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Solutions of the Pell Equations $x^2 - (a^2b^2 + 2b)y^2 = N$ when $N \in \{\pm 1, \pm 4\}$

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Abstract

Let a and b be natural number and $d = a^2b^2 + 2b$. In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and we get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

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1 Introduction

The quadratic Diophantine equation of the form $x^2 - dy^2 = 1$ where d is a positive square-free integer is called a Pell Equation after the English mathematician John Pell. The equation $x^2 - dy^2 = 1$ has infinitely many solutions (x, y) whereas the negative Pell equation $x^2 - dy^2 = -1$ does not always have a solution. Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$. It can be seen that the equation $x^2 - 15y^2 = -1$ has no positive integer solutions. To find all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$, one first determines a fundamental solution. In this paper, after the Pell equations are described briefly, the fundamental solution to the Pell equations $x^2 - (a^2b^2 + 2b)y^2 = \pm 1$ are calculated by means of the convergent of continued fraction of $\sqrt{a^2b^2+2b}$. Moreover, all positive integer solutions of $x^2-(a^2b^2+2b)y^2=\pm 4$ and $x^2-(a^2b^2+2b)y^2=\pm 1$ are given in terms of the generalized Fibonacci and Lucas sequences. Especially, all positive integer solutions of the equations $x^2-(k^2+2)y^2=\pm 4$ and $x^2-(k^2+2)y^2=\pm 1$ are discovered.

Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let k and s be two nonzero integers with $k^2+4s>0$. Generalized Fibonacci sequence is defined by

$$U_0(k,s)=0, U_1(k,s)=1$$

and

$$U_{n+1}(k,s)=kU_n(k,s)+sU_{n-1}(k,s)$$

for $n\geq 1$ and generalized Lucas sequence is defined by

$$V_0(k,s)=2, V_1(k,s)=k$$

and

$$V_{n+1}(k,s)=kV_n(k,s)+sV_{n-1}(k,s)$$

for $n\geq 1$, respectively. It is well known that

$$U_n(k,s)=\frac{\alpha^n-\beta^n}{\alpha-\beta} \tag{1}$$

and

$$V_n(k,s)=\alpha^n+\beta^n \tag{2}$$

where $\alpha=(k+\sqrt{k^2+4s})/2$ and $=(k-\sqrt{k^2+4s})/2$. The above identities are known as Binet's formula. Clearly, $\alpha+\beta=k$, $\alpha-\beta=\sqrt{k^2+4s}$, and $\alpha\beta=-s$.

For more information about generalized Fibonacci and Lucas sequences, one can consult [14],[7],[13],[9] and [10].

2 Preliminary Notes

Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2-dy^2=N$ is known as Pell equation. For $N=\pm 1$, the equations $x^2-dy^2=1$ and $x^2-dy^2=-1$ are known as classical Pell equation. If $a^2-db^2=N$, we say that (a,b) is a solution to the Pell equation $x^2-dy^2=N$. We use the notations (a,b) and $a+b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2-dy^2=N$. Also, if a and b are both positive, we say that $a+b\sqrt{d}$ is a positive solution to the equation $x^2-dy^2=N$. Among these there is a least solution $a_1+b_1\sqrt{d}$, in which a_1 and b_1 have their least positive values. Then the number $a_1 + b_1\sqrt{d}$ is called the fundamental solution of the equation $x^2 - dy^2 = N$. Recall that if $a + b\sqrt{d}$ and $r + s\sqrt{d}$ are two solutions to the equation $x^2 - dy^2 = N$, then $a = r$ if and only if $b = s$, and $a + b\sqrt{d} < r + s\sqrt{d}$ if and only if $a < r$ and $b < s$.

Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Let d be a positive integer that is not a perfect square. Then there is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = [a_0, \overline{a_1, a_2, ..., a_{l-1}, 2a_0}]$ where l is the period length and the a_j's are given by the recursion formulas;

$$\alpha_0 = \sqrt{d}, a_k = \lfloor \alpha_k \rfloor$$

and

$$\alpha_{k+1} = \frac{1}{\alpha_k - a_k}, k = 0, 1, 2, 3, ...$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The n^{th} convergent of $\sqrt{d}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}}.$$

By means of the k^{th} convergent of $\sqrt{d}$, we can give the fundamental solution of the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$.

If we know fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$, then we can give all positive integer solutions to these equations. For more information about Pell equation, one can consult [12] and [15].

Now we give the fundamental solution of the equations $x^2 - dy^2 = \pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d}$.

Lemma 2.1 *Let l be the period length of continued fraction expansion of $\sqrt{d}$. If l is even, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by*

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no integer solutions. If l is odd, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}$$

and the fundamental solution to the equation $x^2 - dy^2 = -1$ is given by

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}.$$

Theorem 2.2 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 1$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$.

Theorem 2.3 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = -1$. Then all positive integer solutions of the equation $x^2 - dy^2 = -1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$$

with $n \geq 1$.

Now we give the following two theorems from [15]. See also [4].

Theorem 2.4 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 4$. Then all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}$$

with $n \geq 1$.

Theorem 2.5 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = -4$. Then all positive integer solutions of the equation $x^2 - dy^2 = -4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}$$

with $n \geq 1$.

From now on, we will assume that k, a, b are positive integers. We give continued fraction expansion of $\sqrt{d}$ for $d = a^2b^2 + 2b$ and $d = a^2b^2 + b$. The proofs of the following two theorems are easy and they can be found many text books on number theory as an exercise.

Theorem 2.6 *Let $d = a^2b^2 + 2b$. Then*

$$\sqrt{d} = [ab, \overline{a, 2ab}].$$

Theorem 2.7 *Let $d = a^2b^2 + b$. If $b \neq 1$ then*

$$\sqrt{d} = [ab, \overline{2a, 2ab}]$$

and if $b = 1$ then

$$\sqrt{d} = [a, \overline{2a}].$$

Corollary 2.8 *Let $d = a^2b^2 + 2b$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = a^2b + 1 + a\sqrt{d},$$

and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof The continued fraction expansion of $\sqrt{d} = a^2b^2 + 2b$ is 2 by Theorem 2.6. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1\sqrt{d}$ by Lemma 2.1. Since

$$\frac{p_1}{q_1} = ab + \frac{1}{a} = \frac{a^2b + 1}{a},$$

the proof follows. Moreover, the period length of continued fraction expansion of $\sqrt{a^2b^2 + 2b}$ is always even by Theorem 2.6. Thus by Lemma 2.1, it follows that the equation $x^2 - dy^2 = -1$ has no positive integer solutions

Corollary 2.9 *Let $d = a^2b^2 + b$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = 2a^2b + 1 + 2a\sqrt{d}.$$

Moreover, when $b \neq 1$, the equation $x^2 - dy^2 = -1$ has no positive integer solutions and when $b = 1$, the fundamental solution to the equation $x^2 - dy^2 = -1$ is $x_1 + y_1\sqrt{d} = a + \sqrt{d}$.

Proof When $b \neq 1$, the period length of the continued fraction expansion of $\sqrt{a^2b^2 + b}$ is 2 by Theorem 2.7. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1\sqrt{d}$ by Lemma 2.1. Since

$$\frac{p_1}{q_1} = ab + \frac{1}{2a} = \frac{2a^2b + 1}{2a},$$

the proof follows. When $b = 1$, the period length of the continued fraction expansion of $\sqrt{a^2 + 1}$ is 1 by Theorem 2.7. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1\sqrt{d}$ by Lemma 2.1. Since

$$\frac{p_1}{q_1} = a + \frac{1}{2a} = \frac{2a^2 + 1}{2a},$$

the proof follows. Moreover, when $b \neq 1$, the period length of continued fraction expansion of $\sqrt{a^2b^2 + b}$ is always even by Theorem 2.7. Thus, by Lemma 2.1, it follows that the equation $x^2 - dy^2 = -1$ has no positive integer solutions. When $b = 1$, it can be seen that the fundamental solution to the equation $x^2 - dy^2 = -1$ is $a + \sqrt{d}$ by Lemma 2.1 and Theorem 2.7.

3 Main Results

Theorem 3.1 *Let $d = a^2b^2 + 2b$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = (V_n(2a^2b + 2, -1)/2, aU_n(2a^2b + 2, -1))$$

with $n \geq 1$.

Proof By Corollary 2.8 and Theorem 2.2, all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = (a^2b + 1 + a\sqrt{d})^n$$

with $n \geq 1$. Let $\alpha = a^2b+1+a\sqrt{d}$ and $\beta = a^2b+1-a\sqrt{d}$. Then $\alpha+\beta = 2a^2b+2$, $\alpha - \beta = 2a\sqrt{d}$ and $\alpha\beta = 1$. Therefore

$$x_n + y_n\sqrt{d} = \alpha^n$$

and

$$x_n - y_n\sqrt{d} = \beta^n.$$

Thus it follows that

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(2a^2b + 2, -1)}{2}$$

and

$$y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}} = a\frac{\alpha^n - \beta^n}{2a\sqrt{d}} = a\frac{\alpha^n - \beta^n}{\alpha - \beta} = aU_n(2a^2b + 2, -1)$$

by (1) and (2). Then the proof follows. Now we give all positive integer solutions of the equations $x^2 - (a^2b^2 + 2b)y^2 = \pm 4$. Before giving all solutions of the equations $x^2 - dy^2 = \pm 4$, we give the following theorems from [5].

Theorem 3.2 *Let $d \equiv 2(mod4)$ or $d \equiv 3(mod4)$. Then the equation $x^2 - dy^2 = -4$ has positive integer solution if and only if the equation $x^2 - dy^2 = -1$ has positive integer solutions.*

Theorem 3.3 *Let $d \equiv 0(mod4)$. If fundamental solution to the equation $x^2 - (d/4)y^2 = 1$ is $x_1 + y_1\sqrt{d/4}$, then fundamental solution to the equation $x^2 - dy^2 = 4$ is $(2x_1, y_1)$.*

Theorem 3.4 *Let $d \equiv 1(mod4)$ or $d \equiv 2(mod4)$ or $d \equiv 3(mod4)4$. If fundamental solution to the equation $x^2 - dy^2 = 1$ is $x_1 + y_1\sqrt{d}$, then fundamental solution to the equation $x^2 - dy^2 = 4$ is $(2x_1, 2y_1)$.*

Theorem 3.5 *Let* $d = a^2b^2 + 2b$. *Then the fundamental solution to the equation* $x^2 - dy^2 = 4$ *is*

$$x_1 + y_1\sqrt{d} = 2(a^2b + 1) + 2a\sqrt{d}.$$

Proof Assume that b is even. Then $d \equiv 0(mod 4)$. Let $b = 2k$ for some $k \in Z$. Then $d/4 = a^2k^2 + k$. Thus, by Corollary 2.9, it follows that the fundamental solution to the fundamental solution to the equation $x^2 - (a^2k^2 + k)y^2 = 1$ is $(2a^2k + 1, 2a)$. Then, by Theorem 3.3, the fundamental solution to the equation $x^2 - dy^2 = 4$ is $(4a^2k + 2, 2a)$. Since $b = 2k$, the fundamental solution to the equation $x^2 - dy^2 = 4$ is $2a^2b + 2 + 2a\sqrt{d}$. Assume that b is odd. If a is odd, then $d \equiv 3(mod 4)$ and if a is even, then $d \equiv 2(mod 4)$. Thus, by Theorem 3.4 and Corollary 2.8, it follows that the fundamental solution to the equation $x^2 - dy^2 = -4$ is $(2(a^2b + 1), 2a)$. Then the proof follows.

Theorem 3.6 *Let* $d = a^2b^2 + 2b$. *Then the equation* $x^2 - dy^2 = -4$ *has no positive integer solutions.*

Proof Assume that b is odd. If a is odd, then $d \equiv 3(mod 4)$ and if a is even, then $d \equiv 2(mod 4)$. Thus, by Theorem 3.2 and Corollary 2.8, it follows that the equation $x^2 - dy^2 = -4$ has no positive integer solutions. Assume that b is even and $m^2 - dn^2 = -4$ for some positive integer m and n. Then d is even and therefore m is even. Let $b = 2k$. Then

$$m^2 - (4a^2k^2 + 4k)n^2 = -4$$

and this implies that

$$(m/2)^2 - (a^2k^2 + k)n^2 = -1.$$

This is impossible by Corollary 2.9. Then the proof follows.

Theorem 3.7 *All positive integer solutions of the equation* $x^2 - (a^2b^2 + 2b)y^2 = 4$ *are given by*

$$(x, y) = (V_n(2a^2b + 2, -1), 2abU_n(2a^2b + 2, -1))$$

with $n \geq 1$.

Proof By Theorem 3.5, the fundamental solution to the equation $x^2 - (a^2b^2 + 2b)y^2 = 4$ is $2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b}$. Therefore, by Theorem 2.4, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b})^n}{2^{n-1}} = 2((2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b})/2)^n.$$

Let $\alpha = (2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b})/2$ and $\beta = (2a^2b + 2 - 2a\sqrt{a^2b^2 + 2b})/2$. Then $\alpha + \beta = 2a^2b + 2$, $\alpha - \beta = 2a\sqrt{d}$ and $\alpha\beta = 1$. Thus it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha^n$$

and

$$x_n - y_n\sqrt{d} = 2\beta^n.$$

Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(2a^2b + 2, -1)$$

and

$$y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 2a\frac{\alpha^n - \beta^n}{2a\sqrt{d}} = 2a\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2aU_n(2a^2b + 2, -1)$$

by (1) and (2). Then the proof follows.

Let $a = k$ and $b = 1$. Then $d = a^2b^2 + 2b = k^2 + 2$. Thus we can give the following corollaries.

Corollary 3.8 *Let $d = k^2 + 2$. Then*

$$\sqrt{k^2 + 2} = [k, \overline{k, 2k}].$$

Corollary 3.9 *Let $d = k^2 + 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = k^2 + 1 + k\sqrt{d}.$$

and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Corollary 3.10 *Let $d = k^2 + 2$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = (V_n(2k^2 + 2, -1)/2, kU_n(2k^2 + 2, -1))$$

with $n \geq 1$.

Corollary 3.11 *All positive integer solutions of the equation $x^2 - (k^2 + 2)y^2 = 4$ are given by*

$$(x, y) = (V_n(2k^2 + 2, -1), 2kU_n(2k^2 + 2, -1))$$

with $n \geq 1$ and the equation $x^2 - (k^2 + 2)y^2 = -4$ has no positive integer solution.

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