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Two types of join preserving operators

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Abstract

In this paper, we investigate two types of join preserving maps in generalized residuated lattices. Two join preserving maps induces two types of isotone and antitone Galois connections. Moreover, we study the relations between join preserving maps and fuzzy relations.

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1 Introduction

Noncommutative structures play an important role in metric spaces, algebraic structures (groups, rings, quantales, pseudo-BL-algebras)[3-9]. Georgescu and Iorgulescu [7] introduced pseudo MV-algebras as the generalization of the MV-algebras. Georgescu and Leustean [6] introduced generalized residuated lattice as a noncommutative structure. On the other hand, Kim [11] investigated that join preserving maps induce formal, attribute oriented and object oriented concept on a complete residuated lattices.

In this paper, we investigate two types of join preserving maps in generalized residuated lattices. Two join preserving maps $\phi^{\rightarrow}$ and $\phi^{\Rightarrow}$ are investigated under the conditions $\phi^{\rightarrow}(\alpha\odot A) = \alpha\odot\phi^{\rightarrow}(A)$ and $\phi^{\Rightarrow}(A\odot\alpha) = \phi^{\Rightarrow}(A)\odot\alpha$ and the weak conditions. Two join preserving maps induces two types of isotone and antitone Galois connections. Moreover, we study the relations between join preserving maps and fuzzy relations.

2 Preliminaries

Definition 2.1 [4,5] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(GR2) $(L, \odot, 1)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \bot$ and $a^* = a \Rightarrow \bot$.

Remark 2.2 [4-8] (1) A generalized residuated lattice is a residuated lattice $(\rightarrow=\Rightarrow)$ iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) Let $(L, \leq, \odot)$ be a quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee\{z \in L \mid z \odot x \leq y\}, \quad x \Rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$ iff $y \leq (x \Rightarrow z)$. Hence $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is a generalized residuated lattice with the law of double negation and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

Lemma 2.3 [4-8] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(4) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(7) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ and $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

(8) $x \odot (x \rightarrow y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.

(9) $(x \Rightarrow y) \odot (y \Rightarrow z) \le x \Rightarrow z$ and $(y \to z) \odot (x \to y) \le x \to z$.
(10) $(x \odot y)^0 = x \to y^0$ and $(x \odot y)^* = y \Rightarrow x^*$.
(11) $(x \to y) \le (y \Rightarrow z) \to (x \Rightarrow z)$ and $(y \Rightarrow z) \le (x \to y) \Rightarrow (x \Rightarrow z)$
(12) $x_i \to y_i \le (\bigwedge_{i\in\Gamma} x_i) \to (\bigwedge_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.
(13) $x_i \to y_i \le (\bigvee_{i\in\Gamma} x_i) \to (\bigvee_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.
(14) $x \to y = \top$ iff $x \le y$.
(15) $x \to y = y^0 \Rightarrow x^0$ and $x \Rightarrow y = y^* \to x^*$.
(16) $x \odot y = (x \to y^0)^*$ and $(x \Rightarrow y^*)^0 = y \odot x$.
(17) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.
(18) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ and $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

3 Two types of join preserving operators

Definition 3.1 Let X and Y be two sets. Let $\omega^\to, \phi^\to : L^X \to L^Y$ and $\omega^\leftarrow, \phi^\leftarrow : L^Y \to L^X$ be operators.

(1) The pair $(\omega^\to, \omega^\leftarrow)$ is called an *antitone Galois connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $B \le \omega^\to(A)$ iff $A \le \omega^\leftarrow(B)$.

(2) The pair $(\phi^\to, \phi^\leftarrow)$ is called an *isotone Galois connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $\phi^\to(A) \le B$ iff $A \le \phi^\leftarrow(B)$.

Definition 3.2 An operator $\phi^\to : L^X \to L^Y$ is called a join preserving operator, denoted by $\phi^\to \in J(X,Y)$, if it satisfies

(J) $\phi^\to(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \phi^\to(A_i)$, for $\{A_i\}_{i\in\Gamma} \subset L^X$.

An operator $\psi^\to : L^X \to L^Y$ is called a meet preserving operator, denoted by $\psi^\to \in M(X,Y)$, if it satisfies

(M) $\psi^\to(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \psi^\to(A_i)$, for $\{A_i\}_{i\in\Gamma} \subset L^X$.

Theorem 3.3 *Let $\phi^\to : L^X \to L^Y$ be a join preserving operator. Define functions $\omega^\to_\phi, \xi^\to_\phi : L^X \to L^Y$ and $\phi^\leftarrow, \omega^\leftarrow_\phi, \xi^\leftarrow_\phi : L^Y \to L^X$ as follows: , for all $A \in L^X, B \in L^Y$,*

$$\begin{array}{ll} \phi^\leftarrow(B) & = \bigvee\{A \in L^X \mid \phi^\to(A) \le B\}, \\ \omega^\to_\phi(A) & = (\phi^\to(A))^0, \quad \omega^\leftarrow_\phi(B) = \phi^\leftarrow(B^*), \\ \xi^\leftarrow_\phi(B) & = \bigwedge\{A \in L^X \mid \phi^\to(A^*) \le B^*\}, \\ \xi^\to_\phi(A) & = \bigvee\{B \in L^Y \mid \xi^\leftarrow_\phi(B) \le A\}. \end{array}$$

Then the following properties hold:

(1) *The pair $(\phi^\to, \phi^\leftarrow)$ is an isotone Galois connection with $\bigwedge_{i\in\Gamma} \phi^\leftarrow(B_i) = \phi^\leftarrow(\bigwedge_{i\in\Gamma} B_i)$.*

(2) *$\alpha \odot \phi^\to(A) \le \phi^\to(\alpha \odot A)$ for $A \in L^X$ iff $\alpha \Rightarrow \phi^\leftarrow(B) \le \phi^\leftarrow(\alpha \Rightarrow B)$ for $B \in L^Y$.*

(3) *$\phi^\to(\alpha \odot A) \le \alpha \odot \phi^\to(A)$ for $A \in L^X$ iff $\phi^\leftarrow(\alpha \Rightarrow B) \le \alpha \Rightarrow \phi^\leftarrow(B)$ for $B \in L^Y$.*

(4) *The pair* $(\omega_\phi^{\rightarrow}, \omega_\phi^{\leftarrow})$ *is an antitone Galois connection with* $\omega_\phi^{\leftarrow}(\bigvee_{i\in\Gamma} B_i) = \bigwedge \omega_\phi^{\leftarrow}(B_i)$ *and* $\omega_\phi^{\rightarrow}(\bigvee_{i\in\Gamma} A_i) = \bigwedge \omega_\phi^{\rightarrow}(A_i)$.

(5) $\alpha \odot \phi^{\rightarrow}(A) \leq \phi^{\rightarrow}(\alpha \odot A)$ *iff* $\omega_\phi^{\rightarrow}(\alpha \odot C) \leq \alpha \rightarrow \omega_\phi^{\rightarrow}(C)$.

(6) $\phi^{\rightarrow}(\alpha \odot A) \leq \alpha \odot \phi^{\rightarrow}(A)$ *iff* $\alpha \rightarrow \omega_\phi^{\rightarrow}(C) \leq \omega_\phi^{\rightarrow}(\alpha \odot C)$.

(7) $\alpha \Rightarrow \phi^{\leftarrow}(A) \leq \phi^{\leftarrow}(\alpha \Rightarrow A)$ *iff* $\omega_\phi^{\leftarrow}(B \odot \alpha) \geq \alpha \Rightarrow \omega_\phi^{\leftarrow}(B)$.

(8) $\alpha \Rightarrow \phi^{\leftarrow}(A) \geq \phi^{\leftarrow}(\alpha \Rightarrow A)$ *iff* $\omega_\phi^{\leftarrow}(B \odot \alpha) \leq \alpha \Rightarrow \omega_\phi^{\leftarrow}(B)$.

(9) $\xi_\phi^{\leftarrow}(B) = (\phi^{\leftarrow}(B^*))^0$ *with* $\xi_\phi^{\leftarrow}(\bigvee_{i\in\Gamma}(B_i)) = \bigvee_{i\in\Gamma} \xi_\phi^{\leftarrow}(B_i)$ *and*

$$\phi^{\rightarrow}(A) \leq B \Leftrightarrow A \leq \phi^{\leftarrow}(B) \Leftrightarrow \xi_\phi^{\leftarrow}(B^0) \leq A^0$$

(10) $\alpha \Rightarrow \phi^{\leftarrow}(A) \leq \phi^{\leftarrow}(\alpha \Rightarrow A)$ *iff* $\xi_\phi^{\leftarrow}(B \odot \alpha) \leq \xi_\phi^{\leftarrow}(B) \odot \alpha$.

(11) $\alpha \Rightarrow \phi^{\leftarrow}(A) \geq \phi^{\leftarrow}(\alpha \Rightarrow A)$ *iff* $\xi_\phi^{\leftarrow}(B \odot \alpha) \geq \xi_\phi^{\leftarrow}(B) \odot \alpha$.

(12) $\xi_\phi^{\rightarrow}(A) = (\phi^{\rightarrow}(A^*))^0$ *with* $\xi_\phi^{\rightarrow}(\bigwedge_{i\in\Gamma}(A_i)) = \bigwedge_{i\in\Gamma} \xi_\phi^{\rightarrow}(A_i)$ *and*

$$\phi(A) \leq B \Leftrightarrow A \leq \phi^{\leftarrow}(B) \Leftrightarrow \xi_\phi^{\leftarrow}(B^*) \leq A^* \Leftrightarrow B^* \leq \xi_\phi^{\rightarrow}(A^*)$$

(13) $\alpha \odot \phi^{\rightarrow}(A) \leq \phi^{\rightarrow}(\alpha \odot A)$ *for* $A \in L^X$ *iff* $\alpha \rightarrow \xi_\phi^{\rightarrow}(B) \geq \xi_\phi^{\rightarrow}(\alpha \rightarrow B)$ *for* $B \in L^Y$.

(14) $\alpha \odot \phi^{\rightarrow}(A) \geq \phi^{\rightarrow}(\alpha \odot A)$ *for* $A \in L^X$ *iff* $\alpha \rightarrow \xi_\phi^{\rightarrow}(B) \leq \xi_\phi^{\rightarrow}(\alpha \rightarrow B)$ *for* $B \in L^Y$.

(15) *The pair* $(\xi_\phi^{\leftarrow}, \xi_\phi^{\rightarrow})$ *is an isotone Galois connection.*

(16) *If* $\phi^{\rightarrow}(A(x) \odot \top_{\{x\}}) = B_x$ *for all* $x \in X$, *then* $\phi^{\rightarrow}(A) = \bigvee_{z\in X} B_z$.

(17) *If* $\phi_1^{\rightarrow}(\alpha \odot \top_{\{x\}}) = \phi_2^{\rightarrow}(\alpha \odot \top_{\{x\}})$ *for all* $x \in X$ *and* $\phi_1^{\rightarrow}, \phi_2^{\rightarrow} \in J(X,Y)$, *then* $\phi_1^{\rightarrow} = \phi_2^{\rightarrow}$.

Proof (1) Since ϕ is a join preserving map and $\phi^{\leftarrow}(B) = \bigvee\{A \in L^X \mid \phi^{\rightarrow}(A) \leq B\}$, we have

$$\phi^{\rightarrow}(A) \leq B \Leftrightarrow A \leq \phi^{\leftarrow}(B).$$

Hence $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galois connection and $\phi^{\leftarrow}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} \phi^{\leftarrow}(B_i)$ from

$$\begin{aligned} \bigwedge_{i\in\Gamma} \phi^{\leftarrow}(B_i) \geq A \quad & \Leftrightarrow \phi^{\leftarrow}(B_i) \geq A, \ \ \forall i \in \Gamma \Leftrightarrow \phi^{\rightarrow}(A) \leq B_i, \ \ \forall i \in \Gamma \\ & \Leftrightarrow \phi^{\rightarrow}(A) \leq \bigwedge_{i\in\Gamma} B_i, \Leftrightarrow \phi^{\leftarrow}(\bigwedge_{i\in\Gamma} B_i) \geq A. \end{aligned}$$

(2) $(\Rightarrow)$

$$\begin{aligned} \alpha \Rightarrow \phi^{\leftarrow}(B) \leq \alpha \Rightarrow \phi^{\leftarrow}(B) \quad & \text{iff } \alpha \odot (\alpha \Rightarrow \phi^{\leftarrow}(B)) \leq \phi^{\leftarrow}(B) \\ & \text{iff } \phi^{\rightarrow}(\alpha \odot (\alpha \Rightarrow \phi^{\leftarrow}(B))) \leq B. \end{aligned}$$

Since $\alpha \odot \phi^{\rightarrow}(A) \leq \phi^{\rightarrow}(\alpha \odot A)$ for $A \in L^X$, then $\alpha \odot \phi^{\rightarrow}(\alpha \Rightarrow \phi^{\leftarrow}(B)) \leq \phi^{\rightarrow}(\alpha \odot (\alpha \Rightarrow \phi^{\leftarrow}(B))) \leq B$. Thus $\phi^{\rightarrow}(\alpha \Rightarrow \phi^{\leftarrow}(B)) \leq \alpha \Rightarrow B$ iff $\alpha \Rightarrow \phi^{\leftarrow}(B) \leq \phi^{\leftarrow}(\alpha \Rightarrow B)$.

(⇐)

$$\begin{aligned}\phi^{\rightarrow}(\alpha\odot A)\leq\phi^{\rightarrow}(\alpha\odot A)\ &\text{ iff } \alpha\odot A\leq\phi^{\leftarrow}(\phi^{\rightarrow}(\alpha\odot A))\\ &\text{ iff } A\leq\alpha\Rightarrow\phi^{\leftarrow}(\phi^{\rightarrow}(\alpha\odot A))\end{aligned}$$

Since $\alpha\Rightarrow\phi^{\leftarrow}(B)\leq\phi^{\leftarrow}(\alpha\Rightarrow B)$, then

$$A\leq\alpha\Rightarrow\phi^{\leftarrow}(\phi^{\rightarrow}(\alpha\odot A))\leq\phi^{\leftarrow}(\alpha\Rightarrow\phi^{\rightarrow}(\alpha\odot A)).$$

Hence $\phi^{\rightarrow}(A)\leq\alpha\Rightarrow\phi^{\rightarrow}(\alpha\odot A)$ iff $\alpha\odot\phi^{\rightarrow}(A)\leq\phi^{\rightarrow}(\alpha\odot A)$.

(3)(⇒)

$$\begin{aligned}\phi^{\leftarrow}(\alpha\Rightarrow B)\leq\phi^{\leftarrow}(\alpha\Rightarrow B)\ &\text{ iff } \phi^{\rightarrow}(\phi^{\leftarrow}(\alpha\Rightarrow B))\leq\alpha\Rightarrow B\\ &\text{ iff } \alpha\odot\phi^{\rightarrow}(\phi^{\leftarrow}(\alpha\Rightarrow B))\leq B.\end{aligned}$$

Since $\phi^{\rightarrow}(\alpha\odot A)\leq\alpha\odot\phi^{\rightarrow}(A)$ for $A\in L^X$, $\phi^{\rightarrow}(\alpha\odot\phi^{\leftarrow}(\alpha\Rightarrow B))\leq B$ iff $\alpha\odot\phi^{\leftarrow}(\alpha\Rightarrow B)\leq\phi^{\leftarrow}(B)$ iff $\phi^{\leftarrow}(\alpha\Rightarrow B)\leq\alpha\Rightarrow\phi^{\leftarrow}(B)$.

(⇐)

$$\begin{aligned}\alpha\odot\phi^{\rightarrow}(A)\leq\alpha\odot\phi^{\rightarrow}(A)\ &\text{ iff } \phi^{\rightarrow}(A)\leq\alpha\Rightarrow\alpha\odot\phi^{\rightarrow}(A)\\ &\text{ iff } A\leq\phi^{\leftarrow}(\alpha\Rightarrow\alpha\odot\phi^{\rightarrow}(A)).\end{aligned}$$

Since $\phi^{\leftarrow}(\alpha\Rightarrow B)\leq\alpha\Rightarrow\phi^{\leftarrow}(B)$, then $A\leq\alpha\Rightarrow\phi^{\leftarrow}(\alpha\odot\phi^{\rightarrow}(A))$ iff $\alpha\odot A\leq\phi^{\leftarrow}(\alpha\odot\phi^{\rightarrow}(A))$ iff $\phi^{\rightarrow}(\alpha\odot A)\leq\alpha\odot\phi^{\rightarrow}(A)$.

(4) The pair $(\omega_\phi^{\rightarrow},\omega_\phi^{\leftarrow})$ is an antitone Galois connection from:

$$\begin{aligned}&B\leq\omega_\phi^{\rightarrow}(A)\text{ iff }B\leq(\phi^{\rightarrow}(A))^0\text{ iff }\phi^{\rightarrow}(A)\leq B^*\\ &\text{ iff }A\leq\phi^{\leftarrow}(B^*)=\omega_\phi^{\leftarrow}(B).\end{aligned}$$

Moreover, $\omega_\phi^{\rightarrow}(\bigvee_{i\in\Gamma}A_i)=\bigwedge\omega_\phi^{\rightarrow}(A_i)$ from:

$$\begin{aligned}\bigwedge\omega_\phi^{\rightarrow}(A_i)\geq C\ &\Leftrightarrow\omega_\phi^{\rightarrow}(A_i)\geq C,\ \forall i\in\Gamma\Leftrightarrow\omega_\phi^{\leftarrow}(C)\geq A_i,\ \forall i\in\Gamma\\ &\Leftrightarrow\omega_\phi^{\leftarrow}(C)\geq\textstyle\bigvee_{i\in\Gamma}A_i,\Leftrightarrow\omega_\phi^{\rightarrow}(\bigvee_{i\in\Gamma}A_i)\geq C.\end{aligned}$$

Other case is similarly proved.

(5) Let $\alpha\odot\phi^{\rightarrow}(A)\leq\phi^{\rightarrow}(\alpha\odot A)$ be given. Then $\omega_\phi^{\rightarrow}(\alpha\odot C)=(\phi^{\rightarrow}(\alpha\odot C))^0\leq(\alpha\odot\phi^{\rightarrow}(C))^0=\alpha\rightarrow\phi^{\rightarrow}(C)^0=\alpha\rightarrow\omega_\phi^{\rightarrow}(C)$.

Let $\omega_\phi^{\rightarrow}(\alpha\odot C)\leq\alpha\rightarrow\omega_\phi^{\rightarrow}(C)$ be given. Then $\phi^{\rightarrow}(\alpha\odot C)=(\omega_\phi^{\rightarrow}(\alpha\odot C))^*\geq(\alpha\rightarrow\omega_\phi^{\rightarrow}(C))^*=(\alpha\rightarrow(\phi^{\rightarrow}(C))^0)^*=\alpha\odot\phi^{\rightarrow}(C)$ from Lemma 2.3(16). So, $\alpha\odot\phi^{\rightarrow}(A)\leq\phi^{\rightarrow}(\alpha\odot A)$.

(7) Let $\alpha\Rightarrow\phi^{\leftarrow}(A)\leq\phi^{\leftarrow}(\alpha\Rightarrow A)$ be given. Then $\omega_\phi^{\leftarrow}(B\odot\alpha)=\phi^{\leftarrow}((B\odot\alpha)^*)=\phi^{\leftarrow}(\alpha\Rightarrow B^*)\geq\alpha\Rightarrow\phi^{\leftarrow}(B^*)=\alpha\Rightarrow\omega_\phi^{\leftarrow}(B)$.

Let $\omega_\phi^{\leftarrow}(B\odot\alpha)\geq\alpha\Rightarrow\omega_\phi^{\leftarrow}(B)$ be given. Since $\phi^{\leftarrow}(\alpha\Rightarrow B)=\phi^{\leftarrow}((\alpha\odot B^0)^*)=\omega_\phi^{\leftarrow}(\alpha\odot B^0)\geq\alpha\Rightarrow\omega_\phi^{\leftarrow}(B^0)=\alpha\Rightarrow\phi^{\leftarrow}(B)$, then $\alpha\Rightarrow\phi^{\leftarrow}(B)\leq\phi^{\leftarrow}(\alpha\Rightarrow B)$.

(6) and (8) are similarly proved as in (5) and (7), respectively.

(9) By Lemma 2.3(17), we have

$$\begin{array}{ll} \xi_\phi^{\leftarrow}(B) & = \bigwedge\{A \in L^X \mid \phi^{\rightarrow}(A^*) \leq B^*\} \\ & = \Big(\bigvee\{A^* \in L^X \mid A^* \leq \phi^{\leftarrow}(B^*)\}\Big)^0 \quad = (\phi^{\leftarrow}(B^*))^0. \end{array}$$

It follows $\xi_\phi^{\leftarrow}(\bigvee_{i\in\Gamma}(B_i)) = (\phi^{\leftarrow}(\bigwedge_{i\in\Gamma}(B_i)^*))^0 = \bigvee_{i\in\Gamma}(\phi^{\leftarrow}(B_i^*))^0 = \bigvee_{i\in\Gamma}\xi_\phi^{\leftarrow}(B_i)$ and $\phi^{\rightarrow}(A) \leq B \Leftrightarrow A \leq \phi^{\leftarrow}(B) \Leftrightarrow \xi_\phi^{\leftarrow}(B^0) \leq A^0$.

(10) Let $\alpha \Rightarrow \phi^{\leftarrow}(A) \leq \phi^{\leftarrow}(\alpha \Rightarrow A)$ be given. By Lemma 2.3(10), we have:

$$\begin{array}{ll} \xi_\phi^{\leftarrow}(B \odot \alpha) & = (\phi^{\leftarrow}((B \odot \alpha)^*))^0 = (\phi^{\leftarrow}(\alpha \Rightarrow B^*))^0 \\ & \leq (\alpha \Rightarrow \phi^{\leftarrow}(B^*))^0 = (\phi^{\leftarrow}(B^*))^0 \odot \alpha \\ & = \xi_\phi^{\leftarrow}(B) \odot \alpha. \end{array}$$

Other case and (11) are similarly proved.

(12)

$$\begin{array}{ll} \xi_\phi^{\rightarrow}(A) & = \bigvee\{B \in L^Y \mid \xi_\phi^{\leftarrow}(B) \leq A\} \\ & = \bigvee\{B \in L^Y \mid \phi^{\rightarrow}(A^*) \leq B^*\} = (\phi^{\rightarrow}(A^*))^0. \end{array}$$

Other cases are similarly proved as (1) and (5).

(13) Let $\alpha \odot \phi^{\rightarrow}(A) \leq \phi^{\rightarrow}(\alpha \odot A)$ be given. Then

$$\begin{array}{ll} \xi_\phi^{\rightarrow}(\alpha \rightarrow A) & = (\phi^{\rightarrow}((\alpha \rightarrow A)^*))^0. \\ & = (\phi^{\rightarrow}(\alpha \odot A^*))^0 \leq (\alpha \odot \phi^{\rightarrow}(A^*))^0 \\ & = \alpha \rightarrow (\phi^{\rightarrow}(A^*))^0 = \alpha \rightarrow \xi_\phi^{\rightarrow}(A). \end{array}$$

Other case and (14) are similarly proved.

(15) Since $\xi_\phi^{\leftarrow}(B) \leq A$ iff $B \leq \xi_\phi^{\rightarrow}(A)$ from the definition of $\xi_\phi^{\rightarrow}$, then the pair $(\xi_\phi^{\leftarrow}, \xi_\phi^{\rightarrow})$ is an isotone Galois connection.

(16) For all $A \in L^X$, we write $A = \bigvee_{z\in X} A(z) \odot 1_{\{z\}}$. Thus,

$$\begin{array}{ll} \phi(A) & = \phi(\bigvee_{z\in X} A(z) \odot 1_{\{z\}}) \\ & = \bigvee_{z\in X} \phi(A(z) \odot 1_{\{z\}}) \\ & = \bigvee_{z\in X} B_z. \end{array}$$

(17) For $A = \bigvee_{z\in X} A(z) \odot 1_{\{z\}}$, we have

$$\begin{array}{ll} \phi_1(A) & = \bigvee_{z\in X} \phi_1(A(z) \odot 1_{\{z\}}) \\ & = \bigvee_{z\in X} \phi_2(A(z) \odot 1_{\{z\}}) \\ & = \phi_2(A). \end{array}$$

Example 3.4 Let X and Y be sets and $R \in L^{X\times Y}$. Define a function $\phi_R^{\rightarrow} : L^X \to L^Y$ as $\phi_R^{\rightarrow}(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y))$.

(1) $\phi_R^{\rightarrow}$ is join preserving because

$$\begin{aligned}\phi_R^{\rightarrow}(\bigvee_i A_i)(y) &= \bigvee_{x\in X}(\bigvee_i A_i(x)) \odot R(x,y) \\ &= \bigvee_i(\bigvee_{x\in X} A_i(x) \odot R(x,y)) \\ &= \bigvee_i \phi_R^{\rightarrow}(A_i)(y).\end{aligned}$$

By Theorem 3.3, we obtain $\phi_R^{\leftarrow}$ as follows

$$\begin{aligned}\phi_R^{\leftarrow}(B)(x) &= \bigvee\{A(x) \mid \phi_R^{\rightarrow}(A) \le B\} \\ &= \bigvee\{A(x) \mid \bigvee_{x\in X}(A(x) \odot R(x,y)) \le B(y)\} \\ &= \bigvee\{A(x) \mid A(x) \le \bigwedge_{y\in Y}(R(x,y) \to B(y))\} \\ &= \bigwedge_{y\in Y}(R(x,y) \to B(y)).\end{aligned}$$

Thus, $(\phi_R^{\rightarrow}, \phi_R^{\leftarrow})$ is an isotone Galois connection with $\phi_R^{\leftarrow} \in M(Y,X)$ from Theorem 3.3(1).

(2) Since $\alpha \odot \phi_R^{\rightarrow}(A) = \phi_R^{\rightarrow}(\alpha \odot A)$, by Theorem 3.3(2,3), $\alpha \Rightarrow \phi_R^{\leftarrow}(B) = \phi_R^{\leftarrow}(\alpha \Rightarrow B)$.

(3)

$$\begin{aligned}\omega_{\phi_R}^{\rightarrow}(C)(y) &= (\phi_R^{\rightarrow}(C))^0(y) = (\bigvee_{x\in X} C(x) \odot R(x,y))^0 \\ &= \bigwedge_{x\in X}(C(x) \odot R(x,y))^0 \\ &= \bigwedge_{x\in X}(C(x) \to R^0(x,y)). \text{ (by Lemma 2.3(10))}\end{aligned}$$

$$\begin{aligned}\omega_{\phi_R}^{\leftarrow}(B)(x) &= \phi_R^{\leftarrow}(B^*)(x) = \bigwedge_{y\in Y}(R(x,y) \to B^*(y)) \\ &= \bigwedge_{y\in Y}(B(y) \Rightarrow R^0(x,y)). \text{ (by Lemma 2.3(15))}\end{aligned}$$

The pair $(\omega_{\phi_R}^{\rightarrow}, \omega_{\phi_R}^{\leftarrow})$ is an antitone Galois connection.

(4) Since $\alpha \odot \phi_R^{\rightarrow}(A) = \phi_R^{\rightarrow}(\alpha \odot A)$, by Theorem 3.3(5-8), then $\omega_{\phi_R}^{\rightarrow}(\alpha \odot C) = \alpha \to \omega_{\phi_R}^{\rightarrow}(C)$ and $\omega_{\phi_R}^{\leftarrow}(B \odot \alpha) = \alpha \Rightarrow \omega_{\phi_R}^{\leftarrow}(B)$.

(5) By Lemma 2.3(10,15), we have

$$\begin{aligned}\xi_{\phi_R}^{\leftarrow}(B)(x) &= \left(\phi_R^{\leftarrow}(B^*)\right)^0(x) = (\bigwedge_{y\in Y}(R(x,y) \to B^*(y)))^0 \\ &= \bigvee_{y\in Y}((B(y) \Rightarrow R(x,y)^0))^0 = \bigvee_{y\in Y}((R^{00}(x,y) \odot B(y))^*)^0 \\ &= \bigvee_{y\in Y}(R^{00}(x,y) \odot B(y)).\end{aligned}$$

Since $\xi_{\phi_R}^{\rightarrow}(A) = \left(\phi_R^{\rightarrow}(A^*)\right)^0$ from Theorem 3.3(12), by Lemma 2.3(15), we have from:

$$\begin{aligned}\xi_{\phi_R}^{\rightarrow}(A) &= \left(\phi_R^{\rightarrow}(A^*)\right)^0 = \left(\bigvee_{x\in X}(A^*(x) \odot R(x,y))\right)^0 \\ &= \bigwedge_{x\in X}(A^*(x) \odot R(x,y))^0 = \bigwedge_{x\in X}(A^*(x) \to R(x,y)^0) \\ &= \bigwedge_{x\in X}(R^{00}(x,y) \Rightarrow A(x)).\end{aligned}$$

The pair $(\xi_{\phi_R}^{\leftarrow}, \xi_{\phi_R}^{\rightarrow})$ is an isotone Galois connection.

(6) Since $\alpha \odot \phi_R^{\rightarrow}(A) = \phi_R^{\rightarrow}(\alpha \odot A)$, by Theorem 3.3(10,11, 13,14), $\xi_{\phi_R}^{\leftarrow}(B \odot \alpha) = \xi_{\phi_R}^{\leftarrow}(B) \odot \alpha$ and $\xi_{\phi_R}^{\rightarrow}(\alpha \to A) = \alpha \to \xi_{\phi_R}^{\rightarrow}(A)$.

Theorem 3.5 *Let $\phi^{\rightarrow} : L^X \to L^Y$ be a join preserving operator. Define functions $\omega_\phi^{\Rightarrow}, \xi_\phi^{\Rightarrow} : L^X \to L^Y$ and $\phi^{\Leftarrow}, \omega_\phi^{\Leftarrow}, \xi_\phi^{\Leftarrow} : L^Y \to L^X$ as follows: , for all $A \in L^X, B \in L^Y$,*

$$\begin{array}{ll} \phi^{\Leftarrow}(B) & = \bigvee\{A \in L^X \mid \phi^{\Rightarrow}(A) \leq B\}, \\ \omega_\phi^{\Rightarrow}(A) & = (\phi^{\Rightarrow}(A))^*, \quad \omega_\phi^{\Leftarrow}(B) = \phi^{\Leftarrow}(B^0), \\ \xi_\phi^{\Leftarrow}(B) & = \bigwedge\{A \in L^X \mid \phi^{\Rightarrow}(A^0) \leq B^0\}, \\ \xi_\phi^{\Rightarrow}(A) & = \bigvee\{B \in L^Y \mid \xi_\phi^{\Leftarrow}(B) \leq A\}. \end{array}$$

Then the following properties hold:

(1) *$\phi^{\Rightarrow}(A) \odot \alpha \leq \phi^{\Rightarrow}(A \odot \alpha)$ for $A \in L^X$ iff $\alpha \to \phi^{\Leftarrow}(B) \leq \phi^{\Leftarrow}(\alpha \to B)$ for $B \in L^Y$.*

(2) *$\phi^{\Rightarrow}(A) \odot \alpha \geq \phi^{\Rightarrow}(A \odot \alpha)$ for $A \in L^X$ iff $\alpha \to \phi^{\Leftarrow}(B) \geq \phi^{\Leftarrow}(\alpha \to B)$ for $B \in L^Y$.*

(3) *The pair $(\omega_\phi^{\Rightarrow}, \omega_\phi^{\Leftarrow})$ is an antitone Galois connection with $\omega_\phi^{\Rightarrow}(\bigvee_{i\in\Gamma} A_i) = \bigwedge \omega_\phi^{\Rightarrow}(A_i)$ and $\omega_\phi^{\Leftarrow}(\bigvee_{i\in\Gamma} B_i) = \bigwedge \omega_\phi^{\Leftarrow}(B_i)$.*

(4) *$\phi^{\Rightarrow}(A) \odot \alpha \leq \phi^{\Rightarrow}(A \odot \alpha)$ iff $\omega_\phi^{\Rightarrow}(C \odot \alpha) \leq \alpha \Rightarrow \omega_\phi^{\Rightarrow}(C)$.*

(5) *$\phi^{\Rightarrow}(A \odot \alpha) \leq \phi^{\Rightarrow}(A) \odot \alpha$ iff $\alpha \Rightarrow \omega_\phi^{\Rightarrow}(C) \leq \omega_\phi^{\Rightarrow}(C \odot \alpha)$.*

(6) *$\alpha \to \phi^{\Leftarrow}(A) \leq \phi^{\Leftarrow}(\alpha \to A)$ iff $\omega_\phi^{\Leftarrow}(\alpha \odot B) \geq \alpha \to \omega_\phi^{\Leftarrow}(B)$.*

(7) *$\alpha \to \phi^{\Leftarrow}(A) \geq \phi^{\Leftarrow}(\alpha \to A)$ iff $\omega_\phi^{\Leftarrow}(\alpha \odot B) \leq \alpha \to \omega_\phi^{\Leftarrow}(B)$.*

(8) *$\xi_\phi^{\Leftarrow}(B) = (\phi^{\Leftarrow}(B^0))^*$ with $\xi_\phi^{\Leftarrow}(\bigvee_{i\in\Gamma}(B_i)) = \bigvee_{i\in\Gamma} \xi_\phi^{\Leftarrow}(B_i)$ and*

$$\phi^{\Rightarrow}(A) \leq B \Leftrightarrow A \leq \phi^{\Leftarrow}(B) \Leftrightarrow \xi_\phi^{\Leftarrow}(B^*) \leq A^*$$

(9) *$\alpha \to \phi^{\Leftarrow}(A) \leq \phi^{\Leftarrow}(\alpha \to A)$ iff $\xi_\phi^{\Leftarrow}(\alpha \odot B) \leq \alpha \odot \xi_\phi^{\Leftarrow}(B)$.*

(10) *$\alpha \to \phi^{\Leftarrow}(A) \geq \phi^{\Leftarrow}(\alpha \to A)$ iff $\xi_\phi^{\Leftarrow}(\alpha \odot B) \geq \alpha \odot \xi_\phi^{\Leftarrow}(B)$.*

(11) *$\xi_\phi^{\Rightarrow}(A) = (\phi^{\Rightarrow}(A^0))^*$ with $\xi_\phi^{\Rightarrow}(\bigwedge_{i\in\Gamma}(A_i)) = \bigwedge_{i\in\Gamma} \xi_\phi^{\Rightarrow}(A_i)$ and*

$$\phi(A) \leq B \Leftrightarrow A \leq \phi^{\Leftarrow}(B) \Leftrightarrow \xi_\phi^{\Leftarrow}(B^0) \leq A^0 \Leftrightarrow B^0 \leq \xi_\phi^{\Rightarrow}(A^0)$$

(12) *$\phi^{\Rightarrow}(A) \odot \alpha \leq \phi^{\Rightarrow}(A \odot \alpha)$ for $A \in L^X$ iff $\alpha \Rightarrow \xi_\phi^{\Rightarrow}(B) \geq \xi_\phi^{\Rightarrow}(\alpha \Rightarrow B)$ for $B \in L^Y$.*

(13) *$\phi^{\Rightarrow}(A) \odot \alpha \geq \phi^{\Rightarrow}(A \odot \alpha)$ for $A \in L^X$ iff $\alpha \Rightarrow \xi_\phi^{\Rightarrow}(B) \leq \xi_\phi^{\Rightarrow}(\alpha \Rightarrow B)$ for $B \in L^Y$.*

(14) *The pair $(\xi_\phi^{\Leftarrow}, \xi_\phi^{\Rightarrow})$ is an isotone Galois connection.*

(15) *If $\phi^{\Rightarrow}(\top_{\{x\}} \odot A(x)) = B_x$ for all $x \in X$, then $\phi^{\Rightarrow}(A) = \bigvee_{z\in X} B_z$.*

(16) *If $\phi_1^{\rightarrow}(\top_{\{x\}} \odot \alpha) = \phi_2^{\rightarrow}(\top_{\{x\}} \odot \alpha)$ for all $x \in X$ and $\phi_1^{\rightarrow}, \phi_2^{\rightarrow} \in J(X,Y)$, then $\phi_1^{\rightarrow} = \phi_2^{\rightarrow}$.*

Proof (1) ($\Rightarrow$)

$$\begin{array}{ll} \alpha \to \phi^{\Leftarrow}(B) \leq \alpha \to \phi^{\Leftarrow}(B) & \text{iff } (\alpha \Rightarrow \phi^{\Leftarrow}(B)) \odot \alpha \leq \phi^{\Leftarrow}(B) \\ & \text{iff } \phi^{\Rightarrow}((\alpha \Rightarrow \phi^{\Leftarrow}(B)) \odot \alpha) \leq B. \end{array}$$

Since $\phi^{\Rightarrow}(A)\odot\alpha \le \phi^{\Rightarrow}(A\odot\alpha)$ for $A\in L^X$, $\phi^{\Rightarrow}(\alpha\to\phi^{\Leftarrow}(B))\odot\alpha\le B$ iff $\phi^{\Rightarrow}(\alpha\to\phi^{\Leftarrow}(B))\le\alpha\to B$ iff $\alpha\to\phi^{\Leftarrow}(B)\le\phi^{\Leftarrow}(\alpha\to B)$.

($\Leftarrow$)

$$\begin{array}{ll}\phi^{\Rightarrow}(A\odot\alpha)\le\phi^{\Rightarrow}(A\odot\alpha) & \text{iff } A\odot\alpha\le\phi^{\Leftarrow}(\phi^{\Rightarrow}(A\odot\alpha))\\ & \text{iff } A\le\alpha\to\phi^{\Leftarrow}(\phi^{\Rightarrow}(A\odot\alpha)).\end{array}$$

Since $\alpha\to\phi^{\Leftarrow}(B)\le\phi^{\Leftarrow}(\alpha\to B)$, $A\le\phi^{\Leftarrow}(\alpha\to\phi^{\Rightarrow}(A\odot\alpha))$ iff $\phi^{\Rightarrow}(A)\le\alpha\to\phi^{\Rightarrow}(A\odot\alpha)$ iff $\phi^{\Rightarrow}(A)\odot\alpha\le\phi^{\Rightarrow}(A\odot\alpha)$.

(3) It follows from

$$B\le\omega_\phi^{\Rightarrow}(A)\Leftrightarrow B\le(\phi^{\Rightarrow}(A))^*\Leftrightarrow\phi^{\Rightarrow}(A)\le B^0\Leftrightarrow A\le\phi^{\Leftarrow}(B^0)=\omega_\phi^{\Leftarrow}(B).$$

(4) Let $\phi^{\Rightarrow}(A)\odot\alpha\le\phi^{\Rightarrow}(A\odot\alpha)$ be given. Then $\omega_\phi^{\Rightarrow}(C\odot\alpha)=(\phi^{\Rightarrow}(C\odot\alpha))^*\le(\phi^{\Rightarrow}(C)\odot\alpha)^*=\alpha\Rightarrow\phi^{\Rightarrow}(C)^*=\alpha\Rightarrow\omega_\phi^{\Rightarrow}(C)$.

Since $\omega_\phi^{\Rightarrow}(\alpha\odot C)=(\phi^{\Rightarrow}(C\odot\alpha))^*\le\alpha\Rightarrow\omega_\phi^{\Rightarrow}(C)=\alpha\Rightarrow(\phi^{\Rightarrow}(C))^*=(\phi^{\Rightarrow}(C)\odot\alpha)^*$. So, $\phi^{\Rightarrow}(C)\odot\alpha\le\phi^{\Rightarrow}(C\odot\alpha)$.

(6) Let $\alpha\to\phi^{\Leftarrow}(B)\le\phi^{\Leftarrow}(\alpha\to B)$ be given. Then $\omega_\phi^{\Leftarrow}(\alpha\odot B)=\phi^{\Leftarrow}((\alpha\odot B)^0)=\phi^{\Leftarrow}(\alpha\to B^0)\ge\alpha\to\phi^{\Leftarrow}(B^0)=\alpha\to\omega_\phi^{\Leftarrow}(B)$.

Let $\omega_\phi^{\Leftarrow}(\alpha\odot B)\ge\alpha\to\omega_\phi^{\Leftarrow}(B)$ be given. Since $\phi^{\Leftarrow}(\alpha\to B)=\phi^{\Leftarrow}((\alpha\odot B^*)^0)=\omega_\phi^{\Leftarrow}(\alpha\odot B^*)\ge\alpha\to\omega_\phi^{\Leftarrow}(B^*)=\alpha\to\phi^{\Leftarrow}(B)$, then $\alpha\to\phi^{\Leftarrow}(B)\le\phi^{\Leftarrow}(\alpha\to B)$.

(9) By Lemma 2.3(10), we have:

$$\begin{array}{ll}\xi_\phi^{\Leftarrow}(\alpha\odot B) & =(\phi^{\Leftarrow}((\alpha\odot B)^0))^*=(\phi^{\Leftarrow}(\alpha\to B^0))^*\\ & \le(\alpha\to\phi^{\Leftarrow}(B^0))^*=\alpha\odot(\phi^{\Leftarrow}(B^0))^*\\ & =\alpha\odot\xi_\phi^{\Leftarrow}(B).\end{array}$$

(12) $\phi^{\Rightarrow}(A)\odot\alpha\le\phi^{\Rightarrow}(A\odot\alpha)$ for $A\in L^X$ iff $\alpha\to\xi_\phi^{\Rightarrow}(B)\le\xi_\phi^{\Rightarrow}(\alpha\to B)$ for $B\in L^Y$.

$$\begin{array}{ll}\xi_\phi^{\Rightarrow}(\alpha\Rightarrow A) & =(\phi^{\Rightarrow}((\alpha\Rightarrow A)^0))^*.\\ & =(\phi^{\Rightarrow}(A^0\odot\alpha))^*\le(\phi^{\Rightarrow}(A^0)\odot\alpha)^*\\ & =\alpha\Rightarrow(\phi^{\Rightarrow}(A^0))^*=\alpha\Rightarrow\xi_\phi^{\Rightarrow}(A).\end{array}$$

(15) and (16) follow that for all $A\in L^X$, $A=\bigvee_{z\in X}(\top_{\{z\}}\odot A(z))$.

Other cases are similarly proved as same methods in Theorem 3.3.

Example 3.6 Let X and Y be sets and $R\in L^{X\times Y}$. Define a function $\phi_R^{\Rightarrow}:L^X\to L^Y$ as $\phi_R^{\Rightarrow}(A)(y)=\bigvee_{x\in X}(R(x,y)\odot A(x))$. Since $\phi_R^{\Rightarrow}$ is join preserving, by Theorem 3.5, we obtain $\phi_R^{\Leftarrow}$ as follows

$$\begin{array}{ll}\phi_R^{\Leftarrow}(B)(x) & =\bigvee\{A(x)\in L^X\mid\phi_R^{\Rightarrow}(A)\le B\}\\ & =\bigvee\{A(x)\in L^X\mid A(x)\le\bigwedge_{y\in Y}(R(x,y)\Rightarrow B(y))\}\\ & =\bigwedge_{y\in Y}(R(x,y)\Rightarrow B(y))\end{array}$$

Thus, $(\phi^{\Rightarrow}_R, \phi^{\Leftarrow}_R)$ is an isotone Galois connection.

(1) Since $\phi^{\Rightarrow}_R(A) \odot \alpha = \phi^{\Rightarrow}_R(A \odot \alpha)$, by Theorem 3.5(1,2), $\alpha \to \phi^{\Leftarrow}_R(B) = \phi^{\Leftarrow}_R(\alpha \to B)$.

(2)

$$\begin{array}{ll} \omega^{\Rightarrow}_{\phi_R}(C)(y) & = (\phi^{\Rightarrow}_R(C))^*(y) = (\bigvee_{x\in X} R(x,y) \odot C(x))^* \\ & = \bigwedge_{x\in X}(R(x,y) \odot C(x))^* \\ & = \bigwedge_{x\in X}(C(x) \Rightarrow R^*(x,y)). \text{ (by Lemma 2.3(10))} \end{array}$$

$$\begin{array}{ll} \omega^{\Leftarrow}_{\phi_R}(B)(x) & = \phi^{\Leftarrow}_R(B^0)(x) = \bigwedge_{y\in Y}(R(x,y) \Rightarrow B^0(y)) \\ & = \bigwedge_{y\in Y}(B(y) \to R^*(x,y)). \text{ (by Lemma 2.3(15)).} \end{array}$$

The pair $(\omega^{\Rightarrow}_{\phi_R}, \omega^{\Leftarrow}_{\phi_R})$ is an antitone Galois connection.

(3) Since $\phi^{\Rightarrow}_R(A) \odot \alpha = \phi^{\Rightarrow}_R(A \odot \alpha)$, by Theorem 3.5(4-7), $\omega^{\Rightarrow}_{\phi_R}(C \odot \alpha) = \alpha \Rightarrow \omega^{\Rightarrow}_{\phi_R}(C)$ and $\omega^{\Leftarrow}_{\phi_R}(\alpha \odot B) = \alpha \to \omega^{\Leftarrow}_{\phi_R}(B)$.

(4) By Lemma 2.3 (10,15), we have

$$\begin{array}{ll} \xi^{\Leftarrow}_{\phi_R}(B)(x) & = \left(\phi^{\Leftarrow}_R(B^0)\right)^*(x) = (\bigwedge_{y\in Y}(R(x,y) \Rightarrow B^0(y)))^* \\ & = \bigvee_{y\in Y}((B(y) \to R(x,y)^*))^* = \bigvee_{y\in Y}((B(y) \odot R^{**}(x,y))^0)^* \\ & = \bigvee_{y\in Y}(B(y) \odot R^{**}(x,y)) \end{array}$$

Since $\xi^{\Rightarrow}_{\phi_R}(A) = \left(\phi^{\Rightarrow}_R(A^0)\right)^*$ from Theorem 3.5(11), by Lemma 2.3(15), we have from:

$$\begin{array}{ll} \xi^{\Rightarrow}_{\phi_R}(A) & = \left(\phi^{\Rightarrow}_R(A^0)\right)^* = \left(\bigvee_{x\in X}(R(x,y) \odot A^0(x))\right)^* \\ & = \bigwedge_{x\in X}(R(x,y) \odot A^0(x))^* = \bigwedge_{x\in X}(A^0(x) \Rightarrow R(x,y)^*) \\ & = \bigwedge_{x\in X}(R^{**}(x,y) \to A(x)). \end{array}$$

The pair $(\xi^{\Leftarrow}_{\phi_R}, \xi^{\Rightarrow}_{\phi_R})$ is an isotone Galois connection.

(5) Since $\phi^{\Rightarrow}_R(A)\odot\alpha = \phi^{\Rightarrow}_R(A\odot\alpha)$, by Theorem 3.5(9,10,12,13), $\xi^{\Leftarrow}_{\phi_R}(\alpha\odot B) = \alpha \odot \xi^{\leftarrow}_{\phi_R}(B)$ and $\xi^{\to}_{\phi_R}(\alpha \Rightarrow A) = \alpha \Rightarrow \xi^{\to}_{\phi_R}(A)$.

Theorem 3.7 *Let $\xi^{\Leftarrow}_\phi, \xi^{\leftarrow}_\phi \in J(Y,X)$ be given in Theorems 3.3 and 3.5. Then the following properties hold:*

(1) $\xi^{\leftarrow}_{\xi^{\Leftarrow}_\phi} = \phi^{\Leftarrow}$ *and* $\xi^{\Leftarrow}_{\xi^{\leftarrow}_\phi} = \phi^{\leftarrow}$.

(2) $\xi^{\to}_{\xi^{\Rightarrow}_\phi} = \phi^{\Rightarrow}$ *and* $\xi^{\Rightarrow}_{\xi^{\to}_\phi} = \phi^{\to}$.

(3) $\omega^{\to}_{\xi^{\Leftarrow}_\phi} = \omega^{\leftarrow}_\phi$ *and* $\omega^{\Rightarrow}_{\xi^{\leftarrow}_\phi} = \omega^{\Leftarrow}_\phi$.

(4) $\omega^{\leftarrow}_{\xi^{\Rightarrow}_\phi} = \omega^{\Rightarrow}_\phi$ *and* $\omega^{\Leftarrow}_{\xi^{\to}_\phi} = \omega^{\to}_\phi$.

Proof (1) $\xi^{\leftarrow}_{\xi^{\Leftarrow}_{\phi}}(B) = \xi^{\Leftarrow}_{\phi}(B^*)^0 = (\phi^{\Leftarrow}(B^{0*}))^{*0} = \phi^{\Leftarrow}(B)$.
(2) $\xi^{\rightarrow}_{\xi^{\Rightarrow}_{\phi}}(A) = \xi^{\Rightarrow}_{\phi}(A^*)^0 = (\phi^{\Rightarrow}(A^{0*}))^{*0} = \phi^{\Rightarrow}(A)$.
(3) $\omega^{\rightarrow}_{\xi^{\Leftarrow}_{\phi}}(B) = (\xi^{\Leftarrow}_{\phi}(B))^0 = (\phi^{\Leftarrow}(B^0))^{*0} = \phi^{\Leftarrow}(B^0) = \omega^{\leftarrow}_{\phi}(B)$.
(4) $\omega^{\leftarrow}_{\xi^{\Rightarrow}_{\phi}}(A) = \xi^{\Rightarrow}_{\phi}(A^*) = (\omega^{\Rightarrow}_{\phi}(A^{*0}))^* = (\omega^{\Rightarrow}_{\phi}(A))^* = \omega^{\Rightarrow}_{\phi}(A)$.

Example 3.8 Let $\xi^{\leftarrow}_{\phi_R}(B)(x) = \bigvee_{y\in Y}(R^{00}(x,y) \odot B(y))$ and $\xi^{\Leftarrow}_{\phi_R}(B)(x) = \bigvee_{y\in Y}(B(y) \odot R^{**}(x,y))$ be given in Examples 3.4 and 3.6. We obtain

$$\begin{aligned}
\xi^{\leftarrow}_{\xi^{\Leftarrow}_{\phi_R}}(B)(x) &= \phi^{\Leftarrow}_R(B)(x) = \textstyle\bigwedge_{y\in Y}(R(x,y) \to B(y)),\\
\xi^{\Leftarrow}_{\xi^{\leftarrow}_{\phi_R}}(B)(x) &= \phi^{\leftarrow}_R(B)(x) = \textstyle\bigwedge_{y\in Y}(R(x,y) \Rightarrow B(y)),\\
\xi^{\rightarrow}_{\xi^{\Rightarrow}_{\phi_R}}(A)(y) &= \phi^{\Rightarrow}_R(A)(y) = \textstyle\bigvee_{x\in X}(R(x,y) \odot A(x)),\\
\xi^{\Rightarrow}_{\xi^{\rightarrow}_{\phi_R}}(A)(y) &= \phi^{\rightarrow}_R(A)(y) = \textstyle\bigvee_{x\in X}(A(x) \odot R(x,y)),\\
\omega^{\rightarrow}_{\xi^{\Leftarrow}_{\phi_R}}(B)(x) &= \omega^{\leftarrow}_{\phi_R}(B)(x) = \textstyle\bigwedge_{y\in Y}(B(y) \Rightarrow R^0(x,y)),\\
\omega^{\Rightarrow}_{\xi^{\leftarrow}_{\phi_R}}(B)(x) &= \omega^{\Leftarrow}_{\phi_R}(B)(x) = \textstyle\bigwedge_{y\in Y}(B(y) \to R^*(x,y)),\\
\omega^{\leftarrow}_{\xi^{\Rightarrow}_{\phi_R}}(A)(y) &= \omega^{\Rightarrow}_{\phi_R}(A)(y) = \textstyle\bigwedge_{y\in Y}(A(x) \Rightarrow R^*(x,y)),\\
\omega^{\Leftarrow}_{\xi^{\rightarrow}_{\phi_R}}(A)(y) &= \omega^{\rightarrow}_{\phi_R}(A)(y) = \textstyle\bigwedge_{y\in Y}(A(x) \to R^0(x,y)).
\end{aligned}$$

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