Notes on Fuzzy Functions and an Application in Fuzzy Control

Luiz F. J. Maia

Federal University of Santa Catarina, Florianopolis, SC, Brazil

maia@inf.ufsc.br

***Abstract*-This paper proposes fuzzy pertinence functions, fuzzy disjunctive and conjunctive functions defined in a geometrical real positive space, a mean operator defined in the (-1,1) fuzzy space, and examples of possible applications. The general class of fuzzy operators is defined in the geometrical space. The general class of pertinence functions and the mean operator makes possible the control of an inverted pendulum with only one control rule and a smooth control surface.**

Keywords-Fuzzy sets; Pertinence function; Fuzzy conjunction; Fuzzy disjunction; Fuzzy norms; Fuzzy concepts; fuzzy combinations; fuzzy control; inverted pendulum control

I. INTRODUCTION

The theory of fuzzy sets, first introduced by Zadeh [1] in 1965, coined the continuous interval (0,1) to represent the graded pertinence degree of an object to a fuzzy set. In 1980 Dubois and Prade [2] published an exhaustive research on fuzzy set theory and its applications, which is a reference on the field, with many alternatives to express pertinence functions and conjunctive and disjunctive combinations of fuzzy values.

The most used triangular and trapezoidal fuzzy pertinence functions have the inconvenient of limited range of signification and lack of discrimination between different objects with the same pertinence degree to the concept.

Dombi [4] has published many papers in membership functions and his work on De Morgan general class of fuzzy operators, which will be used in the sequence. The Dombi's operators are well expressed in the proposed geometrical space, and only in the limit are reduced to the *max* and *min* operators originally proposed by Zadeh and widely used.

The alternative continuous interval (-1,1) to represent pertinence degrees to fuzzy sets, has greater significance instead of the classical interval (0,1), where positive values indicates graded pertinence to the set and negatives values graded no pertinence, being zero the indifference point.

In fuzzy control applications, each input variable are converted in pertinence degrees to several fuzzy sets, usually triangular, and the rule base must contain a number of fuzzy rules that is equal to the products of the number of fuzzy sets defined for each input variable. The number of fuzzy rules increases the computational time to obtain the values of the control variables and results in a control surface irregular and absolutely not smooth.

II. FUZZY AND GEOMETRIC DEGREES

In either interval (0,1) or (-1,1) of the fuzzy space, if z is the fuzzy pertinence degree of an object to a fuzzy set, a transform can be defined from z to a pertinence degree in a geometric positive real space, denoted by y.

If z_A is the pertinence degree of an object to the fuzzy set A, in the fuzzy space (0, 1), then

$$z_{\neg A} = 1 - z_A$$

$$y_A = \Gamma\ z_A = z_A / \ 1 - z_A$$

and in the fuzzy space (-1,1),

$$z_{\neg A} = -\ z_A$$

$$y_A = \gamma\ z_A = \ 1 + z_A \ / \ 1 - z_A$$

The inverse transforms will be given by

$$z = \Phi\ y\ = y / \ y + 1$$

to the interval (0,1), and

$$z = \varphi\ y = \ y - 1 \ / \ y + 1$$

to the interval (-1,1).

In the geometric space, y=1 represents the point of indifference to the concept, y>1 graded agreements to the concept, and y<1 graded disagreements to the concept. The complement in the geometric space is given by

$$y_{\neg A} = 1 / y_A \ .$$

In the sequence it will be used geometric degrees to propose pertinence and combination functions. The resulting geometric pertinence degrees can be transformed into fuzzy degrees, in any of the intervals, by the transforms φ and Φ.

III. PERTINENCE FUNCTIONS

The pertinence of objects to some concepts that can be related to a mensurable geometric attribute, measured by a real positive number x, can be expressed by

$$y = \ x / Q \ ^W$$

where Q and W determines the relevant interval of the attribute in the particular context [3].

The parameter Q sets the value of x which corresponds to the indifference point to the concept, and the parameter W the range of values of x that are relevant to the concept.

DOI 10.5963/JCET0201003

As an example, Fig. 1 is the plot of the pertinence degree of persons to the concept “heavy adult person”, obtained from the weight *x* of the person, in Kg, with Q=70 and W=8.

$$z=\varphi\left[x/70\right]^{8}$$

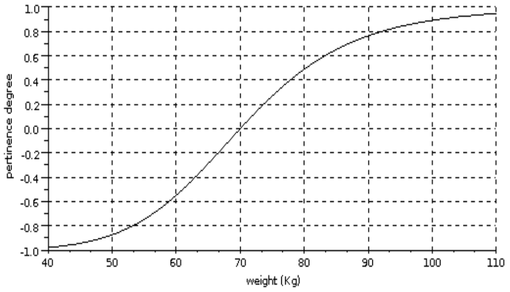

Fig. 1 Concept “Heavy adult person”

In different contexts, other concepts related to the qualifier heavy, such as “heavy molecule” or “heavy star”, can be defined by the same expression, with different values of the Q and W parameters. There are no information loss, because an inverse function can trivially be derived, transforming back the fuzzy grade *z* in the physical measure *x*.

If the physical attribute of the object related to the concept to be expressed by values *x* assuming positive and negative values in the real space, like for instance the financial balance of an entity, the pertinence function can be expressed by

$$y=\left[e^{x}\ /\ e^{q}\right]^{W}$$

where *q* and *w* are context dependent.

The defined pertinence functions are increasing with the measure of the physical attribute, with W positive, and decreasing with W negative. The parameters Q or q establishes the point of indifference and W the relevant range to the concept in the particular context.

IV. PSYCHOLOGICAL PERCEPTIONS

In measuring psychological variables, the proposed pertinence function is in agreement with Stevens' power law, as in an experiment related by Lindsay and Norman [5], on the perception of sweetness of a solution, using a confusion scaling method. A standard solution is prepared with one teaspoon of sugar dissolved in a glass of distilled water, and other ten solutions, with less and more sugar dissolved in a glass of distilled water. The subjects are asked to judge what is the sweetest of two exemplars, one being always the standard solution, presented in random order.

The results are presented in table 1, where the line labelled C presents the concentrations of the solutions, in teaspoons of sugar, and the line labelled x shows the times, in percentages, that the solution was judged sweeter than the standard. Dividing the percentages by 100 it is inferred to be the fuzzy grades of membership of the concept “sweet” in this particular context, in the scale (0, 1).

The line labelled z shows the values calculated in function of C, with Q=1 and W=5, in the same space (0, 1), by the expression $z=\varphi\left[C\right]^{5}$, with a difference inferior to 10% to the inferred values by the confused percentages.

TABLE I PERCEPTION OF SWEETNESS OF A SOLUTION

C	0,5	0,6	0,7	0,8	0,9	1	1,1	1,2	1,3	1,4	1,5
x	4	8	13	28	37	50	60	68	75	88	97
P	0,04	0,08	0,13	0,28	0,37	0,50	0,60	0,68	0,75	0,88	0,97
z	0,03	0,07	0,14	0,25	0,37	0,50	0,62	0,71	0,79	0,84	0,88

V. CONCEPT MODIFICATION

A concept can be modified by a similar expression

$$y_m=\left[y/R\right]^{S}$$

to transform the concept “heavy adult person” to concepts like “heavy star”, “heavy molecule” or “heavy child”, with appropriate parameters R and S.

The R parameter performs a translation of the curve, and the S parameter modifies the contrast, increasing if |S|>1 and decreasing if |S|<1. If S<0 the effect is the complement of the concept.

The expression of the pertinence function to a concept as a modification of the value of the pertinence function to another concept, defined in a different context but dependent of the same measurable geometric attribute, is possible because there is no loss of information in the process.

VI. FUZZY COMBINATIONS

If y_A and y_B are geometric pertinences of an object to the fuzzy sets *A* and *B*, the pertinence of the object to a combination of the sets *A* and *B* can be proposed by the general expression

$$y_{A\alpha B}=\left[y_A^{\alpha}+y_B^{\alpha}\right]^{1/\alpha}$$

defining fuzzy disjunction with α>0 and fuzzy conjunction with α<0 [4].

Proper disjunction is defined with α=1 and proper conjunction with α=-1, resulting in

$$y_{A\vee B}=\left[y_A+y_B\right]$$

an additive aggregation, and

$$y_{A\wedge B}=\left[y_A^{-1}+y_B^{-1}\right]^{-1}$$

an harmonic aggregation. The general expression is commutative and associative but not idempotent.

The fuzzy disjunction will be greater than any of their arguments, and the fuzzy conjunction will be less than any of their arguments. In the limits of plus and minus infinity to α, the function reduces to the *max* and *min* operators originally proposed by Zadeh [1], a particular and extreme case of the general expression. For all k>0, the disjunctive function with α=k and the conjunctive function with α=--k satisfies De Morgan's laws.

$$\neg\left(A\cup B\right)=\neg A\cap\neg B$$
$$\neg\left(A\cap B\right)=\neg A\cup\neg B$$

DOI 10.5963/JCET0201003

VII. COMPOUND CONCEPTS FALLACY

In 1981, Osherson and Smith [6], concluded that the prototype theory is not a valid theory of concepts because some conceptual combinations, when expressed by conjunctive or disjunctive fuzzy operations, leads to results that do not agree with common intuition.

Their example of the concept (financial) wealth as a disjunction of liquidity and investment is in perfect agreement with common intuition if used the proper disjunctive combination function, not the *max* operator, considered by them as the only possible operation in the theory of fuzzy sets.

The flaw in their interpretation of some compound concepts as disjunctions, as in “pet fish”, a disjunction of “pet” and “fish”, or in “stripped apple”, as a disjunction of “stripped” and “apple”, lead to wrong conclusions. Some compound concepts cannot be interpreted as a disjunction, like “lead soldier”, where good exemplars of the compound concept will be poor exemplars of the concept “soldier” or even as a leading object.

If some theory don't agree with the reality, it's not the reality that is wrong, but the theory that intends to model the reality.

VIII. FUZZY MEAN

In the fuzzy sets A, B and C, if A induces C, A→C (¬A→¬C) and B also induces C, B→C (¬B→¬C), the influences of A and B on C can be compensated, when a positive pertinence to A and a negative pertinence to B can result in a null pertinence to C.

In the fuzzy interval (-1,1), if z_A and z_B are the pertinence degrees of some object to the fuzzy sets A and B, the pertinence degree of the object to the fuzzy set C can be expressed by

$$z_C = 1/2 \;\; z_A + z_B$$

the arithmetic mean, that can be generalized for the grades of satisfaction z_i of *n* concepts that contributes to the satisfaction of the compound concept H by

$$z_H = 1/n \; \Sigma z_i$$

The weighted mean can be produced by assigning positive weights S_i to the z_i by concept modifications

$$\gamma \;\; z_i^{\;S_i}$$

increasing the contribution of z_i, with $S_i>1$ and decreasing the contribution when $S_i<1$.

If a new concept z_{n+1} is added to influence the new concept H*, the grade of satisfaction to this new concept will be

$$z_H = [1/ \; n+1 \;] \; n.z_H + z_{n+1}$$

It is trivial to show that the defined fuzzy mean is independent of the order of its constituents, despite their more complex equivalent expressions in the (0, 1) fuzzy space or in the geometric space.

IX. CLASSICAL FUZZY CONTROL

Fuzzy control is an alternative to control complex non-linear systems, and it was object of intense research in the past decades, with many publications like the work of Passino and Yurkovich [7], always with the same steps of fuzzyfication of the input variables, the application of control rules of a rule base and a defuzzyfication last step to produce the values of the control variables.

In the control of an inverted pendulum that is conducted by a cart that moves in the same oscillation plane of the pendulum, the target is to move the cart to balance the pendulum in the vertical position. The input variables are the angular deviation of the pendulum in relation to the vertical and its angular velocity, and the output variable is the intensity of the force that might be pushed upon the cart.

Taking as positive the direction of deviation in relation to the vertical and the angular velocity both to the right, or clockwise, and also as positive the force applied to the cart to the right, the classical approach proceeds by defining fuzzy sets to the input and output variables, like for instance, [*positive large; positive medium; positive small; zero; negative small; negative medium; negative large*], defined normally as triangular fuzzy sets.

The fuzzyfication process transforms the numerical values of the input variables to pertinence degrees in the defined fuzzy sets.

The rule base, as defined in Passino and Yurkovich [7], consists of control rules like:

IF deviation is *positive medium* **AND** velocity is *negative small* **THEN** force is *positive small*.

The antecedent of every rule is evaluated and the result is the level of activation of the fuzzy set of the consequent.

The activations of the fuzzy sets are combined in an appropriate defuzzyfication procedure which results in a numerical value to the output variable.

In the case of the example, with 7 fuzzy sets defined for each of the variables, the rule base will contain 49 control rules, and the obtained control surface is not smooth, according to the authors former cited.

X. PROPOSED FUZZY CONTROL

Being *d* the pendulum deviation, in radians, and *v* its velocity, in radians per second, the pertinence functions for the concepts “positive deviation” and “positive velocity” can be expressed, with q=0 and W=16 for both, as:

$$y_d = [\exp \; d \;]^{16}$$

$$z_d = \varphi \;\; y_d$$

for the deviation, and

$$y_v = [\exp \; v \;]^{16}$$

$$z_v = \varphi \;\; y_v$$

for the velocity.

DOI 10.5963/JCET0201003

Fig. 2 shows the pertinence function to the concept "positive deviation", being identical to the pertinence function to the concept "positive velocity".

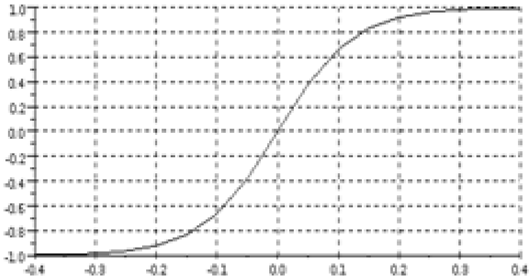

Fig. 2 Concepts "Positive deviation" (rad) and "Positive velocity" (rad/s)

The control can be established with only one control rule

IF deviation is positive **AND** velocity is positive,

THAN force is positive.

The AND in the control rule must not be interpreted as the intersection of the fuzzy values of deviation and velocity, since there must be pairs of values of positive deviation and negative velocity which will result in a null force. Then the aggregation of the inputs must be the mean of the pertinence degree of the input variables to the concepts "positive deviation" and "positive velocity", defining the force that must be applied to the cart, in the range (-1,1). The force applied to the cart must be an aggregation of the deviation and velocity.

The control rule can then be stated as

IF deviation is positive, **THEN** force is positive,

and also

IF velocity is positive, **THEN** force is positive.

The result is showed in Fig. 3, which presents a smooth control surface, without the irregularities observed in the classical approach.

The obtained control surface is possible only with the pertinence functions defined, which preserve the original information of the values of the input variables. In the classical approach [7], similar control surfaces only can be reached with many fuzzy sets for the input variables, with an exponential growth of the rule base and a very large computational cost in the fuzzyfication and defuzzyfication operations.

Dynamic adjustments can be made in the *W* parameters of the input variables, to weight their relative importance, and in the multiplicative factor of the output variable, to fit the real dynamic properties of the system.

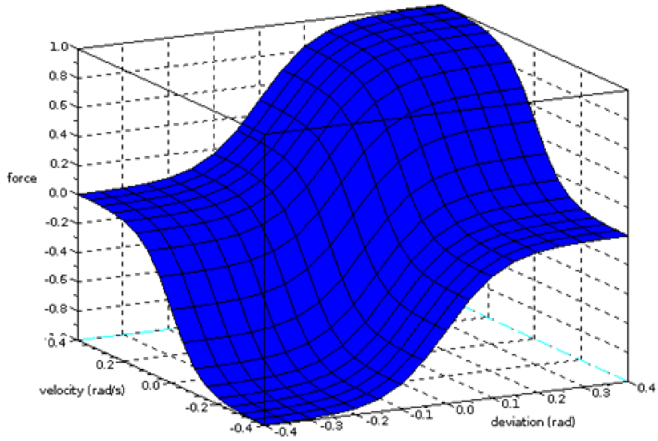

Fig. 3 Control Surface for the Inverted Pendulum

XI. CONCLUSIONS

The geometric pertinence space simplifies the expression of De Morgan's class of fuzzy operators proposed by Dombi [4], and is also very useful for the characterization of pertinence functions, in accordance with psychological perceptions.

The pertinence functions maintain all the information of the values of the input variables *deviation* and *velocity* of the inverted pendulum, allowing their control with only one control rule.

The fuzzy mean, defined in the fuzzy interval (-1,1), can implement fuzzy control in very simple and more effective approach than the classical procedures as described in [7].

The control system can be implemented with analogical electronic devices, affording a true real-time control system.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets", *Inf. Control* vol. 8, pp. 338-352, 1965.

[2] D. Dubois and H. Prade, *Fuzzy Sets and Systems*, Academic Press, 1980..

[3] L. F. J. Maia, "Concepts characterization and recognition", Dr. Eng. thesis, State University of Campinas, SP, Brazil, 1991.

[4] Dombi: "A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators", *Fuzzy Sets and Systems*, vol. 8, pp. 149-163, 1982.

[5] P. H. Lindsay and D. A. Norman, *Human Information Processing,* 2nd ed., Academic Press, 1977.

[6] K. M. Passino and S. Yurkovich, *Fuzzy Control*, Addison-Wesley, 1997.

Luiz F. J. Maia was born in Bagé, RS, Brasil, in 1939. He obtained a degree in Electrical and Civil Engineering by the Federal University of Rio Grande do Sul, in 1962, a Master Degree in Electrical Engineering by the Federal University of Rio de Janeiro, in 1968, and a Doctoral Degree in Electrical Engineering by the State University of Campinas, SP, in 1991.

He was Professor in several Universities in Brazil and does consulting work for several institutions. As a retired researcher in the Federal University of Santa Catarina, Brazil, his present major interest is in fuzzy computational models.

DOI 10.5963/JCET0201003