Mathematica Aeterna, Vol. 2, 2012, no. 4, 287 - 295

The cosine-function method and the modified extended tanh method to generalized Zakharov system

Nasir Taghizadeh

Department of Mathematics, Faculty of Sciences,
university of Guilan, P.O. Box 1914, Rasht, Iran
taghizadeh@guilan.ac.ir

Mohammadali Mirzazadeh

Department of Mathematics, Faculty of Sciences,
university of Guilan, P.O. Box 1914, Rasht, Iran
mirzazadehs2@guilan.ac.ir

Ameneh Samiei Paghaleh

Department of Mathematics, Faculty of Sciences,
university of Guilan, P.O. Box 1914, Rasht, Iran
ameneh.samiei@gmail.com

Abstract

The cosine-function method and the modified extended tanh method are efficient methods for obtaining exact soliton solutions of nonlinear partial differential equations. These methods can be applied to nonintegrable equations as well as to integrable ones. In this paper, we look for exact soliton solutions of generalized Zakharov equation.

Mathematics Subject Classification: 47F05; 35B05; 35Dxx.

Keywords: Cosine-function method; Modified extended tanh method; Generalized Zakharov equation.

1 Introduction

Exact solutions to nonlinear evolution equations play an important role in nonlinear physical science, since these solutions may well describe various natural phenomena, such as vibrations, solitons, and propagation with a finite speed. Recently many new approaches for finding the exact solutions to nonlinear evolution equations have been proposed, such as tanh-sech method [1], [2], [3], extended tanh method [4], [5], [6], hyperbolic function method [7], sine-cosine method [8], [9], [10], Jacobi elliptic function expansion method [11], F-expansion method [12] ,and the first integral method [13], [14]. In recent years, there was interest in obtaining exact solutions of NLPDEs by extended tanh method and cosine-function method. The standard tanh method is developed by Malfliet [1]. Lately, Wazwaz investigated exact solutions of the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations by the extended tanh method [6]. Fan in [16] presented the generalized tanh method for constructing the exact solutions of NLPDEs, such as, the (2 + 1)-dimensional sine-Gordon equation and the double sine-Gordon equation. The generalized Zakharov(GZ)equation [15] is in the following form:

$$iu_t + u_{xx} - 2a|u|^2u + 2uv = 0,$$

$$v_{tt} - v_{xx} + (|u|^2)_{xx} = 0.$$

The aim of this paper is to find exact soliton solutions of generalized Zakharov equation by the cosine-function method and the modified extended tanh method(METM).

2 The cosine-function method

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is the solution of nonlinear partial differential equation Eq. (1). We use the transformations,

$$u(x,t) = f(\xi), \tag{2}$$

where $\xi = x - ct$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -c\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \quad .... \tag{3}$$

Using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{4}$$

The solution of Eq. (4) can be expressed in the form:

$$f(\xi) = \lambda \cos^{\beta}(\mu\xi), \qquad |\xi| \leq \frac{\pi}{2\mu}, \tag{5}$$

where λ, β and μ are unknown parameters which will be determined. Then we have:

$$\frac{\partial f(\xi)}{\partial \xi} = -\lambda\beta\mu \cos^{\beta-1}(\mu\xi)\ sin(\mu\xi),$$

$$\frac{\partial^2 f(\xi)}{\partial \xi^2} = -\lambda\mu^2\beta^2 \cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi). \tag{6}$$

Substituting Eq. (5) and Eq. (6) into the nonlinear ordinary differential equation Eq. (4) gives a trigonometric equation of $cos^{\alpha}(\mu\xi)$ terms.To determine the parameters first balancing the exponents of each pair of cosine to determine α.Then we collect all terms with the same power in $cos^{\beta}(\mu\xi)$ and put to zero their coefficients to get a system of algebraic equations among the unknowns λ, β and μ. Now, the problem is reduced to a system of algebraic equations that can be solved to obtain the unknown parameters λ, β and μ. Hence, the solution considered in Eq. (5) is obtained.

3 Modified extended tanh method

For given a nonlinear equation

$$F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, ...) = 0, \tag{7}$$

when we look for its traveling wave solutions, the first step is to introduce the wave transformation $u(x, y, t) = U(\xi), \quad \xi = x + \gamma y + \lambda t$ and change Eq. (7) to an ordinary differential equation(ODE)

$$H(U, U', U'', U''', ...) = 0. \tag{8}$$

The next crucial step is to introduce a new variable $\phi = \phi(\xi)$, which is a solution of the Riccati equation

$$\frac{d\phi}{d\xi} = k + \phi^2. \tag{9}$$

The modified extended tanh method admits the use of the finite expansion:

$$u(x,y,t) = U(\xi) = \sum_{i=0}^{m} a_i \phi^i(\xi) + \sum_{i=1}^{m} b_i \phi^{-i}(\xi), \tag{10}$$

where the positive integer m is usually obtained by balancing the highest-order linear term with the nonlinear terms in Eq. (8). Expansion (10) reduces to the generalized tanh method [16] for $b_i = 0,\ i = 1, ..., m$. Substituting Eq. (9) and Eq. (10) into Eq. (8) and then setting zero all coefficients of $\phi^i(\xi)$, we can obtain a system of algebraic equations with respect to the constants $k, \gamma, \lambda, a_0, ..., a_m, b_1, ..., b_m$. Then we can determine the constants $k, \gamma, \lambda, a_0, ..., a_m, b_1, ..., b_m$. The Riccati equation (9) has the general solutions:
If $k < 0$ then

$$\begin{aligned} \phi(\xi) &= -\sqrt{-k}\tanh(\sqrt{-k}\xi), \\ \phi(\xi) &= -\sqrt{-k}\coth(\sqrt{-k}\xi). \end{aligned} \tag{11}$$

If $k = 0$ then

$$\phi(\xi) = -\frac{1}{\xi}. \tag{12}$$

If $k > 0$ then

$$\begin{aligned} \phi(\xi) &= \sqrt{k}\tan(\sqrt{k}\xi), \\ \phi(\xi) &= -\sqrt{k}\cot(\sqrt{k}\xi). \end{aligned} \tag{13}$$

Therefore,by the sign test of k, we obtain exact soliton solutions of Eq. (7).

4 Exact solutions of GZ equation by cosine method

Let us consider the generalized Zakharov(GZ)equation [15]:

$$iu_t + u_{xx} - 2a|u|^2u + 2uv = 0, \tag{14}$$

$$v_{tt} - v_{xx} + (|u|^2)_{xx} = 0. \tag{15}$$

We introduce the transformations

$$u(x,t) = e^{i\theta}U(\xi), \quad v(x,t) = V(\xi), \quad \theta = \alpha x + \beta t, \quad \xi = x - 2\alpha t, \tag{16}$$

where α, and β are real constants.Hence,

$$u_t = (i\beta U(\xi) - 2\alpha\frac{\partial U(\xi)}{\partial \xi})e^{i\theta}, \tag{17}$$

$$u_{xx} = (-\alpha^2 U + 2i\alpha\frac{\partial U(\xi)}{\partial \xi} + \frac{\partial^2 U(\xi)}{\partial \xi^2})e^{i\theta}, \tag{18}$$

$$v_{tt} = 4\alpha^2 \frac{\partial^2 V(\xi)}{\partial \xi^2}, \quad v_{xx} = \frac{\partial^2 V(\xi)}{\partial \xi^2}. \tag{19}$$

Substituting (16) into Eqs. (14)-(15), and using (17)-(19), we have the ordinary differential equations (ODEs) for $U(\xi)$ and $V(\xi)$

$$-(\beta + \alpha^2)U(\xi) + U''(\xi) - 2aU^3(\xi) + 2U(\xi)V(\xi) = 0, \tag{20}$$

$$(4\alpha^2 - 1)V''(\xi) + (U^2(\xi))'' = 0. \tag{21}$$

Integrating (21), we get

$$(4\alpha^2 - 1)V'(\xi) + (U^2(\xi))' = \tilde{C},$$

where $\tilde{C}$ is integration constant. Because we will find the special form of exact solutions and,for simplicity purpose, we take $\tilde{C} = 0$ and, integrating this formula once again, we have

$$V(\xi) = \frac{C - U^2(\xi)}{4\alpha^2 - 1}, \tag{22}$$

where C is integration constant.
Substituting the (22) into (20) yields

$$(\beta + \alpha^2 - \frac{2C}{4\alpha^2 - 1})U(\xi) - U''(\xi) + 2(a + \frac{1}{4\alpha^2 - 1})U^3(\xi) = 0. \tag{23}$$

Substituting Eq. (5) and Eq. (6) into (23) gives:

$$(\beta+\alpha^2-\frac{2C}{4\alpha^2 - 1})(\lambda_1 \cos^{\beta_1}(\mu_1\xi))-(-\lambda_1\mu_1^2\beta_1^2 \cos^{\beta_1}(\mu_1\xi)+\lambda_1\mu_1^2\beta_1(\beta_1-1)\cos^{\beta_1-2}(\mu_1\xi))$$

$$+2(a + \tfrac{1}{4\alpha^2-1})(\lambda_1 \cos^{\beta_1}(\mu_1\xi))^3 = 0.$$

By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

$$\begin{gathered}
(\beta + \alpha^2 - \frac{2C}{4\alpha^2 - 1})\lambda_1 + \lambda_1\mu_1^2\beta_1^2 = 0, \\
-\lambda_1\mu_1^2\beta_1(\beta_1 - 1) + 2\lambda_1^3(a + \frac{1}{4\alpha^2 - 1}) = 0, \\
3\beta_1 = \beta_1 - 2.
\end{gathered} \tag{24}$$

Solving the system (24),we obtain:

$$\beta_1 = -1, \quad \mu_1 = \pm\sqrt{\frac{2C - (\beta + \alpha^2)(4\alpha^2 - 1)}{4\alpha^2 - 1}}, \tag{25}$$

$$\lambda_1 = \pm\sqrt{\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{a(4\alpha^2-1)+1}}.$$

Substituting (25) into Eq. (5), we obtain exact soliton solutions of the generalized Zakharov(GZ)equation in the forms

$$u(x,t) = \pm e^{i(\alpha x+\beta t)}\sqrt{\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{a(4\alpha^2-1)+1}}\sec[\sqrt{\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{4\alpha^2-1}}(x-2\alpha t)],$$

$$v(x,t) = \frac{C}{4\alpha^2-1} - \frac{1}{4\alpha^2-1}\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{a(4\alpha^2-1)+1}\times$$
$$\sec^2[\sqrt{\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{4\alpha^2-1}}(x-2\alpha t)],$$

where

$$\frac{2C-(\beta+\alpha^2)(4\alpha^2-1)}{4\alpha^2-1} > 0.$$

5 Exact solutions of GZ equation by METM

As shown before, the equation as follows:

$$-(\beta+\alpha^2)U(\xi) + U''(\xi) - 2aU^3(\xi) + 2U(\xi)V(\xi) = 0, \tag{26}$$

$$(4\alpha^2-1)V''(\xi) + (U^2(\xi))'' = 0. \tag{27}$$

is the transformed ODE of the GZ equation with using the wave variables

$$u(x,t) = e^{i\theta}U(\xi), \quad v(x,t) = V(\xi), \quad \theta = \alpha x + \beta t, \quad \xi = x - 2\alpha t.$$

Substituting the (22) into (26) yields

$$(\beta+\alpha^2 - \frac{2C}{4\alpha^2-1})U(\xi) - U''(\xi) + 2(a + \frac{1}{4\alpha^2-1})U^3(\xi) = 0. \tag{28}$$

Balancing U'' with U^3 in Eq. (28) give

$$m + 2 = 3m,$$

so that $m = 1$.
The modified extended tanh method (10) admits the use of the finite expansion

$$U(\xi) = a_0 + a_1\phi(\xi) + \frac{b_1}{\phi(\xi)}. \tag{29}$$

Substituting (29) into Eq. (28), making use of Eq. (9),collecting the coefficients of $\phi^i(\xi)$ $-3 \leq i \leq 3$, we obtain:

Coefficient of ϕ^3: $-2a_1 + 2(a + \frac{1}{4\alpha^2-1})a_1^3$.
Coefficient of ϕ^2: $6(a + \frac{1}{4\alpha^2-1})a_0a_1^2$.
Coefficient of ϕ^1: $6(a+\frac{1}{4\alpha^2-1})b_1a_1^2+6(a+\frac{1}{4\alpha^2-1})a_0^2a_1-2ka_1+(\beta+\alpha^2-\frac{2C}{4\alpha^2-1})a_1$.
Coefficient of ϕ^0: $12(a + \frac{1}{4\alpha^2-1})a_0a_1b_1 + 2(a + \frac{1}{4\alpha^2-1})a_0^3 + (\beta + \alpha^2 - \frac{2C}{4\alpha^2-1})a_0$.
Coefficient of ϕ^{-1}: $6(a+\frac{1}{4\alpha^2-1})b_1^2a_1-2kb_1+(\beta+\alpha^2-\frac{2C}{4\alpha^2-1})b_1+6(a+\frac{1}{4\alpha^2-1})a_0^2b_1$.
Coefficient of ϕ^{-2}: $6(a + \frac{1}{4\alpha^2-1})a_0b_1^2$.
Coefficient of ϕ^{-3}: $-2k^2b_1 + 2(a + \frac{1}{4\alpha^2-1})b_1^3$.

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following set of solutions:

$$a_0 = a_1 = 0, \quad b_1 = \pm\frac{1}{2}\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{\sqrt{16a\alpha^4 - 8a\alpha^2 + a + 4\alpha^2 - 1}}, \tag{30}$$
$$k = -\frac{1}{2}\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{4\alpha^2 - 1},$$
$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{4\alpha^2 - 1}{4a\alpha^2 - a + 1}}, \quad b_1 = 0, \tag{31}$$
$$k = -\frac{1}{2}\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{4\alpha^2 - 1},$$

where α and β are arbitrary constants.

By using (11) and (29), the sets (30)-(31) give the following solutions:

$$U_1(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{(-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C)(4\alpha^2 - 1)}{16a\alpha^4 - 8a\alpha^2 + a + 4\alpha^2 - 1}}\times$$
$$\coth(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{4\alpha^2 - 1}}\xi),$$
$$U_2(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{4a\alpha^2 - a + 1}}\times$$
$$\tanh(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C}{4\alpha^2 - 1}}\xi),$$

for $k < 0$.
Thus,in (x, t)-variables we have

$$u_1(x, t) = \pm e^{i(\alpha x+\beta t)}\frac{\sqrt{2}}{2}\sqrt{\frac{(-4\beta\alpha^2 + \beta - 4\alpha^4 + \alpha^2 + 2C)(4\alpha^2 - 1)}{16a\alpha^4 - 8a\alpha^2 + a + 4\alpha^2 - 1}}\times$$

$$\coth(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+2C}{4\alpha^2-1}}(x-2\alpha t)),$$

$$v_1(x,t)=\frac{C}{4\alpha^2-1}-\frac{(\frac{\sqrt{2}}{2}\sqrt{\frac{(-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+2C)(4\alpha^2-1)}{16a\alpha^4-8a\alpha^2+a+4\alpha^2-1}}\coth(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+2C}{4\alpha^2-1}}(x-2\alpha t)))^2}{4\alpha^2-1}.$$

$$u_2(x,t)=\pm e^{i(\alpha x+\beta t)}\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+1}{4a\alpha^2-a+1}}\times$$

$$\tanh(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+2C}{4\alpha^2-1}}(x-2\alpha t)),$$

$$v_2(x,t)=\frac{C}{4\alpha^2-1}-\frac{(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+1}{4a\alpha^2-a+1}}\tanh(\frac{\sqrt{2}}{2}\sqrt{\frac{-4\beta\alpha^2+\beta-4\alpha^4+\alpha^2+2C}{4\alpha^2-1}}(x-2\alpha t)))^2}{4\alpha^2-1}.$$

References

[1] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60(7) (1992) 650-654.

[2] A.H. Khater, W. Malfliet, D.K. Callebaut and E.S. Kamel, The tanh method, a simple transformation and exact analytical solutions for nonlinear reactiondiffusion equations, Chaos Solitons Fractals 14(3) (2002) 513-522.

[3] A.M. Wazwaz, Two reliable methods for solving variants of the KdV equation with compact and noncompact structures. Chaos Solitons Fractals, 28(2) (2006) 454-462.

[4] S.A. El-Wakil and M.A. Abdou , New exact travelling wave solutions using modified extended tanh-function method, Chaos Solitons Fractals, 31(4) (2007) 840-852.

[5] E.Fan, Extended tanh-function method and its applications to nonlinear equations,Phys. Lett. A 277 (4-5)(2000) 212-218.

[6] A.M. Wazwaz, The tanh-function method: Solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations. Chaos, Solitons and Fractals 25(1) (2005) 55-63.

[7] T. C.Xia, B. Li and H. Q. Zhang, New explicit and exact solutions for the Nizhnik- Novikov-Vesselov equation, Appl. Math. E-Notes, 1(2001),139-142.

[8] A.M. Wazwaz, The sine-cosine method for obtaining solutions with compact and noncompact structures, Appl.Math.Comput. 159(2) (2004) 559-576.

[9] A.M. Wazwaz, A sine-cosine method for handling nonlinear wave equations.Math. Comput. Modelling 40 (5-6) (2004) 499-508.

[10] E.Yusufoglu and A. Bekir, Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method.Internat. J. Comput. Math. 83(12) (2006) 915-924

[11] M. Inc and M. Ergut, Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method, Appl. Math. E-Notes 5 (2005) 89-96.

[12] Zhang Sheng, The periodic wave solutions for the (2 + 1) dimensional Konopelchenko-Dubrovsky equations, Chaos Solitons Fractals, 30 (2006) 1213-1220.

[13] Z.S. Feng, The first integer method to study the Burgers-Korteweg-de Vries equation, J.Phys. A. 35(2) (2002) 343-349.

[14] T.R. Ding and C.Z. Li, Ordinary differential equations. Peking University Press, Peking, (1996).

[15] Y. Zhou, M. Wang and T. Miao, The periodic wave solutions and solitary for a class of nonlinear partial differential equations, Phys. Lett. A 323 (2004) 77-88.

[16] E. Fan, Y.C. Hon, Generalized tanh method Extended to special types of nonlinear equations, Z. Naturforsch 57 (2002) 692-700.

Received: March, 2012