Mathematica Aeterna, Vol. 2, 2012, no. 5, 459 - 472

Basicity of a perturbed system of exponents in generalized Lebesgue spaces[1]

Togrul R. Muradov

Institute of Mathematics and Mechanics
National Academy of Sciences, Azerbaijan, Baku

Abstract

We consider a system of exponents with piecewise continuous phase which can be a set of eigenfunctions of discontinuous differential operators. The basicity of this system in generalized Lebesgue spaces are established under certain conditions.

Mathematics Subject Classification: 34L10; 41A58; 42C15

Keywords: Bases with exponents, Generalized Lebesgue space, Variable exponent, Generalized Hardy classes.

1 Introduction

Consider the following system of exponents

$$\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}, \tag{1}$$

where $\lambda_n(t)$ has the representation $\lambda_n(t) = nt - \alpha(t)\, sign\, n$, Z is the set of all integers and $\alpha(t)$ is a piecewise continuous function on the segment $[-\pi;\pi]$. A great number of papers beginning with classic Theorem of Paley and Wiener [17] on the Riesz basicity in L_2 and the results of Levinson [15] have been devoted to basis properties (basicity, completeness, minimality) of system (1) in classic Lebesgue spaces $L_p \equiv L_p(-\pi;\pi)$, $1 \le p \le +\infty$, $(L_\infty \equiv C[-\pi;\pi])$ when $\alpha(t) = \alpha t$and $\alpha \in R$ is a real parameter. Necessary and sufficient basicity conditions in L_p, $1 < p < \infty$, for a parameter $\alpha \in R$ have been obtained in [16,19]. The most general case has been considered in [2,3].

Recently, in the light of specific problems of mechanics and mathematical physics, there arose a great interest in studying this kind of matters in Lebesgue

[1]This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan - **Grant No EIF/GAM-1-2011-2(4)-26/03/1**.

spaces $L_{p(\cdot)}$ and Sobolev spaces $W^m_{p(\cdot)}$ with variable summability index $p(t)$. Detailed information about these problems can be found in [10,11,14,18,21]. Solving many partial differential equations by the method of separation of variables urges the necessity to study basis properties in the spaces $L_{p(\cdot)}$ and $W^m_{p(\cdot)}$ of the system of root functions of ordinary differential operators, generated by these problems.

The case $\alpha(t) \equiv \alpha t$ was earlier studied in [20] for $\alpha = 0$ and in [5,7] for $\alpha \in R$. The basicity in$L_{p(\cdot)}$of the system (1) when $\lambda_n(t) \equiv -\text{sign}n\,[\alpha t + \beta \text{sign}t]\,,\ t \in [-\pi;\pi]\,,\ \alpha,\beta \in C$ are complex parameters, is established in [6].

The present paper studies the basicity of system (1) and its perturbations in generalized Lebesgue spaces $L_{p(\cdot)}$ with variable summability exponent $p(\cdot)$.

2 Necessary notion and facts. Basic Assumptions

We state some ideas from the theory of $L_{p(\cdot)}$ spaces. Let $p : [-\pi,\pi] \to [1,+\infty)$ be a Lebesgue measurable function. Denote by L_0 the class of all measurable functions on $[-\pi,\pi]$ (with respect to Lebesgue measure). Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\,dt.$$

Let $\mathrm{L} \equiv \{f \in \mathrm{L}_0 : I_p(f) < +\infty\}$ and $p^{\pm} = \sup_{[-\pi,\pi]} vrai\, p(t)^{\pm 1}$. Subject to the condition $1 \le p^- \le p^+ < +\infty$, L turns into a linear space with respect to ordinary linear operations of addition of functions and multiplication of a function by a number. With the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}, f \in \mathrm{L}$$

the space L is a Banach space and we denote it by $L_{p(\cdot)}$. Assume

$$WL_\pi \stackrel{def}{\equiv} \{p : p(\pi) = p(-\pi) \quad and \quad \exists C > 0; \quad \forall t_1,t_2 \in [-\pi,\pi]\,, |t_1 - t_2| \le \frac{1}{2} \Rightarrow$$

$$\left.\Rightarrow |p(t_1) - p(t_2)| \le \frac{C}{-\ln|t_1 - t_2|}\right\}.$$

This is a weakly Lipschitz class of functions periodic on $[-\pi,\ \pi]$. Throughout this paper $q(t)$ will denote the function conjugated to $p(t)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. The following Holder's generalized inequality holds:

$$\int_{-\pi}^{\pi} |f(t)\,g(t)|\,dt \le c\left(p^-;p^+\right) \|f\|_{p(\cdot)}\, \|g\|_{q(\cdot)}\,,$$

where $c(p^-;p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. The following property is valid.

Property A. *If $|f(t)| \le |g(t)|$ a.e. on $(-\pi,\pi)$, then $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.* We'll oftenly use this property. Also the following lemma is easily proved.

Lemma 2.1 *Let $p \in WL_\pi$, $p(t) > 0, \forall t \in [-\pi,\pi]$ and $\{\alpha_i\}_1^m \subset R$ (R is a real axis). The function $\omega(t) = \prod_{i=1}^m |t - t_i|^{\alpha_i}$ belongs to the space $L_{p(\cdot)}$ if $\alpha_i > -\frac{1}{p(t_i)}$, $\forall i = \overline{1,m}$; where $\{t_i\}_1^m \subset [-\pi,\pi]$, $t_i \ne t_j$, for $i \ne j$.*

Detailed information about these results can be found in [10,11,14,18,21]. We'll assume that the function $\alpha(t)$ satisfies the following basic assumptions:

(α) *$\alpha(t)$ is piecewise Holder on $[-\pi,\pi]$ and $\{s_k\} : -\pi = s_0 < s_1 < ... < s_r < s_{r+1} = \pi$ are its discontinuity points on $(-\pi,\pi)$. Let $\{h_k\}_1^r : h_k = \alpha(s_k + 0) - \alpha(s_k - 0)$, $k = \overline{1,r}$ be the jumps of the function $\alpha(t)$ at the points s_k and $h_0 = \frac{\alpha(-\pi)-\alpha(\pi)}{\pi}$.*

$$(\beta) \left\{ \frac{h_k}{\pi} - \frac{1}{p(s_k)} : k = \overline{0,r} \right\} \bigcap Z = \emptyset.$$

Define $\{n_k\}_1^r \subset Z$ by the following relations:

$$-\frac{1}{q(s_k)} < \frac{h_k}{\pi} + n_{k-1} - n_k < \frac{1}{p(s_k)}, n_0 = 0\,,\ k = \overline{1,r}$$

and assume that $\omega_\pi = h_0 + n_r$.

Let $\omega = \{z : |z| < 1\}$ be a unit circle on a complex plane and $\partial\omega$ be a unit circumference. Introduce the Hardy class

$$H_{p(\cdot)}^+ \equiv \left\{ f :\ f \text{ analytic in } \omega \text{ and } \|f\|_{H_{p(\cdot)}^+} < +\infty \right\},$$

where $\|f\|_{H_{p(\cdot)}^+} \equiv \sup\limits_{0<r<1} \|f(re^{it})\|_{p(\cdot)}$. $H_{p(\cdot)}^+$ is a Banach space if $1 \le p^- \le p^+ < +\infty$. Determine the Hardy class ${}_mH_{p(\cdot)}^-$ of functions analytic outside the unit circle of order less than or equal to $m \ge 0$ at infinity. Let $f(z)$ be an analytic function on $C\backslash\bar\omega$ $(\bar\omega = \omega \cup \partial\omega)$, of finite order $m_0 \le m$ at infinity, i.e. $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is a polynomial of degree m_0, and $f_2(z)$ is the right part of the expansion of the function $f(z)$ in Lorents series in the neighborhood of infinite point. We'll say that the function $f(z)$ belongs to the class ${}_mH_{p(\cdot)}^-$, if the function $\varphi(z) = \overline{f_2\left(\frac{1}{\bar z}\right)}$ ($\bar{(\cdot)}$ is a complex conjugation) belongs to the class $H_{p(\cdot)}^+$.

For our investigation we need some basic concepts of the theory of close bases, which are given as follows.

We'll denote a Banach space as B-space, the space conjugated to X is denoted by X^*. N is the set of all positive integers and $Z_+ = \{0\} \cup N$.

Definition 2.2 *The system* $\{x_n\}_{n\in N} \subset X$ *in* B*-space* X *is called* ω*-linearly independent if* $\sum_{n=1}^{\infty} a_n x_n = 0 \Rightarrow a_n = 0$, $\forall n \in N$.

The following lemma holds true.

Lemma 2.3 *Let* X *be a* B*-space with the basis* $\{x_n\}_{n\in N}$ *and* $F : X \to X$ *a Fredholm operator. Then the following properties of the system* $\{y_n = Fx_n\}_{n\in N}$ *in* X *are equivalent:*

1) $\{y_n\}_{n\in N}$ *is complete; 2)* $\{y_n\}_{n\in N}$ *is minimal; 3)* $\{y_n\}_{n\in N}$ *is* ω *- linearly independent; 4)* $\{y_n\}_{n\in N}$ *is basis isomorphic to* $\{x_n\}_{n\in N}$.

Definition 2.4 *The systems* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *in* B*-space* X *with the norm* $\|\cdot\|$ *are called* p*-close if* $\sum_n \|x_n - y_n\|^p < +\infty$.

Definition 2.5 *The minimal system* $\{x_n\}_{n\in N} \subset X$ *in* B*-space* X *is called a* p*-system if for* $\forall x \in X :\ \{x_n^*(x)\}_{n\in N} \in l_p$, *where* $\{x_n^*\}_{n\in N} \subset X^*$ *is its conjugate and* l_p *a usual space of* p*-absolutely summable sequences* $\{a_n\}_{n\in N}$ *normed by* $\left\|\{a_n\}_{n\in N}\right\|_{l_p} = \left(\sum_n |a_n|^p\right)^{\frac{1}{p}}$. *In the case of basicity this system will be called a* p*-basis.*

Detailed information about this kind of facts can be found in the monographs [22,23] and in the papers [1,8]. We also need the following theorem.

Theorem 2.6 ***(Krein-Milman-Rutman [20])*** *Let* X *be a* $B-$*space with norm* $\|\cdot\|$ *and with normalized basis* $\{x_n\}_{n\in N}$ *and* $\{x_n^*\}_{n\in N} \subset X^*$ *be a system biorthogonal to it. If the system* $\{y_n\}_{n\in N} \subset X$ *satisfies the condition* $\sum_{n=1}^{\infty} \|x_n - y_n\| < \eta^{-1}$, *where* $\eta = \sup_n \|x_n^*\|$, *then it forms a basis for* X *isomorphic to* $\{x_n\}_{n\in N}$.

To obtain our main result we'll use the following lemma.

Lemma 2.7 *Let* X *be a* B*-space with the basis* $\{x_n\}_{n\in N}$ *and* $\{x_n^*\}_{n\in N} \subset X^*$ *be a system biorthogonal to* $\{x_n\}_{n\in N}$. *Assume that the system* $\{y_n\}_{n\in N} \subset X$ *differs from* $\{x_n\}_{n\in N}$ *by the finite number of elements, i.e. . Then, if* $\Delta_{n_0} = \det\left(x_n^*(y_k)\right)_{n,k=\overline{1,n_0}} = 0$, *the system* $\{y_n\}_{n\in N}$ *is not minimal in* X.

We'll need the following

Theorem 2.8 *Let* $p \in WL_\pi$, $p^- > 1$, *and* $A^{\pm 1}, B^{\pm 1} \in L_\infty(-\pi, \pi)$. *If the double system of exponents* $\left\{A(t)e^{int};\ B(t)e^{-ikt}\right\}_{n\in Z_+; k\in N}$ *forms a basis for* $L_{p(\cdot)}(-\pi, \pi)$ *it is isomorphic in* $L_{p(\cdot)}$ *to the classic system of exponents* $\{e^{int}\}_{n\in Z}$, *and the isomorphism is given by the operator* S, *where*

$$Sf = A\sum_{0}^{\infty}\left(f, e^{inx}\right)e^{int} + B\sum_{1}^{\infty}\left(f, e^{-inx}\right)e^{-int}, (f, g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

The validity of this statement is proved in [4]. Using this result, the following theorem is easily proved.

Theorem 2.9 *Let the system (1) forms a basis for L_p, $1 < p < +\infty$. Then*
1) If $1 < p \leq 2$ and $f \in L_p$, then $\{f_n\}_{n \in Z} \in l_q$, and the inequality

$$\left\| \{f_n\}_{n \in Z} \right\|_{l_q} \leq m_p \left\| f \right\|_p$$

is fulfilled, where m_p is a constant independent of f and $\|\cdot\|_p$ is the ordinary norm in L_p.

2) let $p > 2$ and let the sequence of numbers $\{a_n\}_{n \in Z}$ belong to l_q. Then $\exists f \in L_p$ such that $f_n = a_n$, $\forall n \in Z$, and the inequality

$$\left\| f \right\|_p \leq M_p \left\| \{f_n\}_{n \in Z} \right\|_{l_q}$$

holds, where M_p is a constant independent of $\{f_n\}_{n \in Z}$.

Consider the Riemann homogeneous problem

$$\begin{array}{l} F^{+}(\tau) - G(\tau) F^{-}(\tau) = 0 \ , \ \tau \in \partial\omega \, , \\ F^{+} \in H^{+}_{p(\cdot)} \, , \ F^{-} \in {}_m H^{-}_{p(\cdot)} \end{array} \qquad (2)$$

The following theorem is true for it.

Theorem 2.10 *Let $p \in WL_{\pi}$, $p^- > 1$, $G(e^{it}) = e^{2i\alpha(t)}$, where $\alpha(t)$ satisfies the condition (α) and the following inequalities be fulfilled:*

$$-\frac{1}{q(\pi)} < \frac{h_0}{\pi} < \frac{1}{p(\pi)}; -\frac{1}{q(s_k)} < \frac{h_k}{\pi} < \frac{1}{p(s_k)}, k = \overline{1, r}.$$

Then the general solution of Riemann homogeneous problem (2) has the form $F(z) = Z(z) P_{m_0}(z)$, $m_0 \leq m$, where $Z(z)$ is a canonical solution.

As immediate consequence of Theorem 2.10 is the following.

Corollary 2.11 *Let all the conditions of theorem 2.10 be fulfilled. Then for $F^-(\infty) = 0$, i.e. in the class $H^+_{p(\cdot)} \times {}_{-1}H^-_{p(\cdot)}$, the homogeneous problem (2) has only a trivial solution.*

Consider the Riemann inhomogeneous problem

$$\begin{array}{l} F^{+}(\tau) - G(\tau) F^{-}(\tau) = f(\tau), \ \tau \in \partial\omega, \\ F^{+} \in H^{+}_{p(\cdot)}, \ F^{-} \in {}_m H^{-}_{p(\cdot)}, \end{array} \qquad (3)$$

where $f \in L_{p(\cdot)}$ and $G(\tau) = e^{2i\alpha(\arg \tau)}$. Consider the Cauchy type integral

$$F_1(z) = \frac{Z(z)}{2\pi i} \int_{\partial\omega} \frac{f(\tau)}{Z^+(\tau)} \frac{d\tau}{\tau - z}, \tag{4}$$

where $Z^+(\tau)$ are non-tangential boundary values of the function $Z(z)$ on $\partial\omega$ inside the unit circle ω.

The following theorem is valid.

Theorem 2.12 *Let* $p \in WL_\pi$, $G(e^{it}) = e^{2i\alpha(t)}$, $\alpha(t)$ *satisfy the condition* (α) *and the following inequality be fulfilled:*

$$-\frac{1}{q(s_k)} < \frac{h_k}{\pi} < \frac{1}{p(s_k)}, \ k = \overline{0, r}. \tag{5}$$

Then the general solution of the Riemann nonhomogeneous problem (3) in the class $H^+_{p(\cdot)} \times {}_mH^-_{p(\cdot)}$ $(m \geq 0)$ *is of the form* $F(z) = Z(z)P_{m_0}(z) + F_1(z)$, *where*

$$F_1(z) = \frac{Z(z)}{2\pi i} \int_{\partial\omega} \frac{f(\tau)}{Z^+(\tau)} \frac{d\tau}{\tau - z}, \tag{6}$$

$Z(z)$ *is a canonical solution of the corresponding homogeneous problem.*

Corollary 2.13 *Let all the conditions of theorem 2.12 be fulfilled. Then, provided* $F(\infty) = 0$, *problem (3) has a unique solution* $F_1(z)$ *defined by (6).*

3 Basicity of a double system of exponents

Consider the system (1). The following theorem is valid:

Theorem 3.1 *Let* $p \in WL_\pi$, $p^- > 1$, *and the function* $\alpha(t)$ *satisfy the condition* (α). *If inequalities (5) are fulfilled, then the system of exponents (1) forms a basis for* $L_{p(\cdot)}$.

Proof. Consider the following Riemann nonhomogeneous problem

$$\begin{array}{l} F^+(\tau) + G(\tau)F^-(\tau) = e^{i\alpha(\arg \tau)} f(\arg \tau), \ \tau \in \partial\omega, \\ F^+ \in H^+_{p(\cdot)} \times {}_{-1}H^-_{p(\cdot)}, \ F(\infty) = 0, \end{array} \tag{7}$$

where $f \in L_{p(\cdot)}$.

Let $p \in WL_\pi$ and $\alpha(t)$ satisfy the condition (α). Suppose that inequalities (5) are fulfilled. Then according to Corollary 2.13, problem (7) has a unique solution of the form (6). According to the results in [20] the system $\{e^{int}\}_{n \in Z}$

forms a basis for $L_{p(\cdot)}$. Denote by $\{g_n\}_{n\in Z}$ a biorthogonal coefficients of the function $g \in L_{p(\cdot)}$ with respect to the system $\{e^{int}\}_{n\in Z}$, i.e.

$$g_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt,\ n \in Z.$$

It is obvious that $F^{\pm}\left(e^{it}\right) \in L_{p(\cdot)}$, and moreover, from $F \in H^{+}_{p(\cdot)} \times {}_{-1}H^{-}_{p(\cdot)}$, $F(\infty) = 0$ it follows that $F^{+}_n = 0,\ \forall n > 0;\ F^{-}_n = 0,\ n \geq 0$. Consequently, the following expansions hold in $L_{p(\cdot)}$:

$$F^{+}\left(e^{it}\right) = \sum_{n=0}^{\infty} F^{+}_n e^{int};\ F^{-}\left(e^{it}\right) = \sum_{n=1}^{\infty} F^{-}_{-n} e^{-int}.$$

Taking into account these expansions in (7), we get that the function f can be expanded as a series with respect to the system (1) in $L_{p(\cdot)}$. Let's prove the uniqueness of such series expansion. Let $e^{-i\alpha(t)}\sum_{n=0}^{\infty} a_n e^{int}+$
$+e^{i\alpha(t)}\sum_{n=1}^{\infty} b_n e^{-int} = 0$ and

$$f(t) = \sum_{n=0}^{\infty} a_n e^{int},\ g(t) = \sum_{n=0}^{\infty} \overline{b_{n+1}} e^{int}.$$

Then, we have

$$A(t)f(t) + B(t)\overline{g(t)} = 0,\ t \in (-\pi, \pi), \tag{8}$$

where $A(t) = e^{-i\alpha(t)}$, $B(t) = e^{-i[\alpha(t)-t]}$. It is obvious that

$$\int_{\partial\omega} f(\arg\tau)\tau^n d\tau = \sum_{k=0}^{\infty} a_k \int_{\partial\omega} \tau^{n+k} d\tau = 0,\ \forall n \geq 0. \tag{9}$$

According to the results of [9], it follows from (9) and $f \in L_1(\partial\omega)$ that $\exists G \in H^{+}_1 :\ G^{+}(\tau) = f(\arg\tau)$, a.e. $\tau \in \partial\omega$. Assume $\Phi_0(z) = \overline{G\left(\frac{1}{\bar z}\right)},\ |z| > 1$. Thus, $\Phi^{-}_0(\tau) = \overline{G^{+}(\tau)},\ \tau \in \partial\omega$, where $\Phi^{-}(\tau)$ are non-tangential boundary values of $\Phi(z)$ on$\partial\omega$ outside the unique circle. It is clear that $G^{+},\ \Phi^{-}_0 \in L_{p(\cdot)}(\partial\omega)$. Then, from Smirnov Theorem [13] it follows that $G \in H^{+}_{p(\cdot)}$ and $\Phi_0 \in {}_0H^{-}_{p(\cdot)}$. If $\Phi(z) = z^{-1}\Phi_0(z)$, it is clear that $\Phi \in {}_{-1}H^{-}_{p(\cdot)}$, i.e. $\Phi(\infty) = 0$. So, we have

$$G^{+}\left(e^{it}\right) = \sum_{n=0}^{\infty} a_n e^{int};\ G^{+}\left(e^{it}\right) = \sum_{n=0}^{\infty} \overline{b_{n+1}} e^{int}.$$

Consequently,

$$\Phi^{-}_0\left(e^{it}\right) = \overline{G^{+}\left(e^{it}\right)} = \sum_{n=0}^{\infty} b_{n+1} e^{-int},\ \Phi^{-}\left(e^{it}\right) = \sum_{n=1}^{\infty} b_n e^{-int}.$$

As a result, from (8) we get the following conjugation problem in the classes $H^+_{p(\cdot)} \times {}_{-1}H^-_{p(\cdot)}$:

$$G^+(\tau) + G(\tau)\Phi^-(\tau) = 0, \ \tau \in \partial\omega,$$
$$(G;\Phi) \in H^+_{p(\cdot)} \times {}_{-1}H^-_{p(\cdot)}.$$

Since all the conditions of Theorem 2.9 are fulfilled, from Corollary 2.11 we get $G(z) \equiv \Phi(z) \equiv 0$, i.e. $a_n = b_{n+1} = 0$, $\forall n \geq 0$. This proves the basicity of the system (1) in $L_{p(\cdot)}$.

4 Perturbed basis with exponents

Consider the following perturbed system of exponents

$$\left\{e^{i\mu_n(t)}\right\}_{n\in Z}, \tag{10}$$

where $\mu_n(t)$ has the following form:

$$\mu_n(t) = nt - \alpha(t)\, sign\, n + \beta_n(t)\,, n \to \infty. \tag{11}$$

Assume that the following condition is fulfilled:

(γ) *the functions* $\{\beta_n\}$ *satisfy the relation*

$$\beta_n(t) = O\left(\frac{1}{n^{\gamma_k}}\right), t \in (s_k, s_{k+1})\,, k = \overline{0,r}; \{\gamma_k\}_1^r \subset (0,+\infty)\,.$$

The following theorem is valid.

Theorem 4.1 *Let the asymptotic formula (11) be true with functions*$\alpha(t)$ *and* $\beta_n(t)$ *satisfying conditions* (α) *and* (γ). *Assume that the following inequalities hold:*

$$-\frac{1}{q(\pi)} < \omega_\pi < \frac{1}{p(\pi)}, \gamma > \frac{1}{\tilde{p}},$$

where $\gamma = \min_k \gamma_k$, $\tilde{p} = \min\{p^-; 2\}$ *and the quantity* ω_π *is determined by the relations* (β). *Then the following properties of the system (10) in* $L_{p(\cdot)}$ *are equivalent:*

1) it is complete;

2) it is minimal;

3) it is ω*-linearly independent;*

4) it forms a basis isomorphic to $\{e^{int}\}_{n\in Z}$.

Proof. We have

$$\left|e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right| = \left|e^{i\beta_n(t)} - 1\right| = \left|\sum_{k=1}^{\infty} \frac{\beta_n^k(t)}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{Mn^{-\gamma}}{k!} = cn^{-\gamma},$$

where $\gamma = \min_k \gamma_k$, and $c$is a constant independent of n. The last inequality follows from the condition (γ). Consider some special cases.
Let $\tilde{p} = \min\{p^-; 2\}$ and $\gamma > \frac{1}{\tilde{p}}$. We have

$$\sum_n \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)}^p \le c_p \sum_n \frac{1}{|n|^{\gamma\tilde{p}}} < +\infty.$$

Suppose that all conditions of Theorem 2.10 and inequalities (5) are fulfilled. Then system (1) forms a basis for $L_{p(\cdot)}$. By Theorem 2.8, it is isomorphic to the classical system with exponents $\{e^{int}\}_{n\in Z}$ in $L_{p(\cdot)}$. As a result, spaces of coefficients of the bases $\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}$ and $\{e^{int}\}_{n\in Z}$ are congruent. Let $T : L_{p(\cdot)} \rightarrow L_{p(\cdot)}$ be a natural automorphism, i.e. $T\left[e^{i\lambda_n(t)}\right] = e^{int}$, $\forall n \in Z$. For all $f \in L_{p(\cdot)}$ let $\{f_n\}_{n\in Z}$ be a biorthogonal coefficients of f with respect to the system $\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}$. Assume that $g = Tf$. Consequently, $\{f_n\}_{n\in Z}$ is a Fourier coefficients of function g with respect to the system $\{e^{int}\}_{n\in Z}$. it follows directly from (11) and from condition (γ)that

$$\sum_{n\in Z} \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)}^{\tilde{p}} < +\infty.$$

Consider the expression $\sum_n \left(e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right) f_n$. We have

$$\sum_{n\in Z} \left\| \left(e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right) f_n \right\|_{p(\cdot)} \le \sum_{n\in Z} \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\| |f_n| \le$$

$$\le \left(\sum_{n\in Z} \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)}^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} \left(\sum_n |f_n|^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}},$$

where $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$. By the Hausdorff-Young theorem [3] we get

$$\left(\sum_n |f_n|^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \le m_1 \|g\|_{\tilde{p}},$$

where m_1 is a constant. From $\tilde{p} \le p^-$ and the continuous embedding $L_{p(\cdot)} \subset L_{\tilde{p}}$ we get

$\|g\|_{\tilde{p}} \le m_2 \|g\|_{p(\cdot)} \le m_2 \|T\| \, \|f\|_{p(\cdot)}$, for some $m_2 > 0$.
As a result we have

$$\left\| \sum_n \left(e^{i\lambda_n(t)} - e^{i\mu_n(t)}\right) f_n \right\|_{p(\cdot)} \le$$

$$\leq m_1 m_2 \left(\sum_n \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)}^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} \|f\|_{p(\cdot)} . \tag{12}$$

Take $n_0 \in N$ such that

$$\delta = m_1 m_2 \|T\| \left(\sum_{|n|>n_0} \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)}^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} < 1$$

and assume that

$$\omega_n(t) = \begin{cases} \lambda_n(t)\,, & |n| > n_0\,, \\ \mu_n(t)\,, & |n| \leq n_0\,. \end{cases}$$

Then it is clear that

$$\left\| \sum_n \left(e^{i\omega_n(t)} - e^{i\lambda_n(t)} \right) f_n \right\|_{p(\cdot)} \leq \delta \|f\|_{p(\cdot)} . \tag{13}$$

It follows directly from (12) that the expression $\sum_n \left(e^{i\omega_n(t)} - e^{i\lambda_n(t)} \right) f_n$ represents a function from $L_{p(\cdot)}$, denote it by $T_0 f$. Taking into account (13), we get $\|T_0\| \leq \delta < 1$. Thus, the operator $F = I + T_0$ is invertible and it is easy to see that $F\left[e^{i\lambda_n(t)}\right] = e^{i\omega_n t}$, $\forall n \in Z$. Consequently, the system $\left\{e^{i\omega_n(t)}\right\}_{n\in Z}$ forms a basis isomorphic to $\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}$ for $L_{p(\cdot)}$. The systems $\left\{e^{i\mu_n(t)}\right\}_{n\in Z}$ and $\left\{e^{i\omega_n(t)}\right\}_{n\in Z}$ differ by the finite number of elements. The further evidence follows directly from Lemma 2.1. Thus, the theorem is proved.

Now we consider the case when $\gamma > 1$. In this case it is obvious that it holds $\sum_{n=-\infty}^{\infty} \|e^{i\lambda_n(t)} - e^{i\mu_n(t)}\|_{p(\cdot)} < +\infty$. Let all the conditions of Theorem 2.12 be fulfilled. Then the system $\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}$ forms a basis for $L_{p(\cdot)}$. Denote by $\{\vartheta_n\}_{n\in Z} \subset L_{q(\cdot)}$ the system biorthogonal to it. Assume $\vartheta = \sup_n \|\vartheta_n\|_{q(\cdot)}$. It is clear that $\exists n_0 \in N$:

$$\sum_{|n|\geq n_0+1} \left\| e^{i\lambda_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)} < \vartheta^{-1}.$$

Consider the following functions:

$$\tilde{\lambda}_n(t) = \begin{cases} \mu_n(t)\,, & |n| > n_0\,, \\ \lambda_n(t)\,, & |n| \leq n_0\,. \end{cases}$$

Thus,

$$\sum_{n=-\infty}^{\infty} \left\| e^{i\tilde{\lambda}_n(t)} - e^{i\mu_n(t)} \right\|_{p(\cdot)} < \vartheta^{-1}.$$

Then it follows from Theorem 2.6 that the system $\left\{e^{i\tilde{\lambda}_n(t)}\right\}_{n\in Z}$ forms a basis isomorphic to $\left\{e^{i\lambda_n(t)}\right\}_{n\in Z}$ for $L_{p(\cdot)}$. System (10) and the basis $\left\{e^{i\tilde{\lambda}_n(t)}\right\}_{n\in Z}$ differ by the finite number of elements. Denote by $\left\{\tilde{\vartheta}_n\right\}_{n\in Z}$ the system biorthogonal to this basis. Consider

$$e^{i\lambda_k(t)} = \sum_{|n|\le n_0} a_{nk} e^{i\lambda_n(t)} + \sum_{|n|>n_0} a_{nk} e^{i\mu_n(t)}, \forall k : |k| \le n_0, \tag{14}$$

where $a_{nk} = \tilde{\vartheta}_n\left(e^{i\lambda_k(t)}\right) = \int_{-\pi}^{\pi} e^{i\lambda_k(t)}\overline{\tilde{\vartheta}_n(t)}dt$. Denote by Δ_{n_0} the following determinant

$$\Delta_{n_0} = \det\left(a_{ij}\right)_{i,j=\overline{-n_0,n_0}}. \tag{15}$$

It is clear that if $\Delta_{n_0} \neq 0$, the elements $e^{i\lambda_k(t)}$, $k = \overline{-n_0, n_0}$, in expansion (14), may be replaced by the elements $e^{i\mu_k(t)}$, $k = \overline{-n_0, n_0}$. Since the system $\left\{e^{i\tilde{\lambda}_n(t)}\right\}_{n\in Z}$ forms a basis for $L_{p(\cdot)}$, then f has the expansion $f = \sum_{n=-\infty}^{\infty} \tilde{\vartheta}_n(f) e^{i\tilde{\lambda}_n(t)}$, $\forall f \in L_{p(\cdot)}$. Hence it follows directly that if $\Delta_{n_0} \neq 0$, then f has the expansion with respect to the system (10) $\forall f \in L_{p(\cdot)}$, i.e. it is complete in $L_{p(\cdot)}$. Consider the operator

$$\tilde{F}f = \sum_{n=-\infty}^{\infty} \tilde{\vartheta}_n(f)\, e^{i\mu_n(t)}.$$

We have

$$\tilde{F}f = \sum_{n=-\infty}^{\infty} \tilde{\vartheta}_n(f)\, e^{i\tilde{\lambda}_n(t)} + \sum_{n=-\infty}^{\infty} \tilde{\vartheta}_n(f)\left[e^{i\mu_n(t)} - e^{i\tilde{\lambda}_n(t)}\right] =$$

$$= f + \sum_{n=-n_0}^{n_0} \tilde{\vartheta}_n(f)\left[e^{i\mu_n(t)} - e^{i\tilde{\lambda}_n(t)}\right] = (I+T)\, f,$$

where $I : L_{p(\cdot)} \to L_{p(\cdot)}$ is an identity operator and T is an operator generated by the second summand above. The Fredholm property of F in $L_{p(\cdot)}$ follows from the finite dimensionality of the operator T. It is clear that $\tilde{F}\left[e^{i\tilde{\lambda}_n}(t)\right] = e^{i\mu_n(t)}$, $\forall n \in Z$. Then from Lemma 2.3 we get the basicity of system (10) in L_p. Conversely, if system (10) forms a basis for $L_{p(\cdot)}$, it follows from Lemma 2.7 that $\Delta_{n_0} \neq 0$. Thus we established that under the given conditions the system (10) forms a basis for $L_{p(\cdot)}$ if the determinant defined by (15) is not equal to zero. So we have just proved the following theorem:

Theorem 4.2 *Let all the conditions of Theorem 3.1 with $\gamma > 1$ be fulfilled and the determinant Δ_{n_0} be defined by expression (15). Then the system (10) forms a basis for $L_{p(\cdot)}$, if $\Delta_{n_0} \neq 0$.*

Now consider the case when ω_π doesn't belong to $\left(-\frac{1}{q(\pi)}, \frac{1}{p(\pi)}\right)$. Let, for example, $\frac{1}{p(\pi)} < \omega_\pi < \frac{1}{p(\pi)} + 1$. In this case, as it follows from Theorem 3.1, the system

$$\left\{e^{i\mu_n(t)}\right\}_{n\in Z} \bigcup \left\{e^{i\alpha(t)}\right\} \tag{16}$$

forms a basis for $L_{p(\cdot)}$. Consider the system

$$\left\{e^{i\lambda_n(t)}\right\}_{n\in Z} \bigcup \left\{g\left(t\right)\right\}, \tag{17}$$

where $g \in L_{p(\cdot)}$. Let the conditions $(\alpha), (\beta)$ and $\gamma > \frac{1}{\tilde{p}}$ be fulfilled for system (10). Then it is easy to see that system (15) and basis (16) are $\tilde{p}$-close in $L_{p(\cdot)}$, where $\tilde{p} = \min\left\{p^-; 2\right\}$. Consequently, the system (10) is not complete in $L_{p(\cdot)}$. The remaining cases of $\omega_\pi > \frac{1}{p(\pi)}$ are proved in the same way.

Consider the case when , for example, $\omega_\pi \in \left(-\frac{1}{q(\pi)} - 1, -\frac{1}{q(\pi)}\right)$. In this case, again by virtue of Theorem 2.12, the system

$$\left\{e^{i\mu_n(t)}\right\}_{n\neq 0}, \tag{18}$$

forms a basis for $L_{p(\cdot)}$. If the conditions (α) and (β) are fulfilled, then the basis (18) and system $\left\{e^{i\lambda_n(t)}\right\}_{n\neq 0}$ are $\tilde{p}$-close in $L_{p(\cdot)}$. Consequently, the system (10) is not minimal in $L_{p(\cdot)}$. The remaining cases of $\omega_\pi < -\frac{1}{q(\pi)}$ can be proved similarly. As a result, we get the following final result for the basicity of system (10) for $L_{p(\cdot)}$.

Theorem 4.3 *Let the asymptotic formula (11) holds with the conditions* (α) *and* (γ) *for the functions* $\alpha\left(t\right)$ *and* $\beta_n(t)$*. Let the quantity* ω_π *be determined by the relations* (β) *and let* $\gamma > \frac{1}{\tilde{p}}$*. Then for* $\omega_\pi < -\frac{1}{q(\pi)}$ *the system (10) is nonminimal in* $L_{p(\cdot)}$ *; for* $\omega_\pi > \frac{1}{p(\pi)}$ *it is not complete in* $L_{p(\cdot)}$ *. For* $-\frac{1}{q(\pi)} < \omega_\pi < \frac{1}{p(\pi)}$ *the following properties of system (10) in* $L_{p(\cdot)}$ *are equivalent:*
1) it is complete in $L_{p(\cdot)}$*;*
2) it is minimal in $L_{p(\cdot)}$*;*
3) it is $\omega-$ *linearly independent in* $L_{p(\cdot)}$*;*
4) it forms a basis isomorphic to $\left\{e^{int}\right\}_{n\in Z}$ *for* $L_{p(\cdot)}$*;*
5) $\Delta_{n_0} \neq 0$*, where* Δ_{n_0}*is determined by (15).*

In fact, equivalence of properties 1)-4) follows directly from Lemma 2.3. As for equivalence of properties 4) and 5), it was proved above.

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Received: June, 2012