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AN INTEGRO-DIFFERENTIAL EQUATION OF VOLTERRA TYPE WITH SUMUDU TRANSFORM

Renu Jain

School of Mathematics and Allied Sciences
Jiwaji University Gwalior-474001, India
Email - renujain3@rediffmail.com

Dinesh Singh

Department of Applied Mathematics
Institute of Technology and Manegement Gwalior-474001, India
Email - dinesh2006_tomar@yahoo.co.in

Abstract

In this paper a closed form solution of a fractional integro-differential equation of Volterra type involving Mittag-Leffler function has been obtained using straight forward technique of Sumudu transform. Some particular cases have also been considered.

Key words: Integro-differential equation of Volterra type, Sumudu transform, Riemann-Liouville fractional integral and differential.

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1. INTRODUCTION

The differential equations have played a central role in every aspect of applied mathematics for a very long time and have assumed a greater significance with the advent of the computers and advanced softwares. There are several transforms which are used to solve differential equations arising in engineering problems. These include Laplace, Mellin, Fourier, and fractional Fourier transform etc. Recently a new integral transform known as Sumudu transform has been introduced by Watugala [11]. Due to its scale and unit preserving properties the Sumudu transform has a great potential of applicability in the areas of engineering mathematics and applied sciences.

Since its introduction, several researchers endeavoured to explore its properties and its prospective uses. Considered as a theoretical dual to Laplace transform, it provides a more effective tool for problem solving without resorting to a new frequency domain. The inverse Sumudu transform was given by Weerakoon [12]. Several fundamental properties of Sumudu transform were derived by Belgacem et al [2, 3]. They also applied this transform to solve an integral production problem. Later on Loonker and Banerji [6] obtained the solution of Abel integral equation using Sumudu transform.

The object of this paper is to present some of the important features of Sumudu transform and a straightforward alternative derivation of the solution of fractional integro-differential equation of Volterra type. The solution of fractional integro-differential equation is demonstrated by many authors, including Barrett [1], Ross and Sachdeva [8], Kilbas, Saigo and Saxena [5], Gupta and Sharma [4] and Saxena [9] and others.

2. DEFINITIONS

The Sumudu transform can be defined [4, 6] for a function of exponential order as follows:

We consider functions in the set A, defined in the form:

$$A=\left\{f(t) \mid \exists M, \tau_1 \text{ and/or } \tau_2 >0, \text{such that } |f(t)|< M e^{|t|/\tau_j}, \text{if } t\in(-1)^j\times[0,\infty)\right\} \quad \dots (1)$$

For a given function *f(t)* in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist, and each may be infinite. Instead of being used as a power to the exponential as in case of the Laplace transform, the variable '*u*' in the Sumudu transform is used to factor the variable '*t*' in the argument of the function *f*. Specifically, for *f(t)* the sumudu transform is defined by

$$G(u) = S[f(t):u] = \begin{cases} \int_0^\infty e^{-t} f(ut)\,dt, & 0 \le u < \tau_2 \\ \int_0^\infty e^{-t} f(ut)\,dt, & -\tau_1 < u \le 0 \end{cases} \quad \dots (2)$$

In other words, the Sumudu transform can be written as

$$G(u)=S[f(t):u]=\frac{1}{u}\int_0^{\infty}e^{-(t/u)}f(t)\,dt\,,u\in(-\tau_1,\tau_2) \quad \text{... (3)}$$

The convolution of Sumudu transform is

$$S(f*g)(\tau)=s\,S[f(\tau)]S[g(\tau)]=s\,F(s)\,G(s),\text{ where } Re(s)>0 \quad \text{... (4)}$$

The inversion formula of Sumudu transform is given by

$$f(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{(t/u)}G(u)\,du \quad \text{... (5)}$$

The generalized Mittag-Leffler function in three parameters *α, β, δ* is defined by, Prabhakar, [7, 10], as

$$E_{\alpha,\beta}^{\rho}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\Gamma(\beta+\alpha k)}\frac{z^k}{k!} \quad \text{... (6)}$$

which in the special case of β = ρ = 1, reduces to well-known Mittag-Leffler function.

The Riemann-Liouville fractional integral of order α is defined in the following form:

$${}_0D_{\tau}^{-\alpha}h(\tau)=\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\mathrm{t})^{\alpha-1}f(t)\,dt\text{ , where } Re(\alpha)>0 \quad \text{... (7)}$$

Similarly the Riemann-Liouville fractional derivative of order η is defined in the following form:

$${}_0D_{\tau}^{\eta}[h(\tau)]=\frac{1}{\Gamma(n-\eta)}\frac{d^n}{d\tau^n}\int_0^{\tau}(\tau-\mathrm{t})^{n-\eta-1}f(t)\,dt\text{, where } \mathrm{Re}(\eta)>0$$

... (8)

The Sumudu transform of fractional derivative of the function *f(t)* is defined as [2, 4]

$$\mathrm{S}[{}_0\mathrm{D}_{\tau}^{\eta}(h(\tau);s)]=s^{-\eta}H(s)-[\sum_{r=1}^{n}s^{-r}\,{}_0\mathrm{D}_{\tau}^{\eta-r}h(\tau)]_{\tau=0} \quad \text{... (9)}$$

Further the Sumudu transform of Mittag-Leffler function is given by

$$\mathrm{S}\,[t^{\beta-1}E_{\alpha,\beta}^{\rho}(t^{\alpha})]=\int_0^{\infty}e^{-t}(st)^{\beta-1}E_{\alpha,\beta}^{\rho}(st)^{\alpha}\,dt=\mathrm{s}^{\beta-1}(1-s^{\alpha})^{-\rho} \quad \text{... (10)}$$

3. SOLUTION OF INTEGRO-DIFFERENTIAL EQUATION

Various physical phenomena like diffusion can be modeled in terms of integro-differential equation. These can be extended to fractional integral equation by replacing ordinary integral to their fractional counter parts. Consider

Volterra type integro-differential involving generalized Mittag-Leffler function and solve by employing Sumudu transform.

Theorem 1: Let $\alpha, \beta, \rho \in C$ and $t \in R$, $Re(\eta) > 0$, $Re(\beta) > 0$, $f(t)$ be assumed to be continuous on every finite interval [0,T], $0 < t < \infty$, and be of exponential order σ when $t \to \infty$ .Then the Cauchy problem for fractional integro-differential equation of Volterra type

$$ {}_0D_\tau^\alpha[h(\tau)] = \eta f(\tau) + k\int_0^\tau t^{\beta-1} E_{\alpha,\beta}^{\rho}(t^\alpha) h(\tau - t)dt \quad \text{... (11)} $$

together with the initial conditions

$$ {}_0D_\tau^{\alpha-k}[h(\tau)] = a_k \ , k = 1.......n = -[-R(\eta)], -1 < \eta \le n], n \in N \quad \text{... (12)} $$

where a_1...... a_k are prescribed constants, there exists a unique continuous solution of Cauchy problem (11) and (12) given by

$$ h(\tau) = \sum_{k=1}^{n} a_k \Omega_k(\tau) + \eta \int_0^\tau \theta(\tau - t) f(t)dt \quad \text{... (13)} $$

where

$$ \Omega_k(\tau) = \sum_{r=0}^{\infty} k^r \tau^{\beta r'-1} E_{\alpha,\beta r'}^{\rho r}(\tau^\alpha) , $$

$$ \beta r' = \beta r + \alpha(1+r) - k + 1 \quad \text{... (14)} $$

and

$$ \theta(\tau) = \sum_{r=0}^{\infty} k^r \tau^{\beta r''-1} E_{\alpha,\beta r''}^{\rho r}(\tau^\alpha) \ , $$

$$ \beta r'' = \beta r + \alpha(1+r) + 1 \quad \text{... (15)} $$

Proof: By applying the Sumudu transform to equation (11), we get

$$ s^{-\alpha}H(s) - \sum_{k=1}^{n} s^{-k} D_\tau^{\alpha-k} h(\tau)\big|_{\tau=0} = ksH(s) \frac{s^{\beta-1}}{(1-s^\alpha)^\rho} + \eta F(s) \quad \text{... (16)} $$

which gives

$$ H(s) = \sum_{k=1}^{n} s^{\alpha-k} a_k \sum_{r=0}^{\infty} k^r \frac{s^{(\alpha+\beta)r}}{(1-s^\alpha)^{\rho r}} + \eta s^\alpha F(s) \sum_{r=0}^{\infty} k^r \frac{s^{(\alpha+\beta)r}}{(1-s^\alpha)^{\rho r}} \quad \text{... (17)} $$

By taking inverse Sumudu transform, we get

$$ S^{-1}[H(s)] = S^{-1}\left[\sum_{k=1}^{n} s^{\alpha-k} a_k \sum_{r=0}^{\infty} k^r \frac{s^{(\alpha+\beta)r}}{(1-s^\alpha)^{\rho r}}\right] + S^{-1}\left[\eta s^\alpha F(s) \sum_{r=0}^{\infty} k^r \frac{s^{(\alpha+\beta)r}}{(1-s^\alpha)^{\rho r}}\right] $$

Now applying the convolution theorem of Sumudu transform,

$$ h(\tau) = \sum_{k=1}^{n} a_k \Omega_k(\tau) + \eta \int_0^\tau \theta(\tau - t) f(t)dt \quad \text{... (18)} $$

where $\Omega_r(\tau)$ and $\Theta(\tau)$ is given by equation (14) and (15) respectively.

SPECIAL CASES: (i) By setting ρ = 1 in (11), we get

$$ {}_0D_\tau^\alpha[h(\tau)] = \eta f(\tau) + k\int_0^\tau t^{\beta-1}E_{\alpha,\beta}^1(t^\alpha)h(\tau-t)dt \quad \text{... (19)}$$

Under the relevant conditions, the unique continuous solution is given by

$$ h(\tau) = \sum_{k=1}^n a_k\Omega_k(\tau) + \eta\int_0^\tau \theta(\tau-t)f(t)dt \quad \text{... (20)}$$

where

$$ \Omega_k(\tau) = \sum_{r=0}^\infty k^r\tau^{\beta r'-1}E_{\alpha,\beta r'}^r(\tau^\alpha),$$

$$ \beta r' = \beta r + \alpha(1+r) - k + 1 \quad \text{... (21)}$$

and

$$ \theta(\tau) = \sum_{r=0}^\infty k^r\tau^{\beta r''-1}E_{\alpha,\beta r''}^r(\tau^\alpha)\ ,$$

$$ \beta r'' = \beta r + \alpha(1+r) + 1 \quad \text{... (22)}$$

(ii) By setting ρ = 1 and β = 1 in Mittag-Leffler function, we have

$$ E_{\alpha,1}^1(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha k)} \quad \text{... (23)}$$

$$ {}_0D_\tau^\alpha[h(\tau)] = \eta f(\tau) + k\int_0^\tau t^{\beta-1}E_{\alpha,1}^1(t^\alpha)h(\tau-t)dt \quad \text{... (24)}$$

Then unique continuous solution under the relevant conditions, is given by

$$ h(\tau) = \sum_{k=1}^n a_k\Omega_k(\tau) + \eta\int_0^\tau \theta(\tau-t)f(t)dt \quad \text{... (25)}$$

where

$$ \Omega_k(\tau) = \sum_{r=0}^\infty k^r\tau^{\beta r'-1}E_{\alpha,\beta r'}^r(\tau^\alpha)\ ,$$

$$ \beta r' = r + \alpha(1+r) - k + 1 \quad \text{... (26)}$$

and

$$ \theta(\tau) = \sum_{r=0}^\infty k^r\tau^{\beta r''-1}E_{\alpha,\beta r''}^r(\tau^\alpha)\ ,$$

$$\beta r'' = r + \alpha(1+r) + 1 \quad \text{... (27)}$$

Theorem 2: If α, β, ρ, λ ∈ C and t ∈ R, Re(η) > 0, Re(β) > 0, and f(t) is assumed to be continuous on every finite interval [0,T], 0 < t < ∞, and is of exponential order σ when t →∞, then the Volterra type integral equation

$${}_0D_\tau^{-\lambda}[h(\tau)] = \eta f(\tau) + k\int_0^\tau t^{\beta-1}E_{\alpha,\beta}^{\rho}(t^\alpha)h(\tau-t)dt \quad \text{... (28)}$$

has its explicit solution

$$h(\tau) = \eta\int_0^\tau \theta(\tau-t)f(t)dt \quad \text{... (29)}$$

where

$$\theta(\tau) = \sum_{r=0}^{\infty} k^r \tau^{\beta r'-1} E_{\alpha,\beta r'}^{\rho r}(\tau^\alpha),$$

$$\beta r' = \beta r + \alpha(1+r) + 1 \quad \text{... (30)}$$

Proof: By applying the Sumudu transform on both sides of Volterra integral equation (28), we get

$$s^{\lambda} H(s) = ksH(s) \frac{s^{\beta-1}}{(1-s^\alpha)^\rho} + \eta F(s) \quad \text{... (31)}$$

and hence

$$H(s) = \eta s^{-\lambda} F(s) \sum_{r=0}^{\infty} k^r \frac{s^{(\beta-\lambda)r}}{(1-s^\alpha)^{\rho r}} \quad \text{... (32)}$$

By taking inverse Sumudu transform, we get

$$S^{-1}[H(s)] = S^{-1}\left[\eta s^\alpha F(s) \sum_{r=0}^{\infty} k^r \frac{s^{(\alpha+\beta)r}}{(1-s^\alpha)^{\rho r}}\right]$$

By application of the convolution theorem of Sumudu transform, we get

$$h(\tau) = \eta\int_0^\tau \theta(\tau-t)f(t)dt \quad \text{... (33)}$$

where *Θ(τ)* is given by equation (30).

It is hoped that the technique used in this chapter will be applicable to a wide range of areas including engineering, mathematical, chemical and physical sciences, in view of frequent occurrence of such integro-differential equations in these subjects.

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