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HOMOTOPY PERTURBATION METHOD FOR SOLVING A MODEL FOR HIV INFECTION OF $CD4^+$ T CELLS

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ABSTRACT

In this article, homotopy perturbation method is implemented to give approximate and analytical solutions of nonlinear ordinary differential equation systems such as a model for HIV infection of $CD4^+$ T cells. A modification of the homotopy perturbation method (HPM), based on the use of Pade´ approximants, is proposed. Some plots are presented to show the reliability and simplicity of the methods.

Keywords: Pade´ Approximants, Homotopy Perturbation Method, A Model for HIV Infection of $CD4^+$ T Cells

$CD4^+$ T HÜCRELERİNİN BİR HIV ENFEKSİYONLU MODELİNİN HOMOTOPY PERTURBATION YÖNTEMİ İLE ÇÖZÜMÜ

ÖZET

Bu makalede, $CD4^+$ T hücrelerinin bir HIV enfeksiyonlu modeli gibi lineer olmayan adi diferensiyel denklem sisteminin yaklaşık analitik çözümünü bulmak için homotopy perturbation yöntemi (HPY) uygulanmıştır. Homotopy perturbation yöntemine pade yaklaşımı uygulanmıştır. Yöntemlerin basitliğini ve doğruluğunu göstermek için birkaç grafik gösterilmiştir.

Anahtar Kelimeler: Pade´ Yaklaşımı, Homotopy Perturbation Yöntemi, $CD4^+$ T Hücrelerinin Bir HIV Enfeksiyonlu Modeli

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1. INTRODUCTION

Dynamics of a model for HIV infection of $CD4^+$ T cells is examined (Liancheng and Michael, 2006) at the study. The components of the basic four-component model are the concentration of $CD4^+$ T cells, the concentration of infected $CD4^+$ T cells by the HIV viruses and free HIV virus particles are denoted respectively by *T(t), I(t),* and *V(t).* These quantities satisfy

$$\begin{cases} \dfrac{dT}{dt} = s - \alpha T + rT\left(1 - \dfrac{T+I}{T_{max}}\right) - kVT, \\ \dfrac{dI}{dt} = kVT - \beta I, \\ \dfrac{dV}{dt} = N\beta I - \gamma V, \end{cases} \tag{1}$$

with the initial conditions:

$T(0) = r_1$, $I(0) = r_2$ and $V(0) = r_3$.

Throughout this paper, we set;

$s = 0.1, \alpha = 0.02, \beta = 0.3, r = 3, \gamma = 2.4, k = 0.0027, T_{\max} = 1500,$
$N = 10.$

The motivation of this paper is to extend the application of the analytic homotopy-perturbation method (HPM) and variational iteration method (He, 1998/a, 1998/b, 1999/a, 2006) to solve a model for HIV infection of $CD4^+$ T cells (1). The homotopy perturbation method (HPM) was first proposed by Chinese mathematician He (1998/a, 1998/b, 1999/a, 1999/b, 2000, 2006). The first connection between series solution methods such as an Adomian decomposition method and Padé approximants was established in. The transmission and dynamics of HTLV-I feature several biological characteristics that are of interest to epidemiologists, mathematicians, and biologists, see for example, Asquith and Bangham (2003), Finlayson (1972), Abdou and Soliman (2005), etc. Like HIV, HTLV-I targets $CD4^+$ T -cells, the most abundant white cells in the immune system, decreasing the body's ability to fight infection. We will use Laplace transform and Pade´ approximant to deal with the truncated series.

2. PADÉ APPROXIMATON

A rational approximation to f(x) on [a, b] is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote

this quotient. The $R_{N,M}(x)$ Padé approximations to a function $f(x)$ are given by Baker (1975).

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \le x \le b. \tag{2}$$

The method of Padé requires that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (2) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_N x^N \tag{3}$$

$$Q_M(x) = 1 + q_1 x + q_2 x^2 + \ldots + q_M x^M \tag{4}$$

The polynomials in (2) and (3) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N+M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N+M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher then $Q_M(x)$.

Notice that the constant coefficient of Q_M is $q_0 = 1$. This is permissible, because it notice be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N+M+1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots, \tag{5}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[\sum_{i=0}^{M} q_i x^i\right] - \left[\sum_{i=0}^{N} p_i x^i\right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i\right], \tag{6}$$

The lower index $j = N+M+1$ in the summation on the right side of (6) is chosen because the first $N+M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (6) is multiplied out and the coefficients of the powers of x^i are set equal to zero for $k = 0, 1, 2, \ldots, N+M$, the result is a system of $N+M+1$ linear equations:

$$
\begin{aligned}
& a_0 - p_0 = 0 \\
& q_1 a_0 + a_1 - p_1 = 0 \\
& q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0 \\
& q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 = 0 \\
& q_M a_{N-M} + q_{M-1} a_{N-M+1} + a_N - p_N = 0
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
& q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \ldots + q_1 a_N + a_{N+2} = 0 \\
& q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \ldots + q_1 a_{N+1} + a_{N+2} = 0 \\
& \cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
& \cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
& \cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
& q_M a_N + q_{M-1} a_{N+1} + \ldots + q_1 a_{N+M+1} + a_{N+M} = 0
\end{aligned} \tag{8}
$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same, and this sum increases consecutively from 0 to $N+M$. The M equations in (8) involve only the unknowns $q_1, q_2, q_3, \ldots, q_M$ and must be solved first. Then the equations in (7) are used successively to find $p_1, p_2, p_3, \ldots, p_N$ (Baker, 1975).

3. HOMOTOPY PERTURBATION METHOD

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He (1999/a, 2000) considered the following non-linear differential equation:

$$A(u) = f(r), \quad r \in \Omega \tag{9}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{10}$$

where A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, Γ is the boundary of the domain Ω and ∂/∂n denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M and N. Therefore, (9) can be rewritten as follows:

$$M\ (u\ )\ +\ N\ (u\ )\ =\ f\ (r\ ),\quad r\ \in\ \Omega \tag{11}$$

He (1999/a, 2000) constructed a homotopy $v(r,p):\Omega x[0,1]\to\Re$ which satisfies

$$H(v,p)=(1-p)\left[M(v)-M(u_0)\right]+p\left[A(v)-f(r)\right]=0, \tag{12}$$

which is equivalent to

$$H(v,p)=M(v)-M(u_0)+pM(v_0)+p\left[N(v)-f(r)\right]=0, \tag{13}$$

where $p\in[0,1]$ is an embedding parameter, and u_0 is an initial approximation of (13). Obviously, we have

$$H(v,0)=M(v)-M(u_0)=0,\quad H(v,1)=A(v)-f(r)=0. \tag{14}$$

The changing process of p from zero to unity is just that of H (v,p) from $M(v)-M(v_0)$ to $A(v)-f(r)$. In topology, this is called deformation and $M(v)-M(v_0)$ and $A(v)-f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of Eq. (12) can be expressed as a series in p in the form

$$v=v_0+pv_1+p^2v_2+p^3v_3+... \tag{15}$$

When $p\to 1$, Eq. (12) corresponds to the original one, Eqs. (11) and (15) become the approximate solution of Eq. (11), i.e.,

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+v_3+... \tag{16}$$

The convergence of the series in Eq. (16) is discussed by He (1999/a and 2000).

4. APPLICATIONS

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (1).

4.1. Homotopy Perturbation Method to A Model for HIV Infection of $CD4^+$ T Cells

According to homotopy perturbation method, we derive a correct functional as follows:

$$\begin{aligned}
&(1-p)(\dot{v}_1-\dot{x}_0)+p\left(\dot{v}_1-s+\alpha v_1-rv_1\left(1-\frac{v_1+v_2}{T_{max}}\right)+kv_1v_3\right)=0,\\
&(1-p)(\dot{v}_2-\dot{y}_0)+p(\dot{v}_2-kv_1v_3+\beta v_2)=0,\\
&(1-p)(\dot{v}_3-\dot{z}_0)+p(\dot{v}_3-N\beta v_2+\gamma v_3)=0,
\end{aligned} \tag{17}$$

where "dot" denotes differentiation with respect to t, and the initial approximations are as follows:

$$\begin{aligned}
&v_{1,0}(t)=x_0(t)=T(0)=r_1,\\
&v_{2,0}(t)=y_0(t)=I(0)=r_2,\\
&v_{3,0}(t)=z_0(t)=V(0)=r_3.
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
v_1&=v_{1,0}+pv_{1,1}+p^2v_{1,2}+p^3v_{1,3}+...,\\
v_2&=v_{2,0}+pv_{2,1}+p^2v_{2,2}+p^3v_{2,3}+...,\\
v_3&=v_{3,0}+pv_{3,1}+p^2v_{3,2}+p^3v_{3,3}+...,
\end{aligned} \tag{19}$$

where $v_{i,j}, i, j=1,2,3,...$ are functions yet to be determined. Substituting Eqs. (18) and (19) into Eq. (17) and arranging the coefficients of "p" powers, we have

$$
\begin{aligned}
&\left(\dot{v}_{1,1} - s + (\alpha - r) r_1 + \frac{r}{T_{\max}} r_1^2 + \frac{r}{T_{\max}} r_1 r_2 + k r_1 r_3 \right) p \\
&+ \left(\begin{array}{l} \dot{v}_{1,2} + (\alpha - r) v_{1,1} + \dfrac{r\left[2 r_1 v_{1,1} \right]}{T_{\max}} + \\ \dfrac{r\left[r_2 v_{1,1} + r_1 v_{2,1} \right]}{T_{\max}} \\ + k\left[r_1 v_{3,1} + r_3 v_{1,1} \right] \end{array} \right) p^2 \\
&+ \left(\begin{array}{l} \dot{v}_{1,3} + (\alpha - r) v_{1,2} + \dfrac{r\left[2 r_1 v_{1,2} + v_{1,1}^2 \right]}{T_{\max}} \\ + \dfrac{r\left[r_2 v_{2,1} + v_{1,1} v_{2,1} + r_1 v_{2,2} \right]}{T_{\max}} \\ + k\left[r_1 v_{3,2} + v_{1,1} v_{3,1} + r_3 v_{1,2} \right] \end{array} \right) p^3 + \ldots = 0, \\
&\left(\dot{v}_{2,1} - k r_1 r_3 + \beta r_2 \right) p \\
&+ \left(\dot{v}_{2,2} - k\left[r_1 v_{3,1} + r_3 v_{1,1} \right] v_{4,1} + \beta v_{2,1} \right) p^2 \\
&+ \left(\dot{v}_{2,3} - k\left[r_1 v_{3,2} + v_{1,1} v_{3,1} + r_3 v_{1,2} \right] v_{4,1} + \beta v_{2,1} \right) p^3 + \ldots = 0, \\
&\left(\dot{v}_{3,1} - N \beta r_2 + \gamma r_3 \right) p + \left(\dot{v}_{3,2} - N \beta v_{2,1} + \gamma v_{3,1} \right) p^2 \\
&+ \left(\dot{v}_{3,3} - N \beta v_{2,2} + \gamma v_{3,2} \right) p^3 + \ldots = 0,
\end{aligned}
\tag{20}
$$

In order to obtain the unknowns $v_{i,j}(t), i, j = 1,2,3,$ we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$v_{i,j}(0) = 0, i, j = 1,2,3,$$

$$
\begin{aligned}
&\dot{v}_{1,1} - s + (\alpha - r) r_1 + \frac{r}{T_{\max}} r_1^2 + \frac{r}{T_{\max}} r_1 r_2 + k r_1 r_3 = 0, \\
&\dot{v}_{1,2} + (\alpha - r) v_{1,1} + \frac{r\left[2 r_1 v_{1,1}\right]}{T_{\max}} + \frac{r\left[r_2 v_{1,1} + r_1 v_{2,1}\right]}{T_{\max}} \\
&\quad + k\left[r_1 v_{3,1} + r_3 v_{1,1}\right] = 0, \\
&\dot{v}_{1,3} + (\alpha - r) v_{1,2} + \frac{r\left[2 r_1 v_{1,2} + v_{1,1}^2\right]}{T_{\max}} \\
&\quad + \frac{r\left[r_2 v_{2,1} + v_{1,1} v_{2,1} + r_1 v_{2,2}\right]}{T_{\max}} \\
&\quad + k\left[r_1 v_{3,2} + v_{1,1} v_{3,1} + r_3 v_{1,2}\right] = 0, \\
&\dot{v}_{2,1} - k r_1 r_3 + \beta r_2 = 0, \\
&\dot{v}_{2,2} - k\left[r_1 v_{3,1} + r_3 v_{1,1}\right] v_{4,1} + \beta v_{2,1} = 0, \\
&\dot{v}_{2,3} - k\left[r_1 v_{3,2} + v_{1,1} v_{3,1} + r_3 v_{1,2}\right] v_{4,1} + \beta v_{2,1} = 0, \\
&\dot{v}_{3,1} - N \beta r_2 + \gamma r_3 = 0, \\
&\dot{v}_{3,2} - N \beta v_{2,1} + \gamma v_{3,1} = 0, \\
&\dot{v}_{3,3} - N \beta v_{2,2} + \gamma v_{3,2} = 0,
\end{aligned}
\tag{21}
$$

From Eq. (16), if the three terms approximations are sufficient, we will obtain:

$$
\begin{aligned}
T(t) &= \lim_{p \to 1} v_1(t) = \sum_{k=0}^{2} v_{1,k}(t), \\
I(t) &= \lim_{p \to 1} v_2(t) = \sum_{k=0}^{2} v_{2,k}(t), \\
V(t) &= \lim_{p \to 1} v_3(t) = \sum_{k=0}^{2} v_{3,k}(t),
\end{aligned}
\tag{22}
$$

therefore

$$T(t) = r_1 + \left[s - (\alpha - r) r_1 - \frac{r r_1^2 + r r_1 r_2}{T_{\max}} + k r_1 r_3 \right] t$$
$$+ \frac{1}{2} \left[\begin{array}{l} r_1 r^2 + r s + r_1 \alpha^2 - \alpha s - 2\alpha r_1 r - k s r_3 + r_1 k^2 r_3^2 \\ -k\beta N r_1 r_2 \, 2(\alpha - r) k r_1 r_3 + k\gamma r_1 r_3 \\ + \dfrac{-3r^2 r_1^2 + 2 r r_3 r_1^2 k + 3\alpha r r_1^2 - 2 r^2 r_1 r_2 - 2 r r_1 s - r r_2 s}{T_{\max}} \\ + \dfrac{2 r r_1 r_2 r_3 k + 2\alpha r_1 r_2 + r r_1 r_2 \beta}{T_{\max}} \\ + \dfrac{r^2 r_1 r_2^2 + 3 r^2 r_1^2 r_2 + 2 r^2 r_1^3}{T_{\max}^2} \end{array} \right] t^2 \qquad (23)$$

$$I(t) = r_2 + \left[k r_1 r_3 - \beta r_2 \right] t$$
$$+ \frac{1}{2} \left[\begin{array}{l} -\beta (k r_1 r_3 - \beta r_2) + k r_1 (N \beta r_2 - \gamma r_3) \\ k r_3 \left(s - (\alpha - r) r_1 - \dfrac{r r_1^2 + r r_1 r_2}{T_{\max}} + k r_1 r_3 \right) \end{array} \right] t^2$$

$$V(t) = r_3 + \left[N \beta r_2 - \gamma r_3 \right] t$$
$$+ \frac{1}{2} \left[N \beta (k r_1 r_3 - \beta r_2) - \gamma (N \beta r_2 - \gamma r_3) \right] t^2$$
$$\vdots$$

Here *T(0)* = 0.1, *I(0)* = 0, and *V(0)* = 0.1 for the three-component model. A few first approximations for *T(t)*, *I(t)*, and *V(t)* are calculated and presented below:

Three terms approximations:

$$\begin{aligned} T(t) &= 0.1 + .397953 t + .5928490535 t^2 + .5887187713 t^3, \\ I(t) &= .000027 t + .000017273655 t^2 - .000008405153687 t^3, \\ V(t) &= 0.1 - .24 t + .2880405 t^2 - .2304151263 t^3. \end{aligned} \qquad (24)$$

Four terms approximations:

$$\begin{aligned} T(t) &= 0.1 + .397953t + .5928490535t^2 + .5887187713\,t^3 \\ &\quad + .4382951585t^4, \\ I(t) &= .000027\,t + .000017273655\,t^2 - .000008405153687t^3 \\ &\quad + .6147278168*10^{-5}t^4, \\ V(t) &= 0.1 - .24\,t + .2880405\,t^2 - .2304151263\,t^3 \\ &\quad + .1382427719t^4. \end{aligned} \tag{25}$$

Five terms approximations:

$$\begin{aligned} T(t) &= 0.1 + .397953t + .5928490535\,t^2 + .5887187713\,t^3 \\ &\quad + .4382951585t^4 + .2608632944t^5, \\ I(t) &= .000027\,t + .000017273655\,t^2 - .000008405153687t^3 \\ &\quad + .6147278168*10^{-5}t^4 - .2835861790*10^{-5}t^5, \\ V(t) &= 0.1 - .24\,t + .2880405\,t^2 - .2304151263\,t^3 \\ &\quad + .1382427719t^4 - .06635284216t^5. \end{aligned} \tag{26}$$

Six terms approximations:

$$\begin{aligned} T(t) &= 0.1 + .397953t + .5928490535\,t^2 + .5887187713\,t^3 \\ &\quad + .4382951585t^4 + .2608632944t^5 + .1291947326t^6, \\ I(t) &= .000027\,t + .000017273655\,t^2 - .000008405153687t^3 \\ &\quad + .6147278168*10^{-5}t^4 - .2835861790*10^{-5}t^5 \\ &\quad + .1153299804*10^{-5}t^6, \\ V(t) &= 0.1 - .24\,t + .2880405\,t^2 - .2304151263\,t^3 \\ &\quad + .1382427719t^4 - .06635284216t^5 + .02653971893t^6. \end{aligned} \tag{27}$$

In this section, we apply Laplace transformation to (27), which yields

$$\begin{aligned} L\left(T(s)\right) &= \frac{0.1}{s} + \frac{.397953}{s^2} + \frac{1.185698107}{s^3} + \frac{3.532312628}{s^4} \\ &\quad + \frac{10.5190838}{s^5} + \frac{31.30359533}{s^6} + \frac{93.02020747}{s^7} \end{aligned}$$

$$L(I(s)) = \frac{.000027}{s^2} + \frac{.00003454731}{s^3} - \frac{.00005043092212}{s^4} + \frac{.000147534676}{s^5} - \frac{.0003403034148}{s^6} + \frac{.0008303758589}{s^7} \quad (28)$$

$$L(V(s)) = \frac{0.1}{s} - \frac{.24}{s^2} + \frac{.576081}{s^3} - \frac{1.382490758}{s^4} + \frac{3.317826526}{s^5} - \frac{7.962341059}{s^6} + \frac{19.10859763}{s^7}$$

For simplicity, let $s = \frac{1}{t}$; then

$$L(T(t)) = 0.1t + .397953t^2 + 1.185698107t^3 + 3.532312628t^4 + 10.5190838t^5 + 31.30359533t^6 + 93.02020747t^7$$

$$L(I(t)) = .000027t^2 + .00003454731t^3 - .00005043092212t^4 + .000147534676t^5 - .0003403034148t^6 + .0008303758589t^7 \quad (29)$$

$$L(V(t)) = .1t - .24t^2 + .576081t^3 - 1.382490758t^4 + 3.317826526t^5 - 7.962341059t^6 + 19.10859763t^7$$

Padé approximant [4/4] of (29) and substituting *t = 1/s*, we obtain [4/4] in terms of s. By using the inverse Laplace transformation, we obtain

$$T(t) = -.03352758677e^{-.0009308935117t} + .1335507854e^{2.980494451t} - .00002319863125e^{5.413486763t} - .297067849*10^{-8}e^{15699.6833t}$$

$$I(t) = .000004346124163e^{-2.395967997t} - .00004489696714e^{-..306254013t} + .00004055084298e^{.5835463905t} \quad (30)$$

$$V(t) = .09999780863e^{-2.400133231t} + .00000219137e^{3.679664095t}$$

These results obtained by Padé approximations for *T(t), I(t),* and *V(t)* are calculated and presented follow.

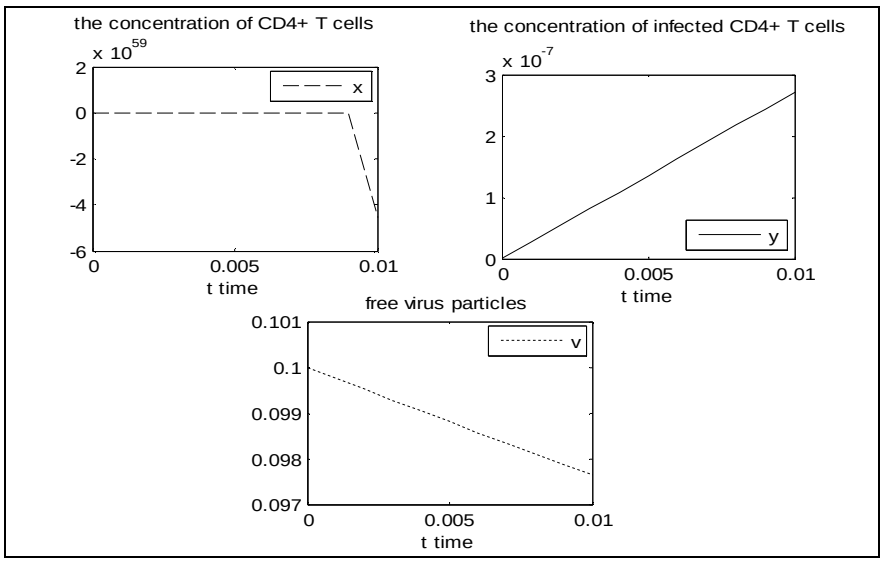

Figure 1. Plots of Padé Approximations for A Model for HIV Infection of $CD4^+$ T Cells

These results obtained by homotopy perturbation method, three, four, five and six terms approximations for *T(t)*, *I(t)*, and *V(t)* are calculated and presented follow.

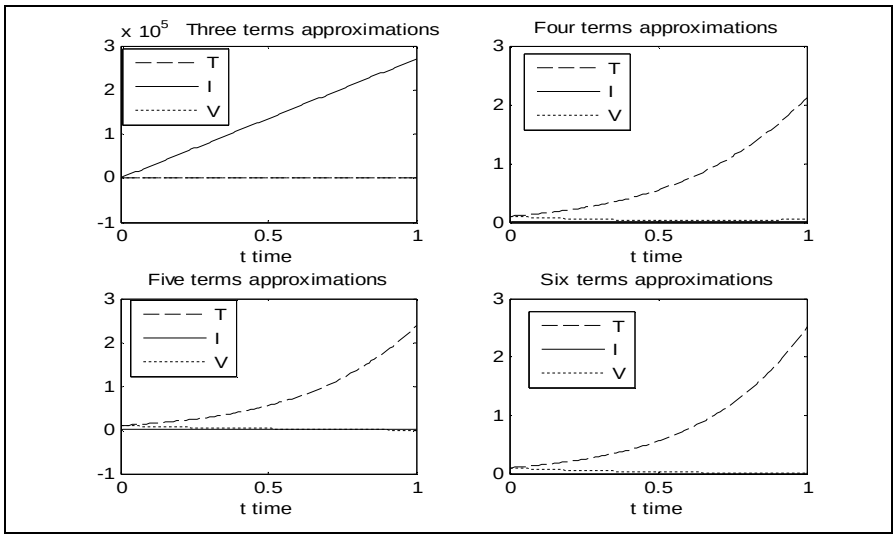

Figure 2. Plots of Three, Four, Five and Six Terms Approximations for A Model for HIV Infection of $CD4^+$ T Cells

5. CONCLUSIONS

In this paper, homotopy perturbation method was used for finding the solutions of nonlinear ordinary differential equation systems such as a model for HIV infection of $CD4^+$ T cells. We demonstrated the accuracy and efficiency of these methods by solving some ordinary differential equation systems. We use Laplace transformation and Padé approximant to obtain an analytic solution and to improve the accuracy of homotopy perturbation method. We apply He's homotopy perturbation method to calculate certain integrals. It is easy and very beneficial tool for calculating certain difficult integrals or in deriving new integration formula.

The computations associated with the examples in this paper were performed using Maple 7 and Matlab 7.

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