRz')\delta_2(TRz,TRz'),\delta_3(z',STRz')d_3(z,z')\} \quad \text{...............}(3)$$

for all *x, x′* in *X*, *y, y′* in *Y* and *z, z′* in *Z* where 0≤ c < 1. Then *RST* has a fixed point *u* in *X*, *TRS* has a fixed point *v* in *Y* and *STR* has a fixed point *w* in *Z*. Further, *Tu = v, Sv = w* and *Rw = u*.

Proof: Let $x=x_1$ be an arbitrary point in X. Define the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in *X*, *Y* and *Z* inductively as follows. Choose a point y_1 in Tx_1, then a point z_1 in Sy_1 and then a point x_2 in Rz_1. In general having chosen x_n in X, y_n in Y and z_n in Z, choose a point x_{n+1} in Rz_n, then a point y_{n+1} in Tx_{n+1} and then a point z_{n+1} in Sy_{n+1} for $n = 1, 2$ .....

Applying inequality *(1)*, we have

$$d_1^{\,2}(x_{n+1},x_{n+2}) \le \delta_1^{\,2}(RSTx_n,RSTx_{n+1}) \le c\max\begin{Bmatrix} \delta_1(x_{n+1},RSTx_{n+1})\delta_2(Tx_n,Tx_{n+1}), \\ \delta_1(x_n,RSTx_n)\delta_1(RSTx_n,RSTx_{n+1}), \\ \delta_1(x_{n+1},RSTx_{n+1})\delta_3(STx_n,STx_{n+1}), \\ d_1(x_n,x_{n+1})\delta_1(x_{n+1},x_{n+2}) \end{Bmatrix}$$

$$\le c\max\begin{Bmatrix} \delta_1(RTSx_n,RTSx_{n+1})\delta_2(TRSy_{n-1},TRSy_n), \\ \delta_1(RSTx_{n-1},RSTx_n)\delta_1(RSTx_n,RSTx_{n+1}), \\ \delta_1(RSTx_n,RTSx_{n+1})\delta_3(STRz_{n-1},STRz_n), \\ \delta_1(RSTx_{n-1},RSTx_n)\delta_1(RSTx_n,RSTx_{n+1}) \end{Bmatrix}$$

$$d_1(x_{n+1},x_{n+2}) \le c\max\begin{Bmatrix} \delta_1(RSTx_{n-1},RSTx_n),\delta_2(TRSy_{n-1},TRSy_n) \\ \delta_3(STRz_{n-1},STRz_n) \end{Bmatrix} \qquad \text{...............}(4)$$

Applying *(2)* we have

$$d_2^2(y_{n+1},y_{n+2}) \le \delta_2^2(TRSy_n,TRSy_{n+1}) \le c\max\begin{Bmatrix} \delta_2(y_{n+1},TRSy_{n+1})\delta_3(Sy_n,Sy_{n+1}), \\ \delta_2(y_n,TRSy_n)\delta_2(TRSy_n,TRSy_{n+1}), \\ \delta_1(RSy_n,RSy_{n+1})\delta_2(y_{n+1},TRSy_{n+1}), \\ \delta_2(y_{n+1},TRSy_{n+1})d_2(y_n,y_{n+1}) \end{Bmatrix}$$

$$\le c\max\begin{Bmatrix} \delta_2(TRSy_n,TRSy_{n+1})\delta_3(STRz_{n-1},STRz_n), \\ \delta_2(TRSy_{n-1},TRSy_n)\delta_2(TRSy_n,TRSy_{n+1}), \\ \delta_1(RSTx_n,RSTx_{n+1})\delta_2(y_{n+1},TRSy_{n+1}), \\ \delta_2(TRSy_n,TRSy_{n+1})\delta_2(TRSy_{n-1},TRSy_n) \end{Bmatrix}$$

$$d_2(y_{n+1},y_{n+2}) \le c\max\begin{Bmatrix} \delta_1(RSTx_n,RSTx_{n+1}),\delta_2(TRSy_{n-1},TRSy_n) \\ \delta_3(STRz_{n-1},STRz_n) \end{Bmatrix}$$

$$\le c\max\begin{Bmatrix} \delta_1(RSTx_{n-1},RSTx_n),\delta_2(TRSy_{n-1},TRSy_n) \\ \delta_3(STRz_{n-1},STRz_n) \end{Bmatrix} \qquad \text{...............}(5)$$

on using inequality *(4)*. Now, applying inequality *(3)* we have

$$d_3^2(z_{n+1},z_{n+2}) \le \delta_3^2(STRz_n,STRz_{n+1}) \le c\max\begin{Bmatrix} \delta_3(z_{n+1},STRz_{n+1})\delta_1(RSTx_n,RSTx_{n+1}), \\ \delta_3(z_{n+1},STRz_{n+1})\delta_3(z_n,STRz_n), \\ \delta_3(z_{n+1},STRz_{n+1})\delta_2(TRSy_n,TRSy_{n+1}), \\ \delta_3(z_{n+1},STRz_{n+1})\delta_3(STRz_{n-1},STRz_n) \end{Bmatrix}$$

$$d_3(z_{n+1},z_{n+2}) \le c\max\begin{Bmatrix} \delta_1(RSTx_n,RSTx_{n+1}),\delta_2(TRSy_{n-1},TRSy_n) \\ \delta_3(STRz_{n-1},STRz_n) \end{Bmatrix}$$

$$\leq c\max\begin{Bmatrix}\delta_1(RSTx_{n-1}, RSTx_n), \delta_2(TRSy_{n-1}, TRSy_n)\\ \delta_3(STRz_{n-1}, STRz_n)\end{Bmatrix} \qquad \text{...............}(6)$$

on using inequality *(4)* and *(5)*.

It follows easily by induction on using inequalities *(4)*, *(5)* and *(6)* that

$d_1(x_{n+1}, x_{n+2}) \leq c^{n-1}\max\{\delta_1(RSTx_1, RSTx_2), \delta_2(TRSy_1, TRSy_2), \delta_3(STRz_1, STRz_2)\}$,

$d_2(y_{n+1}, y_{n+2}) \leq c^{n-1}\max\{\delta_1(RSTx_1, RSTx_2), \delta_2(TRSy_1, TRSy_2), \delta_3(STRz_1, STRz_2)\}$,

$d_3(z_{n+1}, z_{n+2}) \leq c^{n-1}\max\{\delta_1(RSTx_1, RSTx_2), \delta_2(TRSy_1, TRSy_2), \delta_3(STRz_1, STRz_2)\}$.

Then for r = 1,2,3,...........and arbitrary $\in > 0$, we have,

$$d_1(x_{n+1}, x_{n+2}) \leq \delta_1(RSTx_n, RSTx_{n+r})$$
$$\leq \delta_1(RSTx_n, RSTx_{n+1}) + \delta_1(RSTx_{n+1}, RSTx_{n+2}) + ... + \delta_1(RSTx_{n+r-1}, RSTx_{n+r})$$
$$\leq (c^{n-1} + c^n + ... + c^{n+r-2}) \times \max\begin{Bmatrix}\delta_1(RSTx_1, RSTx_2), \delta_2(TRSy_1, TRSy_2),\\ \delta_3(STRz_1, STRz_2)\end{Bmatrix}$$
$$< \in \qquad \text{............ (7)}$$

for *n* > *N*, since *c* < 1. It follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits *u*, *v* and *w* in *X*, *Y* and *Z* respectively. Further inequality (7) gives

$$\delta_1(u, RSTx_n) \leq d_1(u, x_{m+1}) + \delta_1(x_{m+1}, RSTx_n)$$
$$\leq d_1(u, x_{m+1}) + \delta_1(RSTx_m, RSTx_n)$$
$$\leq d_1(u, x_{m+1}) + \in$$

for $m, n \geq N$. Letting *m* tend to infinity, it follows that

$\delta_1(u, RSTx_n) \leq \in$, for $n > N$ and so,

$$\lim_{n\to\infty} RSTx_n = \{u\} = \lim_{n\to\infty} RSy_n \text{ .... } (8)$$

Since $\in$ is arbitrary. Similarly,

$$\lim_{n\to\infty} TRSy_n = \{v\} = \lim_{n\to\infty} TRz_n \text{ .... } (9)$$

$$\lim_{n\to\infty} STRz_n = \{w\} = \lim_{n\to\infty} STx_n \text{ .... } (10)$$

Since *T* and *S* are continuous, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tu = v \text{...............}(11)$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Sy_n = Sv = w \text{................}(12).$$

and then we see that

$$STu = \{w\} \qquad \text{........... (13)}$$

We now show that *u* is affixed point of *RST*. Using inequality *(1)* again, we have,

$$\delta_1^2(RSTu, x_n) \leq \delta_1^2(RSTu, RSTx_{n-1})$$
$$\leq c\max\begin{Bmatrix}\delta_1(x_{n-1}, RSTx_{n-1})\delta_2(Tu, Tx_{n-1}),\\ \delta_1(u, RSTu)\delta_1(RSTu, RSTx_{n-1}),\\ \delta_1(x_{n-1}, RSTx_{n-1})\delta_3(STu, STx_{n-1}),\\ d_1(u, x_{n-1})\delta_1(x_{n-1}, RSTx_{n-1})\end{Bmatrix}$$

Since *T* and *S* are continuous, it follows on letting *n* tend to infinity and using equation (8) that

$\delta_1(RSTu, u) \leq c\delta_1(u, RSTu)$.

Thus *RSTu* ={ *u*}, as *c* < 1, and so *u* is a fixed point of *RST*.

Now, we have,

RSTu =*Rw* ={ *u*}

on using equation (13) and then,

TRSv = *TRSTu* = *Tu* ={ *v*}

and so

STRw = *STRSv* = *Sv* = {*w*}

on using equations (11) and (12).Hence *v* and *w* are fixed points of *TRS* and *STR* respectively. Further we see that *RSv* = {*u*} and *TRw* = {*v*}

This completes the proof.

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