Mathematica Aeterna, Vol. 2, 2012, no. 6, 561 - 572

On a different kind of d-orthogonal polynomials that generalize the Laguerre polynomials

Serhan Varma
Ankara University, Faculty of Science, Department of Mathematics
TR-06100 Tandoğan Ankara, Turkey
svarma@science.ankara.edu.tr

Fatma Taşdelen
Ankara University, Faculty of Science, Department of Mathematics
TR-06100 Tandoğan Ankara, Turkey
tasdelen@science.ankara.edu.tr

Abstract

The purpose of this work is to give an another generalization of the Laguerre polynomials in the context of d-orthogonality by a generating function of a certain form. We derive the d-dimensional functional vector for which the d-orthogonality holds. Some properties of the obtained polynomials are determined: expilicit representation, relation with a known polynomial, a recurrence relation of order-$(d+1)$ and a differential equation of order-$(d+1)$.

Mathematics Subject Classification: 33C45, 42C05.
Keywords: d-Orthogonal polynomials, Generating functions, d-Dimensional functionals, Recurrence relations, Laguerre polynomials.

1 Introduction

Lately, there has been an accelerating interest in extensions of the notion of orthogonal polynomials. One of them is the multiple orthogonality $[1-4]$. This notion has many applications in various fields of mathematics as approximation theory, spectral theory of operators and special functions theory. A subclass of multiple orthogonal polynomials known as d-orthogonal polynomials, introduced by Van Iseghem [23] and Maroni [20].

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial set (PS for short) if and only if $\deg P_n = n$ for all non-negative integer n. We denote by $\langle u, f\rangle$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n\geq 0}$ be a PS in $\mathcal{P}$. The corresponding dual sequence $(u_n)_{n\geq 0}$ is defined by

$$\langle u_n, P_m\rangle = \delta_{n,m}, \qquad n, m = 0, 1, ...$$

$\delta_{n,m}$ being the Kronecker symbol.

Definition 1.1 *(Van Iseghem [23] and Maroni [20]) Let d be a positive integer and let $\{P_n\}_{n\geq 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set (d-OPS, for short) with respect to the d-dimensional functional vector $\Gamma =^t (\Gamma_0, \Gamma_1, ..., \Gamma_{d-1})$ if it satisfies the following orthogonality relations:*

$$\begin{cases} \langle \Gamma_k, P_m P_n\rangle = 0, & m > nd + k, \ n \in \mathbb{N} = \{0, 1, 2, ...\}, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, & n \in \mathbb{N}, \end{cases} \tag{1}$$

for each integer k belonging to $\{0, 1, ..., d-1\}$.

For $d = 1$, we recover the well-known notion of orthogonality.

Theorem 1.2 *(Maroni [20])A PS $\{P_n\}_{n\geq 0}$ is a d-OPS if and only if it satisfies a $(d+1)$-order recurrence relation of the type*

$$xP_n(x) = \sum_{k=0}^{d+1} a_{k,d}(n) P_{n-d+k}(x), \qquad n \in \mathbb{N}, \tag{2}$$

with the regularity conditions $a_{d+1,d}(n)\, a_{0,d}(n) \neq 0$, $n \geq d$, and under the convention $P_{-n} = 0$, $n \geq 1$.

For the particular case $d = 1$, this theorem is reduced to the so-called Favard Theorem [16].

The d-orthogonality notion seems to appear in various domains of mathematics. For instance, there is a closed relationship between 2-orthogonality and the birth and the death process [26]. Furthermore, Vinet and Zhedanov [24] showed that there exists a connection with application of d-orthogonal polynomials and nonlinear automorphisms of the Weyl algebra. Most of the known examples of d-OPSs were introduced by solving characterization problems which consist to find all d-OPSs satisfying a fixed property $[6-11]$, $[13-14]$, $[18-19]$, [25].

In the literature, the pair of Konhauser polynomials $Z_n^\alpha(x; k)$ and $Y_n^\alpha(x; k)$ $(k \in \mathbb{Z}^+)$ discovered by Joseph E. D. Konhauser [17] have studied intensively. These polynomials were the first examples of biorthogonal polynomials. In the context of d-orthogonality, Ben Cheikh and Douak [7] showed that the polynomial set $\left\{Z_n^\alpha\left(dx^{1/d}; d\right)\right\}_{n\geq 0}$ is d-orthogonal. Our inspiration is the generating functions of $Y_n^{\alpha}(x;k)$ polynomials given in [12]. By taking $k=\frac{1}{d}$, we obtain a family of polynomials by means of following generating functions

$$\begin{aligned} G_d(x,t) &= (1-t)^{-(\alpha+1)d}\exp\left\{-x\left[(1-t)^{-d}-1\right]\right\} \\ &= \sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d)\frac{t^n}{n!}, \qquad |t|<1, \end{aligned} \tag{3}$$

where d is a positive integer. Note that with choosing $k=\frac{1}{d}$, PS $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$ is not related with the $Y_n^{\alpha}(x;k)$ polynomials, except the case $d=1$ and obviously $k=1$ which gives the Laguerre polynomials for sure, and the method summarizing at the beginning of following section is applicable for these polynomials.

The purpose of the paper is to investigate the d-orthogonality property of the PS $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$ generated by (3). Also, we explicitly express the corresponding d-dimensional functional vector ensuring the d-orthogonality of these polynomials. In the last section, some properties of the polynomials $P_n^{(\alpha)}(x;d)$ which generalize the Laguerre polynomials in a natural way are obtained: explicit representation, a relation with a known polynomial, recurrence relation of order-$(d+1)$ and $(d+1)$-order differential equation satisfied by each polynomials.

2 d-Orthogonality

In this section, for achieving our main theorem, we need following definitions and results.

Lemma 2.1 *(Freeman [15])Let $F(x,t)=\sum_{n=0}^{\infty} P_n(x)e_n(t)$ where $\{P_n\}_{n\geq 0}$ is a polynomial set in $\mathcal{P}$ and $\{e_n\}_{n\geq 0}$ is a sequence in $\mathbb{C}[t]$; e_n being of order n. Then for every $L:=L_x\in\wedge^{(1)}$ $\left(\text{resp. } M:=M_t\in\vee^{(1)}\right)$, there exists a unique $\hat{L}:=\hat{L}_t\in\vee^{(-1)}$ $\left(\text{resp. } \hat{M}:=\hat{M}_x\in\wedge^{(-1)}\right)$ such that*

$$L_xF(x,t)=\hat{L}_tF(x,t) \quad \left(\text{resp. } M_tF(x,t)=\hat{M}_xF(x,t)\right),$$

where $\wedge^{(1)}$ (resp. $\wedge^{(-1)}$) is the set of operators acting on formal power series that increase (resp. decrease) the degree of every polynomial by one and $\vee^{(1)}$ (resp. $\vee^{(-1)}$) is the set of operators acting on formal power series that increase (resp. decrease) the order of every formal power series by one.

Definition 2.2 *The operator $\hat{L}$ (resp. $\hat{M}$) is called the transform operator of L (resp. M) corresponding to the generating function $F(x,t)$.*

In [10], Ben Cheikh and Zaghouani investigated the following special case

$$e_n(t) = \frac{t^n}{n!}, \qquad L_x = X \qquad \text{and} \qquad M_t = T,$$

where X (resp. T) is the multiplication operator by x (resp. the multiplication operator by t). For this case, the generating function $G(x,t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$ is the eigenfunction of the operator $\hat{X} := \hat{X}_t$ (resp. $\hat{T} := \hat{T}_x$) associated with the eigenvalue x (resp. t)

$$\hat{X}_t G(x,t) = xG(x,t) \quad , \quad \hat{T}_x G(x,t) = tG(x,t). \tag{4}$$

Definition 2.3 *The operator $\hat{T}_x$ is called the lowering operator of $\{P_n\}_{n\geq 0}$.*

Also, in view of these results, they gave the following useful theorem.

Theorem 2.4 *(Ben Cheikh and Zaghouani [10]) Let $\{P_n\}_{n\geq 0}$ be a PS generated by*

$$G(x,t) = A(t)\, G_0(x,t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}, \tag{5}$$

where $G_0(0,t) = 1$. Let $\hat{X}$ and $\sigma := \hat{T}$, respectively, be the transform operator of X and T, the multiplication operator by x and the multiplication operator by t according to the generating function $G(x,t)$. Then
(i) The following assertions are equivalent:
(a) $\{P_n\}_{n\geq 0}$ is a d-OPS.
(b) $\hat{X} \in \vee_{d+2}^{(-1)}$.
(ii) If $\{P_n\}_{n\geq 0}$ is a d-OPS, the d-dimensional functional vector

$$\mathcal{U} =^t (u_0, u_1, ..., u_{d-1})$$

for which the d-orthogonality holds is given by

$$\langle u_i, f\rangle = \frac{1}{i!}\left[\frac{\sigma^i}{A(\sigma)}(f)(x)\right]_{x=0} = \frac{\sigma^i}{i!A(\sigma)} f(0), \qquad i = 0,1,...,d-1, \qquad f \in \mathcal{P}, \tag{6}$$

where $A(t) = \sum_{k=0}^{\infty} a_k t^k$ is a power series and $\vee_r^{(-1)}$ $(r \geq 2)$ is the set of operators $\tau \in \vee^{(-1)}$ such that there exist r complex sequences $\left(b_n^{(m)}\right)_{n\geq 0}$, $m = 0,1,...,r-1$, satisfying the condition $b_n^{(0)} b_n^{(r-1)} \neq 0$ for all non-negative integer n,

$$\tau(t^n) = \sum_{m=0}^{r-1} b_n^{(m)} t^{n+m-1}, \quad n \geq 1 \quad \text{and} \quad \tau(1) = \sum_{m=1}^{r-1} b_{m-1}^{(m)} t^{m-1}.$$

Recall the following corollary from [5],

Lemma 2.5 *Let*

$$A(t)=\sum_{k=0}^{\infty}a_k t^k \quad a_0\neq 0, \qquad C(t)=\sum_{k=0}^{\infty}c_k t^{k+1} \quad c_0\neq 0,$$

be two power series and $C^*(t)$ *is the inverse of* $C(t)$ *such that*

$$C(C^*(t))=C^*(C(t))=t \tag{7}$$

where $C^*(t)=\sum_{k=0}^{\infty}c_k^* t^{k+1}$, *then the lowering operator* $\sigma:=\hat{T}_x$ *of the polynomial set generated by*

$$A(t)\exp(xC(t))=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}$$

is given by

$$\sigma=C^*(D), \quad D=\frac{d}{dx}. \tag{8}$$

Now, we can state our main theorem.

Theorem 2.6 *The PS* $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$, *generated by* (3), *is d-orthogonal with respect to the d-dimensional functional vector* $\mathcal{U}={}^t(u_0,u_1,...,u_{d-1})$ *given by*

$$\langle u_i,f\rangle=\frac{1}{i!}\sum_{r=0}^{i}\frac{(-1)^r\binom{i}{r}}{\Gamma\left(\alpha+\frac{r}{d}+1\right)}\int_0^{\infty}f(x)\,x^{\alpha+\frac{r}{d}}e^{-x}dx, \quad i=0,1,...,d-1,$$

where $f\in\mathcal{P}$ *and* $\alpha\notin\{-1,-2,...\}$.

Proof Taking the derivative of the equality of

$$G_d(x,t)=(1-t)^{-(\alpha+1)d}\exp\left\{-x\left[(1-t)^{-d}-1\right]\right\}$$

with respect to t, we get

$$\left[(\alpha+1)(1-t)^d-d^{-1}(1-t)^{d+1}D_t\right]G_d(x,t)=xG_d(x,t).$$

From Lemma 2.1 and (4), we obtain

$$\hat{X}_t=(\alpha+1)(1-t)^d-d^{-1}(1-t)^{d+1}D_t.$$

According to Theorem 2.4 $\hat{X}_t \in \vee_{d+2}^{(-1)}$, this means that $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$ is d-orthogonal.

Taking into account that (6), (7) and (8), we have for all polynomials $f \in \mathcal{P}$

$$\langle u_i, f\rangle = \frac{1}{i!}\left[\frac{\sigma^i}{A(\sigma)} f(x)\right]_{x=0}$$

where $A(t) = (1-t)^{-(\alpha+1)d}$ and $\sigma = 1-(1-D_x)^{-1/d}$. It follows that if and only if $\alpha \notin \{-1,-2,...\}$

$$\begin{aligned}
\langle u_i, f\rangle &= \frac{1}{i!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\left[(1-D_x)^{-\left(\alpha+\frac{r}{d}+1\right)}(f)(x)\right]_{x=0} \\
&= \frac{1}{i!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\sum_{j=0}^{\infty}\frac{\left(\alpha+\frac{r}{d}+1\right)_j}{j!} f^{(j)}(0) \\
&= \frac{1}{i!}\sum_{r=0}^{i}\frac{(-1)^r\binom{i}{r}}{\Gamma\left(\alpha+\frac{r}{d}+1\right)}\sum_{j=0}^{\infty}\frac{\Gamma\left(\alpha+j+\frac{r}{d}+1\right)}{j!} f^{(j)}(0) \\
&= \frac{1}{i!}\sum_{r=0}^{i}\frac{(-1)^r\binom{i}{r}}{\Gamma\left(\alpha+\frac{r}{d}+1\right)}\int_0^\infty\left(\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j\right)x^{\alpha+\frac{r}{d}}e^{-x}dx \\
&= \frac{1}{i!}\sum_{r=0}^{i}\frac{(-1)^r\binom{i}{r}}{\Gamma\left(\alpha+\frac{r}{d}+1\right)}\int_0^\infty f(x)\, x^{\alpha+\frac{r}{d}}e^{-x}dx
\end{aligned}$$

where $(\alpha)_j$ denotes the Pochhammer's symbol defined by

$$(\alpha)_0 = 1, \qquad (\alpha)_j = (\alpha)(\alpha+1)\ldots(\alpha+j-1), \qquad j = 1,2,3,\ldots.$$

Hence, we get the desired result.

Remark 2.7 *For the cases $d=1$ and $\alpha > -1$, we note that we again meet the orthogonality of the Laguerre polynomials $\left\{n!L_n^{(\alpha)}(x)\right\}_{n\geq 0}$ with respect to the following linear functional from [21]*

$$\langle u_0, f\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)\, x^\alpha e^{-x}dx.$$

In view of Remark 2.7, we obtain an another generalization of the Laguerre polynomials in the context of d-orthogonality notion different from ones given in [7] and [10].

3 Some Properties of $P_n^{(\alpha)}(x;d)$

The PS $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$, generated by (3), has the following explicit representation

$$P_n^{(\alpha)}(x;d)=\sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}((\alpha+s+1)d)_n, \quad \alpha\notin\{-1,-2,...\}. \quad (9)$$

From (3), it is easily to verify the equality (9) with the help of usual serial expansions.

In [22], Srivastava and Singhal introduced the classes of polynomials defined by the generalized Rodrigues' formula

$$G_n^{(\alpha)}(x,r,p,k)=\frac{1}{n!}x^{-\alpha-kn}e^{px^r}\theta^n\left(x^{\alpha}e^{-px^r}\right) \quad (10)$$

and also gave explicit form for these polynomials

$$G_n^{(\alpha)}(x,r,p,k)=\frac{k^n}{n!}\sum_{m=0}^{n}\frac{(px^r)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{\alpha+rj}{k}\right)_n \quad (11)$$

where α, r, p and k are unrestricted in general, $\theta=x^{k+1}D$ and $D=\frac{d}{dx}$. Comparing (9) with (11), we obtain the below relation

$$P_n^{(\alpha)}(x;d)=d^n n!G_n^{(\alpha+1)}\left(x,1,1,d^{-1}\right), \quad \alpha\notin\{-1,-2,...\}, \quad d=1,2,.... \quad (12)$$

From the relation (12), we can deduce that the special case of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,p,k)$, given by (11), is also d-orthogonal. By using the relation (12), we derive the next theorem.

Theorem 3.1 *The PS* $\left\{P_n^{(\alpha)}(x;d)\right\}_{n\geq 0}$ *satisfies the following recurrence relations*

$$\begin{aligned} P_{n+1}^{(\alpha)}(x;d) &= dxDP_n^{(\alpha)}(x;d)+[(\alpha+1)d+n-dx]P_n^{(\alpha)}(x;d) \quad (13)\\ DP_n^{(\alpha)}(x;d) &= P_n^{(\alpha)}(x;d)-P_n^{(\alpha+1)}(x;d) \quad (14)\\ P_{n+1}^{(\alpha)}(x;d) &= [(\alpha+1)d+n]P_n^{(\alpha)}(x;d)-dxP_n^{(\alpha+1)}(x;d) \quad (15)\end{aligned}$$

where $D=\frac{d}{dx}$.

Proof In view of the relation (12), we obtain the recurrence relations (13), (14), and (15) from [22].

Remark 3.2 *The recurrence relations* (13)-(15) *return to the recurrence relations satisfied by Laguerre polynomials when* $d=1$.

Next, we give a $(d+1)$-order recurrence relation satisfied by the PS $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$.

Theorem 3.3 *The PS* $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$, *generated by* (3), *satisfies a* $(d+1)$-*order recurrence relation of the form*

$$xP_n^{(\alpha)}(x;d) = \sum_{k=0}^{d+1} a_{k,d}(n)\, P_{n-k+1}^{(\alpha)}(x;d) \tag{16}$$

where

$$a_{k,d}(n) = (-1)^{k-1}\left[(\alpha+1)\binom{d}{k-1}(n-k+2)_{k-1} + \frac{1}{d}\binom{d+1}{k}(n-k+1)_k\right].$$

Proof The application of the operator D_t to each member of (3) and then multiplication of both sides with $(1-t)^{d+1}$ gives

$$\begin{aligned} &(\alpha+1)\,d\,(1-t)^d \sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d)\frac{t^n}{n!} - xd\sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d)\frac{t^n}{n!} \\ = \;& (1-t)^{d+1}\sum_{n=0}^{\infty} P_{n+1}^{(\alpha)}(x;d)\frac{t^n}{n!}. \end{aligned} \tag{17}$$

By using binomial expansion and shifting indices, (17) leads to

$$\begin{aligned} & x\sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d)\frac{t^n}{n!} \\ = \;& \sum_{n=0}^{\infty}\sum_{k=0}^{d}(\alpha+1)\binom{d}{k}(-1)^k (n-k+1)_k\, P_{n-k}^{(\alpha)}(x;d)\frac{t^n}{n!} \\ & -\sum_{n=0}^{\infty}\sum_{k=0}^{d+1}\frac{1}{d}\binom{d+1}{k}(-1)^k(n-k+1)_k\, P_{n-k+1}^{(\alpha)}(x;d)\frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$\begin{aligned} & xP_n^{(\alpha)}(x;d) \\ = \;& \sum_{k=0}^{d}(\alpha+1)\binom{d}{k}(-1)^k(n-k+1)_k\, P_{n-k}^{(\alpha)}(x;d) \\ & -\frac{1}{d}\sum_{k=0}^{d+1}\binom{d+1}{k}(-1)^k(n-k+1)_k\, P_{n-k+1}^{(\alpha)}(x;d). \end{aligned} \tag{18}$$

After some computations, (18) becomes

$$\begin{aligned} & xP_n^{(\alpha)}(x;d) \\ = & \sum_{k=0}^{d+1}(-1)^{k-1}\left[(\alpha+1)\binom{d}{k-1}(n-k+2)_{k-1}+\frac{1}{d}\binom{d+1}{k}(n-k+1)_k\right] \\ & \times P_{n-k+1}^{(\alpha)}(x;d) \\ = & \sum_{k=0}^{d+1}a_{k,d}(n)\,P_{n-k+1}^{(\alpha)}(x;d) \end{aligned}$$

which finishes the proof.

Remark 3.4 *It is worthy to note that from* (16)*, changing* k *by* $-k+d+1$*, the PS* $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$ *satisfies a* $(d+1)$*-order recurrence relation of the type given by* (2)*. Notice that* $a_{k,d}(n)$ *satisfies the regularity conditions*

$$a_{d+1,d}(n)\,a_{0,d}(n)\neq 0,\ \ n\geq d$$

with

$$\begin{aligned} a_{0,d}(n) &= -\frac{1}{d} \\ a_{d+1,d}(n) &= (-1)^d\left[\left(\alpha+1+\frac{1}{d}(n-d)\right)(n-d+1)_d\right] \end{aligned}$$

under the restriction $\alpha\notin\{-1,-2,...\}$.

Remark 3.5 *For* $d=1$*, the recurrence relation obtained in Theorem 3.3 reduces a well-known three-term recurrence relation for the Laguerre polynomials* $\left\{L_n^{(\alpha)}(x)\right\}_{n\geq 0}$

$$L_{n+1}^{(\alpha)}(x)=\left(2+\frac{\alpha-1-x}{n+1}\right)L_n^{(\alpha)}(x)-\left(1+\frac{\alpha-1}{n+1}\right)L_{n-1}^{(\alpha)}(x).$$

Finally, we give a $(d+1)$-order differential equation satisfied by the PS $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$.

Theorem 3.6 *The PS* $\left\{P_n^{(\alpha)}(.;d)\right\}_{n\geq 0}$*, generated by* (3)*, satisfies a* $(d+1)$*-order differential equation of the type*

$$\left[(\delta - x)\prod_{j=1}^{d}\left(\delta - x + \alpha + \frac{1}{d}(j-1)\right)\right.$$

$$\left. +x\prod_{j=1}^{d}\left(\delta - x + \alpha + 1 + \frac{1}{d}(n+j-1)\right)\right] P_n^{(\alpha)}(x;d) = 0 \qquad (19)$$

where $\delta = x\frac{d}{dx}$.

Proof According to the relation (12), we can obtain the differential equation (19) from [22, Eq. (5.6)].

Remark 3.7 *Taking* $d = 1$ *in the differential equation* (19) *gives well-known differential equation satisfied by the Laguerre polynomials of the type*

$$xy''(x) + (\alpha + 1 - x)\,y'(x) + ny(x) = 0, \qquad y(x) = L_n^{(\alpha)}(x).$$

References

[1] A. I. Aptakarev, Multiple orthogonal polynomials, J. Comput. Appl. Math., 99 (1998), 423-447.

[2] A. I. Aptakarev, A. Branquinho, W. Van Assche, Multiple orthogonal polynomials for classical weights, Trans. Amer. Math. Soc., 355 (2003), 3887-3914.

[3] A. I. Aptakarev, F. Marcellán, I. A. Rocha, Semi classical multiple orthogonal polynomials and the properties of Jacobi-Bessel polynomials, J. Approx. Theory, 90 (1997), 117-146.

[4] J. Arvesú, J. Coussement, W. Van Assche, Some discrete multiple orthogonal polynomials, J. Comput. Appl. Math., 153 (2003), 19-45.

[5] Y. Ben Cheikh, Some results on quasi-monomiality, Appl. Math. Comput., 141 (2003), 63-76.

[6] Y. Ben Cheikh, K. Douak, On the classical d-orthogonal polynomials defined by certain generating functions I, Bull. Belg. Math. Soc., 7 (2000), 107-124.

[7] Y. Ben Cheikh, K. Douak, On the classical d-orthogonal polynomials defined by certain generating functions II, Bull. Belg. Math. Soc., 8 (2001), 591-605.

[8] Y. Ben Cheikh, A. Ouni, Some generalized hypergeometric d-orthogonal polynomial sets, J. Math. Anal. Appl., 343 (2008), 464-478.

[9] Y. Ben Cheikh, A. Zaghouani, Some discrete d-orthogonal polynomial sets, J. Comput. Appl. Math., 156 (2003), 253-263.

[10] Y. Ben Cheikh, A. Zaghouani, d-Orthogonality via generating functions, J. Comput. Appl. Math., 199 (2007), 2-22.

[11] N. Ben Romdhane, d-Orthogonal Faber polynomials, Integral Transforms Spec. Funct., 18 (2007), 663-677.

[12] L. Carlitz, A note on certain biorthogonal polynomials, Pacific J. Math., 24 (1968), 425-430.

[13] K. Douak, P. Maroni, On d-orthogonal Tchebyshev polynomials I, Appl. Numer. Math., 24 (1997), 23-53.

[14] K. Douak, P. Maroni, On d-orthogonal Tchebyshev polynomials II, Methods Appl. Anal., 4 (1997), 404-429.

[15] J. M. Freeman, Transforms of operators on $K[X][t]$, Congr. Numer., 48 (1985), 115-132.

[16] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer Monographs in Mathematics. Springer-Verlag, (Berlin, 2010).

[17] J. D. E. Konhauser, Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math., 21 (1967), 303-314.

[18] I. Lamiri, A. Ouni, d-Orthogonality of Hermite type polynomials, Appl. Math. Comput., 202 (2008), 24-43.

[19] I. Lamiri, A. Ouni, d-Orthogonality of Humbert and Jacobi type polynomials, J. Math. Anal. Appl., 341 (2008), 24-51.

[20] P. Maroni, L'orthogonalitè et les recurrences de polynômes d'ordre superieur à deux, Ann. Fac. Sci. Toulouse, 10 (1989), 105-139.

[21] H. M. Srivastava, Y. Ben Cheikh, Orthogonality of some polynomial sets via quasi-monomiality, Appl. Math. Comput., 141 (2003), 415-425.

[22] H. M Srivastava, J. P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, Ann. Mat. Pura Appl., 90 (1971), 75-85.

[23] J. Van Iseghem, Vector orthogonal relations. Vector QD-algorithm, J. Comput. Appl. Math., 19 (1987), 141-150.

[24] L. Vinet, A. Zhedanov, Automorphisms of the Heisenberg-Weyl algebra and d-orthogonal polynomials, J. Math. Phys., 50 (2009), 19 pp.

[25] A. Zaghouani, Some basic d-orthogonal polynomial sets, Georgian Math. J., 12 (2005), 583-593.

[26] E. Zerouki, A. Boukhemis, On the 2-orthogonal polynomials and the generalized birth and death process, Int. J. Math. Math. Sci., Article ID 28131 (2006), 1-12.

Received: July, 2012