Mathematica Aeterna, Vol. 2, 2012, no. 6, 533 - 540

Some Results in Asymmetric Metric Spaces

Abdolmohammad Aminpour

Faculty of Mathematical Sciences and Computer, Shahid Chamran University, Ahvaz, Iran

aminpour@scu.ac.ir

Sajad Khorshidvandpour

Faculty of Mathematical Sciences and Computer, Shahid Chamran University, Ahvaz, Iran
sajad_khorshidvand@yahoo.com

Mohsen Mousavi

Faculty of Mathematical Sciences and Computer, Shahid Chamran University, Ahvaz, Iran

Mmousavi88@yahoo.com

Abstract

In this paper, we recall some definition and theorem in asymmetric metric space and then prove some theorems and results in these spaces.

Mathematics Subject Classification: 26A15

Keywords: Asymmetric metric, Forward and backward limits

1. Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$.

In the realms of applied mathematics and materials science we find many recent applications of asymmetric metric spaces; for example, in rate-independent models for plasticity [1], shape-memory alloys [2], and models for material failure [3].

There are other applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton–Jacobi equations [4] in mind.

The study of asymmetric metrics apparently goes back to Wilson [5]. Following his terminology, asymmetric metrics are often called quasi-metrics. Author in [6] has discussed completely on asymmetric metric spaces.

In this work we prove some theorems in asymmetric metric spaces. We start with some elementary definitions from [6].

Definition 1.1. A function $d : X \times X \to$ is an *asymmetric metric* and

(X, d) is an *asymmetric metric space* if:

(1) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,

(2) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Henceforth, (X, d) shall be an asymmetric metric space.

Example1.2. Let $\alpha > 0$. Then $d : \times \to \mathbb{R}^{\geq 0}$ defined by

$$d(x, y) = \begin{cases} x - y & x \geq y \\ \alpha(y - x) & y > x \end{cases}$$

is obviously an asymmetric metric.

Definition1.3. The *forward topology* $\tau+$ induced by d is the topology generated by the *forward open balls*

$$B^+(x,\varepsilon) = \{y \in X: d(x,y)<\varepsilon\} \qquad \text{for } x \in X,\ \varepsilon > 0.$$

Likewise, the *backward topology* $\tau-$ induced by d is the topology generated by the *backward open balls*

$$B^-(x,\varepsilon) = \{y \in X: d(y,x)<\varepsilon\} \qquad \text{for } x \in X,\ \varepsilon > 0.$$

Definition 1.4. A sequence $\{x_k\}_{k\in\mathbb{N}}$ *forward converges* to $x_0 \in X$, respectively *backward converges* to $x_0 \in X$ if and only if

$$\lim_{k\to\infty} d(x_0,x_k) = 0, \quad \text{respectively} \quad \lim_{k\to\infty} d(x_k,x_0) = 0.$$

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Example1.5. Let (, d) be an asymmetric space, where d is as in example1.2. It is easy to show that the sequence $\left\{x+\frac{1}{n}\right\}_{n\in\mathbb{N}}$ ($x \in X$) is both forward and backward converges to x.

Definition 1.6 Suppose (X,d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f : X \to Y$ *be a function* . We say f is *forward continuous at* $x \in X$, respectively *backward continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x,\delta)$ implies $f(y) \in B^+(f(x),\varepsilon)$, respectively $f(y) \in B^-(f(x),\varepsilon)$.

However, note that uniform forward continuity and uniform backward continuity are the same.

Definition1.7. A set $S \subset X$ is *forward compact* if every open cover of

S in the forward topology has a finite subcover. We say that S is *forward relatively compact* if $\bar{S}$ is forward compact, where $\bar{S}$ denotes the closure in the forward topology.We say S is *forward sequentially compact* if every sequence has a forward convergent subsequence with limit in S. Finally, $S \subset X$ is *forward complete* if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing 'forward' with 'backward' in each definition.

Lemma1.8. *Let $d : X \times X \to \mathbb{R}^{\geq 0}$ be an asymmetric metric. If (X, d) is forward sequentially compact and $x_n \overset{b}{\to} x_0$ then $x_k \overset{f}{\to} x_0$.*

Notation1.9.We introduce some further notations. Y^X denotes the space of functions from X to Y. The *uniform metric* on Y^X is

$$\bar{\rho}\ (f, g) := \sup\{\bar{d}\ (f(x), g(x)) : x \in X\},$$

where $\bar{d}\,(x, y) := \min\{d(x, y), 1\}$ and d is the asymmetric metric associated with Y.

2 . Main Result

Throughout this section Let (X, d_X) and $(Y\ , d_Y)$ *be* asymmetric metric spaces.

Lemma2.1. *Let Y be forward (backward) complete, then Y^X is so*

.

Proof . Let $\{f_n\} \subseteq Y^X$ *be an arbitrary forward Cauchy sequence. By definition, given* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for* $m \geq n \geq N$, $\bar{\rho}\ (f_n, f_m)$

$<\varepsilon$ *holds. Fix* $x\in X$.*Clearly,* $\{f_n(x)\}$ *is a forward Cauchy sequence in Y.* *Since Y is forward complete so* $\{f_n(x)\}$ *is convergent., say* $f_n(x)\xrightarrow{f} f(x)$. *Thus there exists* $N\in\mathbb{N}$ *such that* $n\geq N$ implies

$$d_Y\big(f(x),f_n(x)\big)<\varepsilon \qquad (1)$$

Since $x\in X$ *was arbitrary, taking sup on* $x\in X$ *in the both side of (1),* *we deduce* $f_n\xrightarrow{f} f$ *in the uniform metric* $\bar{\rho}$. □

Theorem2.2. Let $\mathcal{G}\subseteq Y^X$ be a family of forward continuous functions. Also, suppose that Y be forward complete and forward convergence implies backward convergence in Y . Then $\mathcal{G}$ is forward complete.

Proof. Let $\{f_n\}\subseteq \mathcal{G}$ such that $f_n\xrightarrow{f} f$ .Since Y^X is forward complete [lemma2.1] and $\mathcal{G}\subseteq Y^X$, so it is sufficient to show that $f\in\mathcal{G}$. Given $\varepsilon>0$ and $x\in X$, there exists $\delta>0$ such that for each $y\in X$ which $d(x, y)<\delta$ we have

$$d_Y\big(f_n(x),f_n(y)\big)<\frac{\varepsilon}{3} \qquad (n\in\mathbb{N})$$

Also, there exists $N\in\mathbb{N}$ so that

$$d_Y(f(x),f_n(x))<\frac{\varepsilon}{3}$$

For all $n\geq N$. Since forward convergence implies backward convergence in Y, so

$$d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

Therefore

$$d\,(f(x), f(y)) \leq d_Y\big(f(x), f_n(x)\big) + d_Y\big(f_n(x), f_n(y)\big) + d_Y\big(f_n(x), f(x)\big) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, so the proof is completed.□

Theorem 2.3.Let $\{f_n\}$ be a sequence of forward continuous functions in Y^X which $f_n \xrightarrow{b} f$ uniform in the uniform metric $\bar{\rho}$ corresponding to d_Y. Also, let Y be forward sequentially compact. Then f is forward continuous.

Proof . Fixed $\varepsilon > 0$ and $x \in X$. Choose $\delta > 0$ such that for all $y \in X$ which $d(x, y) < \delta$,

$$d_Y\big(f_n(x), f_n(y)\big) < \frac{\varepsilon}{4} \quad (n \in \mathbb{N})$$

holds. Since $f_n \xrightarrow{b} f$ in the uniform metric $\bar{\rho}$, So $f_n x \xrightarrow{b} f x$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$d_Y(f_n(x), f(x)) < \frac{\varepsilon}{4}$$

For all $n \geq N_1$.

Y is forward sequentially compact. Thus lemma1.8 implies $f_n x \xrightarrow{f} f x$. so there exists $N_2 \in \mathbb{N}$ so that

$$d_Y(f(x), f_n(x)) < \frac{\varepsilon}{4}$$

For all $n \geq N_2$. Set N :={ N_1, N_2}. Then for each y$\in X$ which d(x, y) < δ we have (m ≥n$\geq N$)

d(f(x),f(y)) ≤

$d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_m(x)) + d_Y(f_m(x), f_m(y)) + d_Y(f_m(y), f(y)) <$

ε

As required. □

Corollary2.4. Let $\{f_n\}$ be a sequence of uniformly forward continuous functions in Y^X such that $f_n \xrightarrow{f} f$ in the uniform metric $\bar{\rho}$ corresponding to d_Y. If forward convergence implies backward convergence in Y, then f is uniformly forward continuous.

Proof. Fixed $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon) > 0$ such that for all x, y$\in X$ which d(x, y)< δ implies

$$d_Y(f_n(x), f_n(y)) < \frac{\varepsilon}{3} \quad (n \in \mathbb{N})$$

On the other hand, there exists N$\in \mathbb{N}$ such that

$$\bar{\rho}(f, f_n) < \frac{\varepsilon}{3}$$

For all n≥ N. It is easy to show that

$$d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$$

For all n≥ N. now by assumption we deduce

$$d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

For all n≥ N. Now , if d(x, y)< δ then

$$d(f(x),f(y)) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(x), f(x)) < \varepsilon$$

Which means f is uniformly forward continuous. □

References

[1] A. Mainik, A. Mielke, Existence results for energetic models for rate-independent systems, Calc. Var. Partial Differential Equations 22(1)(2005) 73–99.

[2] A. Mielke, T. Roubíˇcek, A rate-independent model for inelastic behavior of shape-memory alloys, Multiscale Model. Simul. 1 (4) (2003) 571–597 (electronic).

[3] M.O. Rieger, J. Zimmer, Young measure flow as a model for damage, Preprint 11/05, Bath Institute for Complex Systems, Bath, UK, 2005.

[4] A. Mennucci, On asymmetric distances, Technical report, Scuola Normale Superiore, Pisa, 2004.

[5] W.A. Wilson, On quasi-metric spaces, Amer. J. Math. 53 (3) (1931) 675–684.

[6] J. Collins, J. Zimmer, An asymmetric Arzela-Ascoli theorem, Topology and it's Applications. 154(2007)2312-2322.

Received: June, 2012