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ON THE CLOSED SYSTEMS IN BANACH SPACES

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Abstract

Some notion of closeness of systems in Banach spaces that leave the basicity properties of the considered systems are introduced. The obtained results generalize many results known earlier

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1 Introduction

The theorems on basicity of close in this or another sense systems in Banach spaces play a special role for establishing basicity. Apparently, this method originates in the paper of Paley-Wiener [10] on a Riesz basicity of a perturbed system of exponents. In the sequel in this direction considerable results were obtained and different generalizations of the Paley-Wiener theorem were suggested. One can find these or other informations in the review paper [7] and in the monographs [8;9;11]. One variant of closeness of systems in Hilbert space L_2 of functions retaining the basicity property was suggested in [6].

The present paper is devoted to the problems mentioned above. New variants of closeness of systems in Banach spaces retaining the basicity property are suggested. In particular, all the results of the paper [6] are obtained from them. It should be noted that the results of the papers [2-5;12] are closely border with the considered questions.

2 Some general facts and a degenerate systems

We'll need some facts from the theory of bases in Banach spaces. By $L(X;Y)$ we denote a Banach space of bounded operators acting from X to Y. Accept $L(X) \equiv L(X;X)$. Recall that the system $\{y_n\}_{n\in N}$ is said to be ω-linear independent in Y, if $\sum_{n=1}^{\infty} a_n y_n = 0$ is possible only for $a_n = 0, \forall n \in N$. Let $\{x_n\}_{n\in N}$ be a basis in X. If $T \in L(X;Y)$ is invertible, then $\{Tx_n\}_{n\in N}$ also makes a basis in Y with the same space of coefficients of $\{x_n\}_{n\in N}$.

Now, let $F \in L(X;Y)$ be a Fredholm operator, $\{x_n\}_{n\in N} \subset X$ be a complete and minimal system in X and $y_n = Fx_n, \forall n \in N$. If F is invertible, then it is clear that $\{y_n\}_{n\in N}$ is also complete and minimal in Y. It is easy to see that if F is invertible, then $\{y_n\}_{n\in N}$ is ω-linear independent. Conversely, assume that $\{y_n\}_{n\in N}$ is complete in Y. Take $\varphi^* \in KerF^*$ and consider:

$$0 = (F^*\varphi^*)x_n = \varphi^*(Fx_n) = \varphi^*(y_n), \forall n \in N.$$

From the completeness of $\{y_n\}_{n\in N}$ we get that $\varphi^* = 0$, i.e. $KerF^* = \{0\}$. Consequently, F is invertible and so $\{y_n\}_{n\in N}$ is also minimal and ω-linear independent in Y.

We'll use these reasonings in the sequel.

3 Closed systems

Let X be some Banach space and $T \in L(X)$ be a completely continuous operator.

Consider $\Phi_\lambda = I + \lambda T$, where $\lambda \in C$ is a complex parameter. It is known that Φ_λ is a Fredholm operator. If λ is a regular value of T, then Φ_λ is invertible, and consequently it takes any basis $\{x_n\}_{n\in N} \subset X$ to the basis $\{\Phi_\lambda x_n\}_{n\in N}$. But if λ is an eigen value of T, then the system $\{\Phi_\lambda x_n\}_{n\in N}$ is simultaneously non complete and non minimal in X, and it has finite deficiency. The set of such values $\{\lambda_k\}_{k\in N}$ is discrete and $\lim\limits_{k\to\infty} |\lambda_k| = \infty$.

Assume that $S_{\overline{x}} \equiv \{x_n\}_{n\in N} \subset X$ is a basis in a Banach space X and $S^*_{\bar{x}} \equiv \{x^*_n\}_{n\in N} \subset X^*$ is adjoint system, where X^* is a space adjoint to X. Consider the operator $\Phi : X \to X$:

$$\Phi x = \sum_{n=1}^{\infty} x^*_n(x) y_n, \tag{1}$$

where $S_{\bar{y}} \equiv \{y_n\}_{n\in N} \subset X$ is some system. It is obvious that the domain of definition D_Φ of the operator Φ consists of those $x \in X$, for which the series (1) converges in X. Clearly, $\Phi = I + T$, where

$$Tx = \sum_{n=1}^{\infty} x_n^*(x)(y_n - x_n), \ \forall x \in D_\Phi. \tag{2}$$

Introduce the following

Definition 3.1 *A system $S_{\bar{y}}$ is said to be $S_{\bar{x}}^*$ -close to the system $S_{\bar{x}}$, if the series (2) converges for $\forall x \in X$, i.e. $D_T = X$. Therewith, if the operator T, defined by the expression (2), is completely continuous, this closeness is said to be $\sigma S_{\bar{x}}^*$-closeness.*

It is easy to see that if for $\forall x \in X$:

$$\{x_n^*(x)\}_{n\in N} \in l_p \quad \textit{and} \quad \{\|y_n - x_n\|\}_{n\in N} \in l_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1, \ 1 \le p \le +\infty$, then the systems $S_{\bar{y}}$ and $S_{\bar{x}}$ are $\sigma S_{\bar{x}}^*$-close.

So, if the system $S_{\bar{y}}$ is $\sigma S_{\bar{x}}^*$-close to the minimal system $S_{\bar{x}}$, then Φ is a Fredholm operator. In this case the following statement holds.

Statement 3.2 *Let $S_{\bar{x}}$ form a basis in X and $S_{\bar{y}}$ be $\sigma S_{\bar{x}}^*$-close to it. Then the following statements are equivalent:*

1) $S_{\bar{y}}$ is complete in X; 2) $S_{\bar{y}}$ is minimal in X; 3) $S_{\bar{y}}$ is ω-linearly independent in X; 4) $S_{\bar{y}}$ forms a basis in X, isomorphic to the basis $S_{\bar{x}}$; 5) The operator $\Phi = I + T$ is invertible in $L(X)$.

The validity of this statement follows directly from the above-mentioned reasonings and relations. Now, let $\lambda \in \rho(T)$ be a regular value of the operator T. Thus, in this case, the Fredholm operator $\Phi_\lambda = I + \lambda T$ is invertible.We have $\Phi_\lambda x_n = x_n + \lambda(y_n - x_n) = (1-\lambda)x_n + \lambda y_n, \forall n \in N$. But if $0 \ne \lambda \in \sigma(T)$ is an eigen value of the operator T, then the system $S_{\bar{y}}^\lambda \equiv \{x_n + \lambda(y_n - x_n)\}_{n\in N}$ is not simultaneously non complete and non minimal (it is not ω-linearly independent) in X.

It is clear that, on the contrary, if $S_{\bar{y}}^\lambda$ is complete (is minimal or ω-linearly independent) then Φ_λ is invertible. Thus, the following statement is valid.

Theorem 3.3 *Let $S_{\bar{x}}$ form a basis in X and the system $S_{\bar{y}}$ be $\sigma S_{\bar{x}}^*$-close to it. Then the following statements are equivalent:*

1) $S_{\bar{y}}^\lambda$ is complete in X; 2) $S_{\bar{y}}^\lambda$ is minimal in X; 3) $S_{\bar{y}}^\lambda$ is ω-linearly independent in X; 4) $S_{\bar{y}}^\lambda$ forms a basis in X, isomorphic to the basis $S_{\bar{x}}$; 5)λ belongs to the resolvent set T.

4 L_p case. Close bases

Further we'll consider the most specific case, namely, the case when $X = L_p, 1 \leq p < +\infty$. Then $X^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and an arbitrary continuous functional l_g on X is realized by the function $g \in L_q$ and has the expression $l_g(f) = \int_a^b f(t)\overline{g(t)}dt, \forall f \in L_p$. So, let $\{x_n\}_{n \in N} \subset L_p$ be a basis in L_p and $\{x_n^*\}_{n \in N} \subset L_q$ be an appropriate adjoint system. Take $\forall f \in L_p$ and consider the operator

$$Tf = \sum_{n=1}^{\infty} l_{x_n^*}(f)\, z_n, \qquad (3)$$

where $\{z_n\}_{n \in N} \subset L_p$ is some system. If expression (3) generates a completely continuous operator in L_p, then from Statement 3.2 we directly get the following theorem.

Theorem 4.1 *Let $\{x_n\}_{n \in N}$ be a basis in L_p, operator (3) be completely continuous in L_p and $f_n^\lambda = x_n + \lambda z_n$, $\forall n \in N$. Then the following statements are equivalent $\left(\Phi_\lambda \equiv \left\{f_n^\lambda\right\}_{n \in N}\right)$:*

1) Φ_λ is complete in L_p; 2) Φ_λ is minimal in L_p; 3) Φ_λ is ω-linearly independent; 4) Φ_λ forms a basis in L_p; 5) $\lambda \in \rho(T)$.

Further, we have:

$$Tf = \sum_{n=1}^{\infty} \int_a^b f(t) x_n^*(t) dt z_n(x) = \int_a^b K(x,t) f(t) dt, \qquad (4)$$

where

$$K(x,t) = \sum_{n=1}^{\infty} x_n^*(t) z_n(x). \qquad (5)$$

Thus, if integral operator (4) with the kernel $K(x,t)$ generates a completely continuous operator in L_p, then Theorem 3.3 is valid with respect to the system Φ_λ. In particular, for $p = 2$, if $K(x,t) \in L_2([a,b] \times [a,b])$, then we obtain a strong version of the result of the [6]. Considering the expression

$$\int_a^b \int_a^b |K(x,t)|^2 \, dx dt = \sum_{n,k} \int_a^b x_n^*(t) \overline{x_k^*(t)} dt \int_a^b z_n(x) \overline{z_k(x)} dx,$$

we get that if the series $\sum_{n,k} a_{nk} b_{nk}$ converges, where $a_{nk} = \int_a^b x_n^*(t) \overline{x_k^*(t)} dt$, $b_{nk} = \int_a^b z_n(x) \overline{z_k(x)} dx$, then it generates a completely continuous operator in L_p.

Using different sufficient conditions on complete continuity of an integral operator in L_p, we get appropriate conditions on the kernel $K(x,t)$.

Considering this from Theorem 4.1 we directly obtain

Corollary 4.2 *Let $\{x_n(t)\}_{n\in N}$ form a basis in L_p with an adjoint system $\{x_n^*\}_{n\in N}\subset L_q$, $\frac{1}{p}+\frac{1}{q}=1,\ 1\leq p<+\infty$. If the series $\sum_{n=1}^{\infty}\|x_n^*\|_q\|z_n\|_p<+\infty$ converges, then for the system $\{x_n+\lambda z_n\}_{n\in N}$ Theorem 4.1 is valid, where $\|\cdot\|_p$ is an ordinary norm in L_p:*

$$\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{1/p}.$$

Indeed, in this case, kernel (5) provides complete continuity of operator (4) in L_p (see f.i. [1], p.557).

A similar statement holds if $\sum_{n=1}^{\infty}\sup\limits_x|z_n(x)|\,\|x_n^*\|_q<+\infty$ is fulfilled.

Corollary 4.3 *Let $\{x_n\}_{n\in N}$ be a basis in L_p and $\sum_n\|x_n^*\|_r\|z_n\|_r<+\infty$ be fulfilled, where $r=\min\{p;q\}$. Then Theorem 4.1 is valid with respect to the system $\{x_n+\lambda z_n\}_{n\in N}$.*

Corollary 4.4 *Let $\{x_n\}_{n\in N}$ be a bases in L_p, $1\leq p<+\infty$, and there hold the conditions: $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ $\exists r,\sigma>0:1-\frac{\sigma}{p}<\frac{r}{q}$, such that*

$$\sum_n\sup_t|x_n^*(t)|\,\|z_n\|_r<+\infty;\sum_n\sup_x|z_n(x)|\,\|x_n^*\|_\sigma<+\infty.$$

Then Theorem 4.1 is valid for the system $\{x_n+\lambda z_n\}_{n\in N}$.

Validity of Corollaries 4.3 and 4.4 follows from Theorem 4.1 and the results of the monograph [1, p.292].

5 Example

Let $X=L_2(0,\pi)\,,x_n(t)=\sin nt,n\in N$. As the system $\{z_n\}_{n\in N}$ we take $z_n(t)=sin\lambda_nt-\sin nt,n\in N$, where $\{\lambda_n\}\subset R$- some sequence of real numbers. The biorthogonal system to $\{x_n\}_{n\in N}$ is $\{\frac{2}{\pi}\sin nt\}_{n\in N}$. Following the formula (5) we have

$$K(x;t)=\frac{2}{\pi}\sum_{n=1}^{\infty}z_n(x)\sin nt.$$

Hence

$$\int_0^\pi\int_0^\pi|K(x;t)|^2dtdx=c_1\sum_{n=1}^{\infty}\int_0^\pi|z_n(x)|^2dx\leq c_2\sum_{n=1}^{\infty}|\lambda_n-n|^2,$$

where $c_k=1,2$- some constants. Having paid attention to Theorem 4.1 we obtain

Statement 5.1 *Let $\sum_{n=1}^{\infty}|\lambda_n - n|^2 < \infty$. Then conserning system $y_n^{\lambda} = (1-\lambda)\sin nt + +\lambda \sin \lambda_n t, n \in N$, the following statements are equivalent:*

1) $\{y_n^{\lambda}\}_{n\in N}$ is complete in $L_2(0;\pi)$; 2)$\{y_n^{\lambda}\}_{n\in N}$ is minimal in $L_2(0;\pi)$; 3)$\{y_n^{\lambda}\}_{n\in N}$ is $\omega-$ linearly independent; 4)$\{y_n^{\lambda}\}_{n\in N}$ forms a Riesz basis in $L_2(0;\pi)$; 5)$\lambda \in \rho(\tau)$, where operator T defined by formula (4).

Similar results can be received in L_p.

References

[1] Yu.M. Berezanskii, G.F. Us , Z.G. Sheftel, Functional analisis, "Vyshe shkola", Kiev, 1990.

[2] B.T. Bilalov, Bases from some systems of exponents in L_p, Dokl. RAN. 392, 2003, No 5.

[3] B.T. Bilalov, Bases from exponents, cosines and sines being eigen functions of differential operators, Diff. uravn. 39, 2003, No 5, pp.1-5.

[4] B.T. Bilalov , T.R. Muradov, On equivalent bases in Banach spaces, Ukr. Mat. Journ. 59, 2007, No 4, pp. 551-554.

[5] X. He, H. Volkmer, Riesz bases of solutions of Sturm-Lioville equations, J. Fourier Anal. Appl. 7, 2001, No 3, pp. 297-307.

[6] Yu. A. Kaz'min, On bases and complete systems of functions in Hilbert space, Mat. Sb. (N.S.), 42(84):4 (1957), pp. 513-522.

[7] V.D. Mil'man, Geometric theory of Banach spaces, Part I. The theory of basis and minimal systems, Russian Mathematical Surveys, 1970,25(3):111.

[8] I.Singer, Bases in Banach spaces, I, Springer, 1970, 673 p.

[9] I. Singer, Bases in Banach spaces, II, Springer, 1981, 880 p.

[10] N. Wiener , R. Paley, Fourier transforms in the complex domain, Amer., Math. Soc. Colloq. Publ., 19, Amer. Math. Soc., Providence, RI, 1934.

[11] R.M. Young, An introduction to non harmonic Fourier series, AP NYLTSSF, 1980, 246 p.

[12] R.M. Young, On perturbing bases of complex exponentials in $L^2(-\pi,\pi)$, Proc. Amer. Math. Soc., 53:1, (1975), pp. 137-140.

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