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Doubly Fuzzy Preordered Sets

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Abstract

We investigate the properties of doubly fuzzy preordered sets. We show that the family of l-stable fuzzy sets is a bounded lattice. We investigate the relation between the bounded lattice X and (resp. maximal) fuzzy filter-ideal pairs on X.

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1 Introduction

A fuzzy context consists of (X, Y, R) where X is a set of objects, Y is a set of attributes and R is a relation between X and Y. Bělohlávek [2-4] developed the notion of lattice structures with $R \in L^{X\times Y}$ on a complete residuated lattice L. Lattice structures are important mathematical tools for data analysis and knowledge processing [2-4,10]. On the other hand, Urquhart [12] showed that the dual space of a bounded lattice is a doubly ordered topological space. This viewpoint develops many representation theorems for various algebraic structures [1,5,6].

In this paper, we investigate the properties of doubly fuzzy preordered sets. Using their properties, we define l-stable and r-stable fuzzy sets. We show that the family of l-stable fuzzy sets is a bounded lattice. We investigate the relation between the bounded lattice X and (resp. maximal) fuzzy filter-ideal pairs on X.

2 Preliminaries

Definition 2.1 [8,9,11] A triple $(L, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Example 2.2 [8,9,11] (1) Each frame $(L, \leq, \wedge)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a complete residuated lattice.

(3) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = \max\{0, x + y - 1\}$. Then $([0,1], \leq, \odot)$ is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^* : L \to L$ defined by $a^* = a \to 0$ is called a *strong negation* if $(a^*)^* = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *.

Definition 2.3 [8,9,11] Let X be a set. A function $e_X : X \times X \to L$ is called *fuzzy preorder* on X if it satisfies the following conditions:

(E1) $e_X(x, x) = 1$ for all $x \in X$,

(E2) $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

The pair (X, e_X) is a *fuzzy preorder set.*

Let e^1_X, e^2_X be fuzzy preorder on X. A structure (X, e^1_X, e^2_X) is called a doubly fuzzy preordered set. If for all $x, y \in X$, $e^1_X(x, y) = e^2_X(x, y) = 1$ implies $x = y$, (X, e^1_X, e^2_X) is called a doubly fuzzy ordered set.

Lemma 2.4 [8,9,11] *For each $x, y, z, x_i, y_i \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then the following properties hold.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(2) *$x \odot y \leq x \wedge y$ and $x \odot (x \to y) \leq y$.*

(3) *$x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.*

(4) *$(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.*

(5) *$x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$.*

(6) *$(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.*

(7) *$\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.*

(8) *$(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.*

(9) $1 \to x = x$.
(10) $x \leq y$ *iff* $x \to y = 1$.
(11) $(x \to y) \odot (y \to z) \leq x \to z$.
(12) $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2 \to y_1 \odot y_2)$.

Example 2.5 (1) We define a map $e_L : L \times L \to L$ $e_L(x, y) = x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$ and $e_L^{-1}(x, y) = e_L(y, x)$. Then (L, e_L, e_L^{-1}) is a doubly fuzzy ordered set from Lemma 2.4 (10-11).

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(f, g) = \bigwedge_{x \in X}(f(x) \to g(x))$. Then (L^X, e_{L^X}) is a fuzzy preordered set.

(3) If (X, e_X) is a fuzzy preordered set and we define a function $e_X^{-1}(x, y) = e_X(y, x)$, then (X, e_X^{-1}) is a fuzzy preordered set.

3 Doubly Fuzzy Preordered Sets

Definition 3.1 Let e_X^1, e_X^2 be fuzzy preorder on X.
(1) $A \in L^X$ is e_X^1-extensional iff $A(x) \odot e_X^1(x, y) \leq A(y)$.
(2) $B \in L^X$ is e_X^2-extensional iff $B(x) \odot e_X^2(x, y) \leq B(y)$.

The family of e_X^1-extensional (resp. e_X^2-extensional) fuzzy sets is denoted by $E_1(L^X)$ (resp. $E_2(L^X)$).

Definition 3.2 Let (X, e_X^1, e_X^2) be a doubly fuzzy preorderd set. We define maps $l, r : L^X \to L^X$ as, for $A^*(y) = (A(y))^*$,

$$l(A)(x) = \bigwedge_{y \in X} (e_X^1(x, y) \to A^*(y)),$$

$$r(A)(x) = \bigwedge_{y \in X} (e_X^2(x, y) \to A^*(y)).$$

A fuzzy set $A \in L^X$ is called l-stable (resp. l-stable) iff $lr(A) = A$ (resp. rl(A)=A).

The family of all l-stable (resp. r-stable) fuzzy sets will be denoted by $L(L^X)$ (resp. $R(L^X)$).

Theorem 3.3 *Let (X, e_X^1, e_X^2) be a doubly fuzzy preorderd set. We have the following properties.*
(1) $l(A) \in E_1(L^X)$ *and* $l(A) \leq A^*$.
(2) $r(A) \in E_2(L^X)$ *and* $r(A) \leq A^*$.
(3) *If* $A \in E_1(L^X)$, *then* $A \leq lr(A)$.
(4) *If* $A \in E_2(L^X)$, *then* $A \leq rl(A)$.

(5) *If* A *is* $(e^2_X)^{-1}$*-extensional, then* $lr(A) = l(A^*) \leq A$.
(6) *If* A *is* $(e^1_X)^{-1}$*-extensional, then* $rl(A) = r(A^*) \leq A$.
(7) *If* $A \in E_1(L^X)$*, then* $r(A) \in R(L^X)$.
(8) *If* $A \in E_2(L^X)$*, then* $l(A) \in L(L^X)$.
(9) *If* $A \in L(L^X)$*, then* $r(A) \in R(L^X)$.
(10) *If* $A \in R(L^X)$*, then* $l(A) \in L(L^X)$.
(11) *If* $A, B \in L(L^X)$*, then* $r(A) \wedge r(B) \in R(L^X)$.

Proof. (1) By Lemma 2.4(2), we have

$$\begin{aligned} & l(A)(x) \odot e^1_X(x,y) \odot e^1_X(y,z) \leq l(A)(x) \odot e^1_X(x,z) \\ & = \bigwedge_{y \in X}(e^1_X(x,y) \to A^*(y)) \odot e^1_X(x,z) \\ & \leq (e^1_X(x,z) \to A^*(z)) \odot e^1_X(x,z) \leq A^*(z). \end{aligned}$$

Thus, $l(A)(x) \odot e^1_X(x,y) \leq \bigwedge_{y \in X}(e^1_X(y,z) \to A^*(z)) = l(A)(y)$. Furthermore, $l(A)(x) \leq e^1_X(x,x) \to A^*(x) = A^*(x)$.

(3) Since A is e^1_X-extensional, $A(y) \odot e^1_X(y,w) \leq A(w)$ and $A(y) \leq e^1_X(y,w) \to A(w)$. Thus,

$$\begin{aligned} l(r(A))(x) & = \bigwedge_{y \in X}(e^1_X(x,y) \to r(A)^*(y)) \\ & = \bigwedge_{y \in X}(e^1_X(x,y) \to (\bigwedge_{w \in X}(e^2_X(y,w) \to A^*(w)))^*) \\ & = \bigwedge_{y \in X}(e^1_X(x,y) \to \bigvee_{w \in X}(e^2_X(y,w) \odot A(w))) \\ & \geq \bigwedge_{y \in X}(e^1_X(x,y) \to \bigvee_{w \in X}(e^2_X(y,w) \odot A(x) \odot e^1_X(x,w))) \\ & \geq \bigwedge_{y \in X}(e^1_X(x,y) \to (e^2_X(y,y) \odot A(x) \odot e^1_X(x,y))) \\ & \geq A(x). \end{aligned}$$

(5) Since $r(A) \leq A^*$, then $lr(A) \geq l(A^*)$. Moreover, we have:

$$\begin{aligned} l(r(A))(x) & = \bigwedge_{y \in X}(e^1_X(x,y) \to r(A)^*(y)) \\ & = \bigwedge_{y \in X}(e^1_X(x,y) \to (\bigwedge_{w \in X}(e^2_X(y,w) \to A^*(w)))^*) \\ & = \bigwedge_{y \in X}(e^1_X(x,y) \to \bigvee_{w \in X}(e^2_X(y,w) \odot A(w))) \\ & \quad (e^2_X(y,w) \odot A(w) \leq A(y)) \\ & \leq \bigwedge_{y \in X}(e^1_X(x,y) \to A(y)) = l(A^*)(x) \leq A(x). \end{aligned}$$

(7) Let A be e^1_X-extensional. Then $A \leq lr(A)$. Thus $r(A) \geq rlr(A)$. Since $r(A)$ be e^2_X-extensional, by (4), $r(A) \leq rlr(A)$.

(11) We have $rl(r(A) \wedge r(B)) \leq rlr(A) \wedge rlr(B) = r(A) \wedge r(B)$. Moreover, since $r(A), r(B) \in E_2(L^X)$, then $r(A) \wedge r(B) \in E_2(L^X)$. Hence $r(A) \wedge r(B) \leq rl(r(A) \wedge r(B))$.

Other cases are similarly proved.

Theorem 3.4 *Let* (X, e^1_X, e^2_X) *be a doubly fuzzy preorderd set. Define* $r : E_1(L^X) \to E_2(L^X)$ *and* $l : E_2(L^X) \to E_1(L^X)$*. Then* r *and* l *form a Galois connection;i.e.* $B \leq r(A)$ *iff* $A \leq l(B)$.

Proof. Let $B \leq r(A)$. Then $l(B) \geq lr(A) \geq A$ because $A \in E_1(L^X)$.
Let $B \leq l(A)$. Then $r(A) \geq rl(B) \geq B$ because $B \in E_2(L^X)$.

Theorem 3.5 *Let (X, e_X^1, e_X^2) be a doubly fuzzy preorderd set. We define*

$$A \sqcap B = A \wedge B, \;\; A \sqcup B = l(r(A) \wedge r(B)), \;\; A, B \in L(L^X).$$

Then $(L(L^X), \sqcap, \sqcup, \overline{0}, \overline{1})$ is a lattice.

Proof. If $A \leq B$, then $r(A) \geq r(B)$ and $lr(A) \leq lr(B)$. Thus $lr(A \wedge B) \leq lr(A) \wedge lr(B) = A \wedge B$.

Since $A = lr(A)$ and $B = lr(B)$, by Theorem 3.3(1), A and B are e_X^1-extensional. Thus $A \wedge B$ is e_X^1-extensional because

$$(A \wedge B)(x) \odot e_X^1(x, y) \leq (A(x) \odot e_X^1(x, y)) \wedge (B(x) \odot e_X^1(x, y)) \leq (A \wedge B)(y)$$

Thus $lr(A \wedge B) = A \wedge B$;i.e. $A \sqcap B \in L(L^X)$.

Since $l(r(A) \wedge r(B))$ is e_X^1-extensional from Theorem 3.3(1), we have $l(r(A) \wedge r(B)) \leq lrl(r(A) \wedge r(B))$.

Since $r(A)$ and $r(B)$ are e_X^2-extensional from Theorem 3.3(2), $r(A) \wedge r(B)$ are e_X^2-extensional. From Theorem 3.3(2), $r(A) \wedge r(B) \leq rl(r(A) \wedge r(B))$. Thus, $l(r(A) \wedge r(B)) \geq lrl(r(A) \wedge r(B))$. Therefore, $l(r(A) \wedge r(B)) \in L(L^X)$. Let $A \leq C$ and $B \leq C$ for $A, B, C \in L(L^X)$. Then $r(A) \geq C$ and $r(B) \geq C$. Hence $r(A) \wedge r(B) \geq r(C)$. Thus, $l(r(A) \wedge r(B)) \leq l(r(C)) = C$. So, $A \sqcup B$ is the least upper bound.

Example 3.6 Let $X = \{a, b, c\}$ be a set, $(L = [0, 1], \odot)$ with $x \odot y = \max\{0, x + y - 1\}$ and $e_1, e_2 : X \times X \to [0, 1]$ as follows:

e_1	a	b	c
a	1	0.8	1
b	0.3	1	0.5
c	0.7	0.6	1

e_2	a	b	c
a	1	0.9	0.5
b	1	1	0.7
c	0.8	0.7	1

We denote $A = (A(a), A(b), A(c))$.

(1) Since $e_1(x, y) = e_2(x, y) = 1$ implies $x = y$, then (X, e_1, e_2) is a doubly fuzzy ordered set.

(2) $A = (0.5, 0.7, 0.6)$ is e_1-extensional but not e_2-extensional because

$$0.7 = A(b) \odot e_2(b, a) \not\leq A(a) = 0.5.$$

Furthermore, $l(A) = (0.5, 0.3, 0.4)$ and $r(A) = (0.5, 0.3, 0.4)$. $l(r(A)) = A$ and $rl(A) = (0.5, 0.5, 0.6) \neq A$. Hence A is l-stable but not r-stable. Since $0.6 = A(b) \odot e_2^{-1}(b, a) \not\leq A(a) = 0.5$, the converse of Theorem 3.3(5) cannot be true.

(2) For $B = (0.3, 0.3, 0.8)$, $r(B) = (0.7, 0.7, 0.2)$ and $l(r(B)) = B$. Since $r(A) \wedge r(B) = (0.5, 0.4, 0.2)$, we have $A \sqcup B = l(r(A) \wedge r(B)) = (0.5, 0.6, 0.8)$.

4 Representations of Bounded Lattices

Definition 4.1 [7] A map $F : X \to L$ is called a *fuzzy filter* on X if it satisfies the following conditions:

(F1) $F(1) = 1, F(0) = 0$,

(F2) $F(x \wedge y) \geq F(x) \wedge F(y)$,

(F3) if $x \leq y$, then $F(x \wedge y) \geq F(x) \wedge F(y)$.

A map $I : X \to L$ is called a *fuzzy ideal* on X if it satisfies the following conditions:

(I1) $I(1) = 0, I(0) = 1$,

(I2) $I(x \vee y) \geq I(x) \wedge I(y)$,

(I3) if $x \leq y$, then $I(x) \geq I(y)$.

Definition 4.2 Let $(X, \wedge, \vee, 0, 1)$ be a bounded lattice. A pair $P = (F, I)$ is called a *fuzzy filter-ideal* if $F \wedge I = 0$ where F (resp. I) is a fuzzy filter (resp. ideal) on X. We will denote $P(L^X)$ the family of all fuzzy filter-ideal pairs and denote $M(L^X)$ the family of all maximal fuzzy filter-ideal pairs.

Theorem 4.3 *Let $(X, \wedge, \vee, 0, 1)$ is a bounded lattice. Define $e^1_X, e^2_X : P(L^X) \times P(L^X) \to L$ as follows:*

$$e^1_X((F_1, I_1), (F_2, I_2)) = \begin{cases} \bigwedge_{x \in X}(F_1(x) \to F_2(x)), & \text{if } I_1 \leq I_2, \\ 0, & \text{otherwise}, \end{cases}$$

$$e^2_X((F_1, I_1), (F_2, I_2)) = \begin{cases} \bigwedge_{x \in X}(I_2(x) \to I_1(x)), & \text{if } F_2 \leq F_1, \\ 0, & \text{otherwise}, \end{cases}$$

and $\bar{a} : P(L^X) \to L$ as $\bar{a}((F, I)) = F(a) \vee I(a)$.

Then:

(1) $(P(L^X), e^1_X, e^2_X)$ is a doubly fuzzy preordered set.

(2) $\bar{a} = lr(\bar{a})$;i.e. $\bar{a} \in L^{P(L^X)}$.

(3) $(L^{P(L^X)}, \sqcap, \sqcup, \overline{0}, \overline{1})$ is a lattice with, for each $\bar{a}, \bar{b} \in L^{P(L^X)}$,

$$\bar{a} \sqcap \bar{b} = \bar{a} \wedge \bar{b}, \;\; \bar{a} \sqcup \bar{b} = l(r(\bar{a}) \wedge r(\bar{b})).$$

Proof. (1) e^1_X is a fuzzy preorder on $P(L^X)$ from:

$$\begin{aligned} & e^1_X((F_1, I_1), (F_2, I_2)) \odot e^1_X((F_2, I_2), (F_3, I_3)) \\ & = \textstyle\bigwedge_{x \in X}(F_1(x) \to F_2(x)) \odot \bigwedge_{x \in X}(F_2(x) \to F_3(x)) \\ & \leq (F_1(x) \to F_2(x)) \odot (F_2(x) \to F_3(x)) \leq (F_1(x) \to F_3(x)). \end{aligned}$$

Similarly, e^2_X is a fuzzy preorder on $P(L^X)$.

(2) $\bar{a}$ is e^1_X-extensional because

$$\begin{aligned}
&\bar{a}(F,I) \odot e^1_X((F,I),(F_1,I_1)\\
&\le (F(a) \vee I(a)) \odot \textstyle\bigwedge_{x\in X}(F(x) \to F_1(x))\\
&\le F(a) \odot (F(a) \to F_1(a)) \vee I(a)\\
&\le F_1(a) \vee I(a) \le F_1(a) \vee I_1(a) = \bar{a}(F_1,I_1).
\end{aligned}$$

Also, $\bar{a}$ is $(e^2_X)^{-1}$-extensional because

$$\begin{aligned}
&\bar{a}(F,I) \odot e^2_X((F_1,I_1),(F,I))\\
&\le (F(a) \vee I(a)) \odot \textstyle\bigwedge_{x\in X}(I(x) \to I_1(x))\\
&\le F(a) \vee (I(a) \odot (I(a) \to I_1(a)))\\
&\le F(a) \vee I_1(a) \le F_1(a) \vee I_1(a) = \bar{a}(F_1,I_1).
\end{aligned}$$

From Theorem 3.3 (3) and (5), $\bar{a} = lr(\bar{a})$.

(3) It follows from Theorem 3.5.

Theorem 4.4 *If (F,I) is a fuzzy filter-ideal pair, then there exists a maximal fuzzy filter-ideal pair (F_m,I_m) such that $F \le \chi_{F_0} \le F_m$ and $I \le \chi_{I_0} \le I_m$ where χ_{F_0} and χ_{I_0} are characteristic functions with $F_0 = \{x \in X \mid F(x) > 0\}$ and $I_0 = \{x \in X \mid I(x) > 0\}$.*

Proof. We easily show that F_0 and I_0 are classical filter and ideal, respectively. By Zorn's Lemma, there exists maximal filter F_1 and ideal I_1, respectively. Put characteristic functions $F_m = \chi_{F_1}$ and $I_m = \chi_{I_1}$. The results hold.

We denote $[a) = \{x \in X \mid a \le x\}$ and $(b] = \{x \in X \mid x \le b\}$.

Theorem 4.5 *Let $(X,\wedge,\vee,0,1)$ be a bounded lattice. Define $e_1,e_2 : M(L^X)\times M(L^X) \to L$ as follows:*

$$e_1((F_1,I_1),(F_2,I_2)) = \begin{cases} 1, & \text{if } F_1 \le F_2,\\ 0, & \text{otherwise,}\end{cases}$$

$$e_2((F_1,I_1),(F_2,I_2)) = \begin{cases} 1, & \text{if } I_1 \le I_2,\\ 0, & \text{otherwise.}\end{cases}$$

and $\hat{a} : M(L^X) \to L$ as $\hat{a}((F,I)) = F(a)$.

Then:

(1) *$(M(L^X),e_1,e_2)$ is a doubly fuzzy ordered set.*

(2) *$r(\hat{a})(F,I) = I(a)$.*

(3) *$\hat{a} = lr(\hat{a})$;i.e. $\hat{a} \in L^{M(L^X)}$.*

(4) *$(L^{M(L^X)},\sqcap,\sqcup,\overline{0},\overline{1})$ is a lattice with*

$$\hat{a} \sqcap \hat{b} = \hat{a} \wedge \hat{b} = \widehat{a \wedge b},$$

$$\hat{a} \sqcup \hat{b} = l(r(\hat{a}) \wedge r(\hat{b})) = \widehat{a \vee b}.$$

Proof. (1) For $e_1((F_1,I_1),(F_2,I_2)) = e_2((F_1,I_1),(F_2,I_2)) = 1$, then $F_1 \leq F_2$ and $I_1 \leq I_2$. From the above theorem, since F_i and I_i are maximal, $F_1 = F_2$ and $I_1 = I_2$.

(2) Put $I = \chi_A$. If $a \in A$;i.e. $I(a) = \chi_A(a) = 1$ and $e_2((F,I),(F_1,I_1)) = 1$, then $I_1(a) = 1$. Thus $F_1(a) = 0$. It implies

$$I(a) = e_2((F,I),(F_1,I_1)) \to F_1^*(a) = F_1^*(a) = 1.$$

If $a \in A$;i.e. $I(a) = \chi_A(a) = 1$ and $e_2((F,I),(F_1,I_1)) = 0$, then

$$I(a) = e_2((F,I),(F_1,I_1)) \to F_1^*(a) = 0 \to F_1^*(a) = 1.$$

If $a \notin A$;i.e. $I(a) = \chi_A(a) = 0$,

$$0 = I(a) \leq e_2((F,I),(F_1,I_1)) \to F_1^*(a).$$

Thus,

$$I(a) \leq \bigwedge_{(F_1,I_1)\in M(L^X)} e_2((F,I),(F_1,I_1)) \to \hat{a}^*(F_1,I_1) = r(\hat{a})(F,I).$$

Conversely, if $I(a) = 0$, then $\chi_{[a)} \wedge I = 0$. There exists $(F_1,I_1) \in M(L^X)$ with $(\chi_{[a)}, I) \leq (F_1,I_1)$ such that

$$F_1(a) = 1, I \leq I_1.$$

Thus

$$\begin{aligned} r(\hat{a})(F,I) &= \textstyle\bigwedge_{(F_1,I_1)\in M(L^X)} e_2((F,I),(F_1,I_1)) \to \hat{a}^*(F_1,I_1) \\ &\leq e_2((F,I),(F_1,I_1)) \to \hat{a}^*(F_1,I_1) = F_1^*(a) = I(a) = 0 \end{aligned}$$

If $I(a) = 1$, trivially, $r(\hat{a})(F,I) \leq I(a)$. Thus, $r(\hat{a})(F,I) \leq I(a)$.

(3) Since $r(\hat{a})(F,I) = I(a)$, we only show that $lr(\hat{a})(F,I) = l(I(a)) = F(a)$. Put $F = \chi_A$. If $a \in A$;i.e. $F(a) = \chi_A(a) = 1$ and $e_1((F,I),(F_1,I_1)) = 1$, then $F_1(a) = 1$. Thus $I_1(a) = 0$. It implies

$$F(a) = e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1) = I_1^*(a) = 1.$$

If $a \in A$ and $e_1((F,I),(F_1,I_1)) = 0$, then

$$F(a) = e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1) = 0 \to I_1^*(a) = 1.$$

If $a \notin A$;i.e. $F(a) = \chi_A(a) = 0$,

$$0 = F(a) \leq e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1).$$

Thus,

$$F(a) \leq \bigwedge_{(F_1,I_1)\in M(L^X)} e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1) = lr(\hat{a})(F,I).$$

Conversely, if $F(a)=0$, then $\chi_{[a)} \wedge F = 0$. There exists $(F_1,I_1) \in M(L^X)$ with $(F,\chi_{[a)}) \leq (F_1,I_1)$ such that

$$F \leq F_1, I_1(a) = 1.$$

Thus

$$\begin{aligned} lr(\hat{a})(F,I) &= \textstyle\bigwedge_{(F_1,I_1)\in M(L^X)} e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1) \\ &\leq e_1((F,I),(F_1,I_1)) \to r^*(\hat{a})(F_1,I_1) = I_1^*(a) = F(a) = 0. \end{aligned}$$

If $F(a)=1$, trivially, $lr(\hat{a})(F,I) \leq F(a)$. Thus, $lr(\hat{a})(F,I) \leq F(a)$.

(4)

$$\begin{aligned} (\hat{a} \sqcup \hat{b})(F,I) &= l(r(\hat{a})(F,I) \wedge r(\hat{b})(F,I)) = l(I(a) \wedge I(b)) \\ &= l(I(a \vee b)) = l(r(\widehat{a \vee b})) = \widehat{a \vee b}. \end{aligned}$$

$$\begin{aligned} (\hat{a} \sqcap \hat{b})(F,I) &= \hat{a}(F,I) \wedge \hat{b}(F,I)) = F(a) \wedge F(b) \\ &= F(a \wedge b) = \widehat{a \wedge b}. \end{aligned}$$

Theorem 4.6 *Let* $(X,\wedge,\vee,0,1)$ *is a bounded lattice. Define a mapping* $h : X \to L^{M(L^X)}$ *as*

$$h(a)(F,I) = \begin{cases} \hat{a}(F,I), & \text{if } a \notin \{0,1\}, \\ 1, & \text{if } a = 1, \\ 0, & \text{if } a = 0, \end{cases}$$

(1) $r(h(a))(F,I) = I(a)$ *for all* $a \in X$.

(2) $h(a)$ *is* l*-stable for every* $a \in X$.

(3) h *is a lattice embedding.*

Proof. (1) Since $r(h(a))(F,I) = r(\hat{a})(F,I)$, by Theorem 4.5(2), $r(h(a))(F,I) = r(\hat{a})(F,I) = I(a)$.

(2) It follows from Theorem 4.5(3).

(3) Let $h(a) = h(b)$. For $(\chi_{[a)}, I) \in M(L^X)$, we have $h(a)(\chi_{[a)}, I) = \chi_{[a)}(a) = 1 = h(b)(\chi_{[a)}, I) = \chi_{[a)}(b)$. Thus, $b \geq a$. For $(\chi_{[b)}, I) \in M(L^X)$, we have $h(b)(\chi_{[b)}, I) = \chi_{[b)}(b) = 1 = h(a)(\chi_{[b)}, I) = \chi_{[b)}(a)$. Thus, $b \leq a$. Hence $a = b$. Thus, h is injective.

$$\begin{aligned} (h(a) \sqcup h(b))(F,I) &= l(r(h(a))(F,I) \wedge r(h(b))(F,I)) = l(I(a) \wedge I(b)) \\ &= l(I(a \vee b)) = l(r(h(a \vee b))) = h(a \vee b). \end{aligned}$$

$$\begin{aligned}(h(a)\sqcap h(b))(F,I) &= h(a)(F,I)\wedge h(b)(F,I)) = F(a)\wedge F(b)\\ &= F(a\wedge b) = h(a\wedge b).\end{aligned}$$

Example 4.7 Let $X=\{0,a,b,c,1\}$ be a set and $(L=[0,1],\odot)$ with $x\odot y=\max\{0,x+y-1\}$. Let $(X,\wedge,\vee,0,1)$ is a bounded lattice as follows:

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	c	b
c	0	0	c	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	b	1
c	c	1	b	c	1
1	1	1	1	1	1

(1) Put $F(1)=1, F(a)=\frac{1}{2}, F(x)=0$,otherwise and $I(0)=1, I(c)=\frac{1}{3}, I(x)=0$,otherwise. For $(F,I)\in P(L^X)$, there exists $(\chi_{[a)},\chi_{(b]})\in M(L^X)$ such that $F\le\chi_{[a)}$ and $I\le\chi_{(b]}$.

$$\begin{array}{ll}\bar{0}(F,I)=F(0)\vee I(0)=1, & \bar{1}(F,I)=F(1)\vee I(1)=1,\\ \bar{a}(F,I)=F(a)\vee I(a)=\frac{1}{2}, & \bar{b}(F,I)=F(b)\vee I(b)=0,\\ \bar{c}(F,I)=F(c)\vee I(c)=\frac{1}{3}. & \end{array}$$

(2) We obtain $M(L^X)=\{(\chi_{[a)},\chi_{(b]}),(\chi_{[b)},\chi_{(c]}),(\chi_{[c)},\chi_{(a]})\}$. We obtain $h:W\to L^{M(L^X)}$ as follows:

$$\begin{array}{ll}h(a)(\chi_{[a)},\chi_{(b]})=\chi_{[a)}(a)=1, & h(a)(\chi_{[b)},\chi_{(c]})=\chi_{[b)}(a)=0,\\ h(a)((\chi_{[c)},\chi_{(a]}))=\chi_{[c)}(a)=0 & h(b)(\chi_{[a)},\chi_{(b]})=\chi_{[a)}(b)=0,\\ h(b)(\chi_{[b)},\chi_{(c]})=\chi_{[b)}(b)=1, & h(b)((\chi_{[c)},\chi_{(a]}))=\chi_{[c)}(b)=1,\\ h(c)(\chi_{[a)},\chi_{(b]})=\chi_{[a)}(c)=0, & h(c)(\chi_{[b)},\chi_{(c]})=\chi_{[b)}(c)=0,\\ h(c)((\chi_{[c)},\chi_{(a]}))=\chi_{[c)}(c)=1. & \end{array}$$

Furthermore, we have

$$\begin{array}{ll}rh(a)(\chi_{[a)},\chi_{(b]})=\chi_{(b]}(a)=0, & rh(a)(\chi_{[b)},\chi_{(c]})=\chi_{(c]}(a)=0,\\ rh(a)((\chi_{[c)},\chi_{(a]}))=\chi_{(a]}(a)=1, & rh(b)(\chi_{[a)},\chi_{(b]})=\chi_{(b]}(b)=1,\\ rh(b)(\chi_{[b)},\chi_{(c]})=\chi_{(c]}(b)=0, & rh(b)((\chi_{[c)},\chi_{(a]}))=\chi_{(a]}(b)=0,\\ rh(c)(\chi_{[a)},\chi_{(b]})=\chi_{(b]}(c)=1, & rh(c)(\chi_{[b)},\chi_{(c]})=\chi_{(c]}(c)=1,\\ rh(c)((\chi_{[c)},\chi_{(a]}))=\chi_{(a]})(c)=0. & \end{array}$$

Example 4.8 Let $X = \{0, a, b, c, 1\}$ be a set and $(L = [0,1], \odot)$ with $x \odot y = \max\{0, x+y-1\}$. Let $(X, \wedge, \vee, 0, 1)$ be a bounded lattice as follows:

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

Let $(F, I) \in P(L^X)$ as $F(a) = \frac{1}{3}, F(1) = 1, F(x) = 0$ for $x \in \{0, b, c\}$ and $I(b) = \frac{1}{2}, I(0) = 1, I(x) = 0$ for $x \in \{1, a, b\}$. Then there exists $(\chi_{[a)}, \chi_{(b]}) \in M(L^X)$ such that $F \leq \chi_{[a)}$ and $I \leq \chi_{(b]}$. We obtain

$$M(L^X) = \{(\chi_{[a)}, \chi_{(b]}), (\chi_{[a)}, \chi_{(c]}), (\chi_{[b)}, \chi_{(a]}), (\chi_{[b)}, \chi_{(c]}), (\chi_{[c)}, \chi_{(a]}), (\chi_{[c)}, \chi_{(b]})\}.$$

We obtain $h : X \to L^{M(L^X)}$ as follows:

$$\begin{array}{ll} h(a)(\chi_{[a)}, \chi_{(b]}) = h(a)(\chi_{[a)}, \chi_{(c]}) = 1, & h(a)(\chi_{[b)}, \chi_{(a]}) = h(a)(\chi_{[b)}, \chi_{(c]}) = 0, \\ h(a)(\chi_{[c)}, \chi_{(a]}) = h(a)(\chi_{[c)}, \chi_{(b]}) = 0, & h(b)(\chi_{[a)}, \chi_{(b]}) = h(b)(\chi_{[a)}, \chi_{(c]}) = 0, \\ h(b)(\chi_{[b)}, \chi_{(a]}) = h(b)(\chi_{[b)}, \chi_{(c]}) = 1, & h(b)(\chi_{[c)}, \chi_{(a]}) = h(b)(\chi_{[c)}, \chi_{(b]}) = 0, \\ h(c)(\chi_{[a)}, \chi_{(b]}) = h(c)(\chi_{[a)}, \chi_{(c]}) = 0, & h(c)(\chi_{[b)}, \chi_{(a]}) = h(c)(\chi_{[b)}, \chi_{(c]}) = 0, \\ h(c)(\chi_{[c)}, \chi_{(a]}) = h(c)(\chi_{[c)}, \chi_{(b]}) = 0. & \end{array}$$

Similarly, we can obtain $rh(a)(F, I) = I(a)$.

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