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Comparison Theorem for Reflected ABSDEs

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Abstract

In this paper, we give a comparison theorem for reflected anticipated backward stochastic differential equations (ABSDEs).

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1 Introduction

It is well known that one of the achievements of BSDE theory is the comparison theorem. In this paper we are concerned with the comparison theorem for the following reflected ABSDE:

$$\begin{cases} -dY_t^{(j)} &= f_j(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(t)}^{(j)}, Z_{t+\zeta(t)}^{(j)})dt + dK_t^{(j)} - Z_t^{(j)}dB_t, \quad t \in [0,T]; \\ Y_t^{(j)} &= \xi_t^{(j)}, \quad t \in [T, T+C]; \\ Z_t^{(j)} &= \eta_t^{(j)}, \quad t \in [T, T+C]; \\ Y_t^{(j)} &\geq S_t^{(j)}, \quad t \in [0,T], \quad \int_0^T (Y_t^{(j)} - S_t^{(j)})dK_t^{(j)} = 0, \end{cases} \tag{1}$$

where $\delta(\cdot) : [0,T] \to \mathbb{R}^+ \setminus \{0\}$ and $\zeta(\cdot) : [0,T] \to \mathbb{R}^+ \setminus \{0\}$ are continuous functions satisfying

(**a1**) there exists a constant $C \geq 0$ such that for each $t \in [0,T]$,

$$t + \delta(t) \leq T + C, \quad t + \zeta(t) \leq T + C;$$

(**a2**) there exists a constant $M \geq 0$ such that for each $t \in [0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s + \delta(s))ds \leq M \int_t^{T+C} g(s)ds, \quad \int_t^T g(s + \zeta(s))ds \leq M \int_t^{T+C} g(s)ds.$$

For this, we first introduce some notations and preliminaries.

Let $\{B_t; t \geq 0\}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. Denote by $|\cdot|$ the norm in $\mathbb{R}^m$. Given $T > 0$, we will use the following notations:

- $L^2(\mathcal{F}_T; \mathbb{R}^m) := \{\xi \in \mathbb{R}^m \mid \xi$ is an $\mathcal{F}_T$-measurable random variable such that $E|\xi|^2 < +\infty\}$;
- $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) := \{\varphi : \Omega \times [0, T] \to \mathbb{R}^m \mid \varphi$ is an $\mathcal{F}_t$-progressively measurable process such that $E\int_0^T |\varphi_t|^2 dt < +\infty\}$;
- $S^2_{\mathcal{F}}(0, T; \mathbb{R}^m) := \{\varphi : \Omega \times [0, T] \to \mathbb{R}^m \mid \varphi$ is a continuous and $\mathcal{F}_t$-progressively measurable process such that $E[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty\}$;
- $A^2(0, T; \mathbb{R}) = \{K : \Omega \times [0, T] \to \mathbb{R} \mid K$ is an $\mathcal{F}$-adapted continuous increasing process such that $K_0 = 0$ and $K_T \in L^2(\mathcal{F}_T)\}$.

For the terminal condition $(\xi^{(j)}_t, \eta^{(j)}_t)_{t\in[T,T+C]}$, the generator $f_j(\omega, s, y, z, \theta, \phi) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times S^2_{\mathcal{F}}(s, T + C; \mathbb{R}) \times L^2_{\mathcal{F}}(s, T + C; \mathbb{R}^d) \to L^2(\mathcal{F}_s)$, and the "obstacle" $(S^{(j)}_t)_{t\in[0,T]}$, we suppose that

(**H1**) $(\xi^{(j)}, \eta^{(j)}) \in S^2_{\mathcal{F}}(T, T + C; \mathbb{R}) \times L^2_{\mathcal{F}}(T, T + C; \mathbb{R}^d)$.

(**H2**) There exists a constant $L_{f_j} > 0$ such that for each $s \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $\theta, \theta' \in L^2_{\mathcal{F}}(s, T + C; \mathbb{R})$, $\phi, \phi' \in L^2_{\mathcal{F}}(s, T + C; \mathbb{R}^d)$, $r, \bar{r} \in [s, T + C]$, the following holds:

$$|f_j(s, y, z, \theta_r, \phi_{\bar{r}}) - f_j(s, y', z', \theta'_r, \phi'_{\bar{r}})| \leq L_{f_j}(|y-y'| + |z-z'| + E^{\mathcal{F}_s}[|\theta_r - \theta'_r| + |\theta_{\bar{r}} - \phi'_{\bar{r}}|]);$$

(**H3**) $E[\int_0^T |f_j(s, 0, 0, 0, 0)|^2 ds] < +\infty$.

(**H4**) $S^{(j)}_{\cdot}$ is a continuous progressively measurable real-valued process satisfying $E \sup_{0\leq t\leq T}[(S^{(j)}_t)^+]^2 < +\infty$ and $S^{(j)}_T \leq \xi^{(j)}_T$.

Let us review the existence and uniqueness theorem from [2].

Theorem 1.1 *Assume that* (a1), (a2) *and (H1)-(H4) hold, then the reflected ABSDE (1) has a unique solution* $(Y, Z, K) \in S^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \times A^2(0, T; \mathbb{R})$.

The next result plays a key role in the next part. Readers are referred to [1] for its detailed proof.

Lemma 1.2 *Putting* $t_0 = T$*, we define by iteration*

$$t_i := \min\{t \in [0, T] : \min\{s + \delta^{(1)}(s),\ s + \delta^{(2)}(s)\} \geq t_{i-1},\ \text{for all } s \in [t, T]\},\quad i \geq 1.$$

Set $N := \max\{i : t_{i-1} > 0\}$*. Then* N *is finite,* $t_N = 0$ *and*

$$[0, T] = [0, t_{N-1}] \cup [t_{N-1}, t_{N-2}] \cup \cdots \cup [t_2, t_1] \cup [t_1, T].$$

2 Main Results

Consider two reflected ABSDEs (1) for $j = 1, 2$. Then by Theorem 1.1, either of the reflected ABSDEs has a unique adapted solution, denoted by $(Y^{(j)}, Z^{(j)}, K^{(j)})$.

Theorem 2.1 *Let $(Y^{(j)}, Z^{(j)}, K^{(j)}) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times A^2(0,T;\mathbb{R})$ $(j = 1, 2)$ be the unique solutions to reflected ABSDEs (1) respectively. Assume that*

(i) $\xi^{(1)}_s \geq \xi^{(2)}_s, s \in [T, T+C], a.s.$;

(ii) for each $s \in [0,T]$, $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $\theta^{(j)} \in S^2_{\mathcal{F}}(s, T+C;\mathbb{R})$ $(j = 1, 2)$ such that $\theta^{(1)} \geq \theta^{(2)}$, $\{\theta^{(j)}_r\}_{r\in[s,T]}$ is a continuous semimartingale and $\{\theta^{(j)}_r\}_{r\in[T,T+C]} = \{\xi^{(j)}_r\}_{r\in[T,T+C]}$,

$$f_1(s,y,z,\theta^{(1)}_{s+\delta(s)}, \eta^{(1)}_{s+\zeta(s)}) \geq f_2(s,y,z,\theta^{(2)}_{s+\delta(s)}, \eta^{(2)}_{s+\zeta(s)}), \quad a.e., a.s.,$$

$$f_1(s,y,z,\theta^{(1)}_{s+\delta(s)}, \frac{d\langle\theta^{(1)}, B\rangle_r}{dr}|_{r=s+\zeta(s)}) \geq f_2(s,y,z,\theta^{(2)}_{s+\delta(s)}, \frac{d\langle\theta^{(2)}, B\rangle_r}{dr}|_{r=s+\zeta(s)}), \quad a.e., a.s.,$$

$$f_1(s,y,z,\xi^{(1)}_{s+\delta(s)}, \frac{d\langle\theta^{(1)}, B\rangle_r}{dr}|_{r=s+\zeta(s)}) \geq f_2(s,y,z,\xi^{(2)}_{s+\delta(s)}, \frac{d\langle\theta^{(2)}, B\rangle_r}{dr}|_{r=s+\zeta(s)}), \quad a.e., a.s.;$$

(iii) $S^{(1)}_s \geq S^{(2)}_s$, $s \in [0,T]$, a.s..

Then $Y^{(1)}_t \geq Y^{(2)}_t$, $dK^{(1)}_t \leq dK^{(2)}_t$, $t \in [0,T]$, a.s..

Proof. According to Lemma 1.2, consider the problem one time interval by one time interval. For the first step, we consider the case when $t \in [t_1, T]$, then equivalently we can consider the following reflected BSDE over time interval $[t_1, T]$:

$$\begin{cases} Y^{1,(j)}_t = \xi^{(j)}_T + \int_t^T f(s, Y^{1,(j)}_s, Z^{1,(j)}_s, \xi^{(j)}_{s+\delta(s)}, \eta^{(j)}_{s+\zeta(s)})ds + K^{1,(j)}_T - K^{1,(j)}_t - \int_t^T Z^{1,(j)}_s dB_s; \\ Y^{1,(j)}_t \geq S^{(j)}_t, \quad t \in [t_1, T], \quad \int_{t_1}^T (Y^{1,(j)}_t - S^{(j)}_t) dK^{1,(j)}_t = 0. \end{cases}$$

from which we have

$$Z^{1,(j)}_t = \frac{d\langle Y^{1,(j)}, B\rangle_t}{dt}, \quad t \in [t_1, T]. \tag{2}$$

Noticing that $\xi^{(j)} \in S^2_{\mathcal{F}}(T, T+C;\mathbb{R})$ $(j = 1, 2)$ and $\xi^{(1)} \geq \xi^{(2)}$, from (ii), we can get, for $s \in [t_1, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$f_1(s,y,z,\xi^{(1)}_{s+\delta(s)}, \eta^{(1)}_{s+\zeta(s)}) \geq f_2(s,y,z,\xi^{(2)}_{s+\delta(s)}, \eta^{(2)}_{s+\zeta(s)}).$$

According to the comparison result for reflected BSDEs, we can get

$$Y_t^{1,(1)} \geq Y_t^{1,(2)}, \quad dK_t^{1,(1)} \leq dK_t^{1,(2)}, \quad t \in [t_1, T], \quad \text{a.s.},$$

i.e.,

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad dK_t^{(1)} \leq dK_t^{(2)}, \quad t \in [t_1, T+C], \quad \text{a.s..} \tag{3}$$

For the second step, we consider the case when $t \in [t_2, t_1]$. Similarly, we can consider the following reflected BSDE over $[t_2, t_1]$ equivalently:

$$\begin{cases} Y_t^{2,(j)} = Y_{t_1}^{(j)} + \int_t^{t_1} f(s, Y_s^{2,(j)}, Z_s^{2,(j)}, Y_{s+\delta(s)}^{(j)}, Z_{s+\zeta(s)}^{(j)})ds + K_{t_1}^{2,(j)} - K_t^{2,(j)} - \int_t^{t_1} Z_s^{2,(j)} dB_s; \\ Y_t^{2,(j)} \geq S_t^{(j)}, \quad t \in [t_2, t_1], \quad \int_{t_2}^{t_1} (Y_t^{2,(j)} - S_t^{(j)}) dK_t^{2,(j)} = 0. \end{cases}$$

from which we have $Z_t^{2,(j)} = \frac{d\langle Y^{2,(j)}, B\rangle_t}{dt}$ for $t \in [t_2, t_1]$. Noticing (2) and (3), according to (ii), we have, for $s \in [t_2, t_1]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$f_1(s, y, z, Y_{s+\delta(s)}^{(1)}, Z_{s+\zeta(s)}^{(1)}) \geq f_2(s, y, z, Y_{s+\delta(s)}^{(2)}, Z_{s+\zeta(s)}^{(2)}).$$

Applying the comparison result for reflected BSDEs again, we can finally get

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad dK_t^{(1)} \leq dK_t^{(2)}, \quad t \in [t_2, t_1], \quad \text{a.s..}$$

Similarly to the above steps, we can give the proofs for the other cases when $t \in [t_3, t_2]$, $[t_4, t_3]$, $\cdots$, $[t_N, t_{N-1}]$. □

Remark 2.2 *If f_1 and f_2 are independent of the anticipated term of Z, then the three inequalities in (ii) can reduce to one inequality:*

$$f_1(t, y, z, \theta_{t+\delta(t)}^{(1)}) \geq f_2(t, y, z, \theta_{t+\delta(t)}^{(2)}).$$

References

[1] X. M. Xu, Necessary and sufficient condition for the comparison theorem of multidimensional anticipated backward stochastic differential equations, Sci. China Math.,2011, 54(2): 301-310.

[2] X. M. Xu, Reflected solutions of anticipated BSDEs and related optimal stopping time problem, Submitted for publication, 2012.

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