Mathematica Aeterna, Vol. 2, 2012, no. 7, 639 - 649

A Special Partial order on Interval Normed Spaces

Zargham Bahmani

Departmant of mathematics, Payam Noor University of Genaveh, Iran

Bahmani.math@gmail.com

Abstract

In this paper we present some facts on interval normed spaces. Then we define a partial order on interval normed space X. Finally, we prove main theorem by defined partial order.

Mathematics Subject Classification: 46B40

Keywords: Partial Ordered Interval Normed spaces, equivalent subspaces, linear functional.

1 Introduction

The topic of interval analysis has been studied for a long time. For a detailed discussion, one can refer to the books [1_5]. The main issue is to regard the closed intervals as a kind of "points".

Sometimes they will be called "interval numbers". In this case, it is possible to consider the arithmetic of closed intervals (interval numbers). The interval-valued function is defined as the function whose function values are taken from the set of closed intervals, where the closed intervals are regarded as "points" or "number". However, we have to mention that the concept of interval-valued function is completely different from the concept of set-valued function. In set-valued analysis, the function value is still regarded as a set, not "point" or "number". It also says that the interval analysis cannot be regarded as the special case of set- valued analysis. In the paper, a different viewpoint on the interval analysis will be studied based on the viewpoint of function analysis.

The set of all closed intervals in R is not a real vector space. The main reason is that there will be no additive invers element for each interval, which will be presented more clearly in the context of this paper. It is well known that the topic of functional analysis is based on the vector space. There for, we may try to develop the theory functional analysis based on the interval space. The first pioneering work is to create the Hahn-Banach extension theorem based on the interval space, since the conventional Hahn-Banach extension theorem in functional analysis is very useful in nonlinear analysis, vector optimization and mathematical economics.

Therefore, the interval analysis may provide a useful tool to tackle this kind of situation when the problems in nonlinear analysis, vector optimization or mathematical economics are formulated as interval-valued problems. At first, the concepts of interval spaces and nonstandard normed interval spaces are introducted. In Interval normed spaces are spaces that have less known in comparison with other spaces. Recently, Many authors have been interested in such spaces; For example, in [1] have defined interval normed spaces completely and another concepts in these spaces. For details see [1]. In this work we wish to present a special partial order on an interval normed space. By this partial order, we prove a result of Hahn-banach theorem. We begin with following definitions:

Definition1.1 interval spaces and nonstandard normed interval spaces

Let ℓ be the set of closed and bounded intervals in . The addition in ℓ is given by [a ,b] [c ,d] = [a +c, b +d]

and the scalar multiplication in ℓ is given by

$$k\,[a,b] = \begin{cases} [ka, kb] & \text{if } k \geq 0 \\ [kb, ka] & \text{if } k < 0 \end{cases}$$

Then the following properties are not hard to prove.

Proposition 1.2. The following equalities hold true:

(i) $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ *for any* $x, y \in \ell$ *and* $\alpha \in \mathbb{R}$;

(ii) $\alpha(\beta x) = (\alpha\beta)x$ *for any* $x \in \ell$ *and* $\alpha, \beta \in \mathbb{R}$;

(iii) $(\alpha + \beta)x = \alpha x \oplus \beta x$ *for any* $x \in \ell$ $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta \geq 0$.

Let $\theta = [0,0]$. Then we see that $x \oplus \theta = \theta \oplus x$ *for any* $x \in \ell$; that is, θ is the zero element of ℓ. However, for any $[a,b] \in \ell$, we have

$$[a,b] \ominus [a,b] = [a,b] \oplus (-[a,b]) = [a,b] \oplus [-b,-a] = [a-b, b-a]$$
$$= [-(b-a), b-a]$$

This says that ℓ is not a real vector space in the conventional sense, since $[a,b] \ominus [a,b] \neq [0,0]$. However, if we define $\Theta = \{[-k,k]: k \geq 0\}$, then we see that $[a,b] \ominus [a,b] \in \Theta$. Therefore, we call Θ as the null set of ℓ. Let $\hat{\theta} = [-1,1]$. The null set can be rewritten as $\Theta = \{k\hat{\theta} : k \geq 0\} = \{k\hat{\theta} : k \in \mathbb{R}\}$. In this case, the element $\hat{\theta}$ is called the generator of Θ.

Let ℓ be an interval space and be a nonempty subset of ℓ. We say that

is a subspace of ℓ if, for any $y_1, y_2 \in g$ and $\alpha \in \mathbb{R}$, we have

$y_1 \oplus y_2 \in g$ *and* $\alpha y_1 \in g$. Then, we have the following observations.

- By considering the generator $\hat{\theta}$ *of* Θ, it is easy that the null set Θ is

  subspace, for any subspace

- Let Y be a subspace of a vector space X, if $\in Y$. However, in the

  case of interval space, for any subspace of ℓ and for any $y \in g$,

  we just have $g \oplus y \subseteq g$ *and* $g \oplus \zeta \subseteq g \oplus y$, where $\zeta = y \ominus y \in \Theta$. Indeed, the inclusion $g \oplus y \subseteq g$ is obvious by the definition of subspace. On the other hand, given any $\bar{y} \in \mathcal{g}$, we have $\bar{y} \oplus \zeta = (\bar{y} \ominus y) \oplus y \in \mathcal{g} \oplus y$, which implies the

inclusion $\mathcal{g} \oplus \zeta$ y.

Definition1.3. Let be a subspace of ℓ. We say that is a proper

subspace if at least one element in is not degenerated as real number.

In other words, there is $[a, b] \in \mathcal{g}$ *with* $a \neq b$.

Let be a subset of ℓ. We write

$\Theta_{\mathcal{g}} = \{x \ominus x : x \in \mathcal{g}\}$.

Then, we have the following interesting and useful results.

Proposition 1.4. Let ℓ be the interval space. The following statements hold true.

(i) If is a subspace of ℓ, then $\Theta_{g} = \cap\Theta$ and $\theta = [0,0] \epsilon\, \Theta_{g}$

(ii) if is a proper subspace, of ℓ, then Θ , I . e, $\Theta = \Theta_{g}$

Definition 1.5 .let be a subspace of ℓ. We say that $(g, \|.\|)$ is a

nonstandard normed interval space if the nonnegative real-valued function $\|.\|: g \to \mathbb{R}$ defined on satisfies the following conditions:

(i) $\|x\| = 0$ *implies* $x \in \Theta_{g}$;

(ii) $\|\alpha x\| = |\alpha| \|x\|$ *for any* $\alpha \in \mathbb{R}$ *and* $x \in g$;

(iii) $\|x \oplus y\| \leq \|x\| + \|y\|$ *for any* $x, y \in g$;

(iv) $\|x \oplus \zeta\| = \|x\|$ *for any* $x, y \in g$ *and* $\zeta \in \Theta_{g}$

Proposition 1.6. We have $\|x\| = 0 \; if \; and \; only \; if \; x \in \Theta_{g}$

Proposition 1.7. let f be a linear functional on a subspace of ℓ. Then

$f(\zeta) = 0 \; for \; any \; \zeta \in \Theta_{g}$.

Proposition 1.8. let f be a bounded linear functional on a subspace $(g, \|.\|)$. The norm of f can be rewritten as

$$\|f\|_{g} = \sup_{\substack{x \in g \backslash \Theta_{g} \\ \|x\| = 1}} |f(x)| = \sup_{\substack{x \in g \backslash \Theta_{g} \\ \|x\| = 1}} |f(x)|$$

2 Main Results

Before main discuss we mention some useful facts:

(1)Any normed interval space is finite dimension. Indeed, if X is a normed interval space then dim X=2. If we set

(2) $\beta_X = \{[1,0],[0,1]\}$ then it is easy to show that β_X is a base for X.

(3) Hereafter, we set $Y_\alpha = \{\alpha[1,0]: \alpha \in \mathbb{R}\}$ and $Y_\beta = \{\beta[1,0]: \beta \in \mathbb{R}\}$. We accept these spaces as canonical closed subspace of X.

(4) Clearly, any linear operator on Y_α or Y_β has an extension. But this extension isn't unique; For this, let T: $Y_\alpha \to X$ be a linear operator. Then $T_e: X \to X$defined by $T_e = T$ on, $T_e(x) = span\beta_x$ otherwise, is linear (we know that a base for X isn't unique)

(5) Obviously, $X = Y_\alpha \oplus Y_\beta$ where denotes the direct sum

of Y_α and Y_β .

Now we present main ideas:

Let X be an normed interval space. Define the following relation on X:

$$[a,b]\sim[c,d] \leftrightarrow a|c\&b|d$$

Here $a|c$ means there exists $k \in \mathbb{N}$ such that c= ka..

In the following proposition we show that ~ is a partial order on X.

Proposition 2.1. The relation ~ is a partial order on X.

Proof. Clearly ~ is reflexive. Suppose $[a,b] \sim [c,d]$ and $[c,f] \sim [a,b]$.

Then by definition there exist $k_1, k_2 \in \mathbb{N}$ such that $c = k_1 a \;\&\; d = k_2 b$. Also there exist $k'_1, k'_2 \in \mathbb{N}$ such that $a = k'_1 c \;\&\; b = k'_2 d$. Therefore $k_1 k'_1 = l$ and $k_2 k'_2 = 1$. So $k_1 = k'_1 = k_2 = k'_2 = l$. Thus ~ is antisymmetric. We can show the transitive property of ~ is in similar manner.□

Definition 2.2. Let be an normed interval space and A, B be subspace of X. A and B is called to be equivalent if $[a,b] \sim [c,d]$ for all $[a,b] \in A$ and $[c,d] \in B$. In this case we write A~B.

Lemma2.3. Let X be an interval linear space and A,B be nontrivial equivalent subspaces of X . Then $A \cap B \neq \emptyset$.

Proof. Since $= Y_\alpha \oplus Y_\beta$ where denotes the direct sum ofY_α $and,$

thus if we suppose $A \cap B = \emptyset$, $then$ $A \subseteq Y_\alpha, B \subseteq Y_\beta$. Let

$[a,b] \in A \text{ and } [c,d] \in B.$ Then we have

$$[a,b] = \alpha_1[1,0]$$

And

$$[c,d] = \beta_1[0,1]$$

On the other hand we know

$$[a,b] \sim [c,d]$$

Hence

$$[\alpha_1, 0] \sim [0, \beta_1]$$

So $\beta_1 = 0$. this means B=0 which is contradiction.□

Now we want to prove main theorem of this article.

Theorem 2.4. Let X be a partial ordered normed interval space and A, B be subspace of X such that $A \cap B \neq \emptyset$. Then there exists a linear functional on X such that f=0 on A.

proof. A is a closed, since by fact (l) X is finite dimension. Also by previous lemma we conclude that A B. Therefore there exists

$[c,d] \in B$ so that $[c,d] \nsucc [a,b]$ for each $[a,b]$ in A. So A is proper

subspace of X; namly $[c,d] \notin A.$ Hence by a corollary of Hahn-Banach theorem, There exists a linear functional on X such that f=0 on A. furthermore, $f([c,d]) = \delta > 0.$□

References

[1] H. Wu, Hahn-Banach theorems is nonstandard normed interval spaces, Nonlinear Analysis 72 (2010) 469-477.

[2] G. alefeld, J. Herzberger, Introduction to interval Computation, academic press, NY, Mathematisch Centrum, Amsterdam , 1981

[3] R.E Moore, interval Analyses, Prentice-Hall, Englewood Cliffs, NJ, 1966.

[4] R.E Moore, Method and Applications of interval Analysis, SIAM, Philadelphia, 1979.

[5] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, New York, 1990.

Received: August, 2012