Stabilization of a Nonlinear Fluid Structure Interaction via Feedback Controls and Geometry of the Domain

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***Abstract*- Asymptotic stability of *finite energy solutions* to a fluid-structure interaction with a static interface in a bounded domain $\Omega\in\mathbb{R}^n$, n=2, is considered. Nonlinear interior damping and dynamic and static boundary damping are exploited to stabilize the system. It is shown that the undamped model subject to "partial flatness" geometric condition on the interface produces solutions whose *energy* converge *strongly* to zero; while with a stress type feedback control applied on the interface of the structure, the model produces solutions whose *energy* is exponentially stable. An addition of a static damping on the interface produces solutions whose *full* norm in the phase space is exponentially stable. Without a static damping, an interesting phenomenon occurs: steady state solutions (equilibria) might generate genuinely growing in time solutions. This purely nonlinear phenomenon is captured by newly developed techniques amenable to handle instability of steady state solutions arising from nonlinearity.**

Keywords*- *Fluid Strusture Interaction; Interface Control; Navier-Stokes Equation; System of Elasticity; Feedback Boundary Control; Strong Stability; Uniform Stability; Optimal Control; Passive Damping; Active Damping; Dynamic and Static Damping

I. INTRODUCTION

A. *Description of the Problem*

We consider fluid-structure interaction described by a coupled system of partial differential equations (PDEs) comprising of a nonlinear Navier-Stokes equation and a system of elasticity of wave equation. The coupling between two systems occurs on the boundary-interface between two environments: fluid and a solid. This model is well established in the literature and has numerous engineering applications that range from naval and aerospace engineering to cell biology and biomedical engineering [36], [12] [19] [15] [20] [14] and references therein.

However, due to mismatch of regularity between the particular hyperbolic component (dynamic system of elasticity) and parabolic component (fluid) the basic mathematical questions such as well-posedness of *finite energy* physical solutions had not been resolved until recently [8] [9] [17] [18] [13]. In this article, being empowered with the existence theory of finite energy solutions, we address the problem of asymptotic stability of *finite energy* solutions when time *t* goes to infinity.

The main goal is to reduce/control vibrations/oscillations of the body submersed into the fluid.

Our aim is to discuss various types of asymptotic stability which depend on topological and geometric considerations. We will show that stronger (topologically) results are obtained when the "damping" cooperates with the geometry of the domain. This is to say that on the part of the domain where damping is inactive, the geometry of the domain should cooperate with anti-reflection of waves causing natural absorption of the energy. It turns out that geometric properties such as *convexity* and *flatness* are of critical importance. Strong, uniform and exponential stability will be described quantitatively as a function of geometry of the solid body, viscosity of the fluid, and the placement of the interior and boundary damping.

B. *The Model*

The model is defined on a bounded domain $\Omega\in\mathbb{R}^n$, n=2, that describes the interaction between an elastic body and a surrounding incompressible viscous fluid. Ω is a bounded simply connected domain, consisting of two open sub-domains Ω_s and Ω_f, where Ω_f is the exterior domain filled with fluid and Ω_s is the interior domain occupied by the elastic solid. The interaction between the fluid and the solid occurs at the interface Γ_s, the boundary of Ω_s. The boundary of Ω is denoted by Γ_f.

The dynamics of the fluid are described by the nonlinear Navier-Stokes equation and the dynamics of the elastic body is described by an elasto-dynamic system of wave equation. $u(t;\, x)\in R^n$ is a vector-valued function representing the velocity of the fluid and $p(t;\, x)$ is a scaler-valued function representing the pressure vector on Γ_s with respect to the region Ω_s. $w(t;\, x); w_t(t;\, x)\in R^n$ denotes the displacement and the velocity functions of the elastic solid Ω_s. $\vec{n}$ denotes the unit outward normal vector on Γ_s with respect to the region Ω_s.

See Fig. 1.

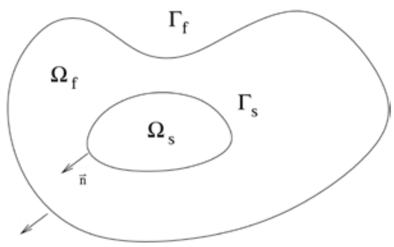

Fig. 1 Geometry of Ω

This leads to the following interactive PDEs defined for the state variables $[u; w; w_t; p]$ [34]:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in}\Omega_f \times (0,\infty) \\ \operatorname{div} u = 0 & \text{in}\Omega_f \times (0,\infty) \\ w_{tt} - \Delta w + \rho g(w_t) = 0 & \text{in}\Omega_s \times (0,\infty) \\ \frac{\partial w}{\partial n} + \alpha w = \frac{\partial w}{\partial n} - p\vec{n} + \frac{1}{2}(u \cdot \vec{n})u \text{on} \Gamma_s \times (0,\infty) & \\ u = w_t + \beta F(w) \text{on} \Gamma_s \times (0,\infty) & \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(0,\cdot) = u_0 \text{in}\Omega_f & \\ w(0,\cdot) = w_0, w_t(0,\cdot) = w_1 \text{in}\Omega_s & \end{cases} \quad (1)$$

where the constants $\alpha, \beta \geq 0$ are arbitrary nonnegative constants.

$F(w) \in L2(0;\infty; L2(\Gamma_s))$ represents feedback boundary control (to be specified later). $\rho(x)g(w_t)$ represents an interior nonlinear damping and the functions $\rho(x)$ and $g(s)$ satisfy the following assumptions

Assumption 1.

(1) $\rho(x) \geq 0$ for almost every x in Ω_s.

(2) $g(s)$is monotone and continuous;

(3) There exists constants $m, M > 0$ such that

$$m|s|^2 \leq g(s) \cdot s \leq M|s|^2, \qquad \text{for}|s| > 1$$

(4) $g(0) = 0$.

Remark 1. The interior damping $\rho(x)g(w_t)$ represents nonlinear effects of the friction impacting the vibrations of the solid. Note that Assumption 1 does not imply that this term will have any stabilizing effect on the structure. For instance, if$\rho(x) = 0$ a.e. in Ω_s, the interior damping does not provide any dissipation of the energy. The stability analysis depends heavily on the strength of the *boundary feedback controls* and the *geometry of the domain*.

The model considered accounts for small but rapid oscillations of the elastic displacements [14]. This allows one to assume that the interface is static. The main goal of this article is to establish the best possible asymptotic stability result of System (1) under various geometric configurations of the domain and constraints imposed on the control actions. Actuators are typically implemented via the appropriate frictional mechanisms applied to the boundary or on the interface. Two notions are commonly used to describe the long time behavior of a dynamical system (or a C_0 −semigroup describing the semi-flow). These are *strong stability* and *uniform stability*.

The latter is typically associated with the rates of convergence to equilibria. To recall, we say that a semigroup $S(t)$ defined on a Hilbert space H is *strongly stable* iff for all $x \in H$ we have$|S(t)x|_H \to 0$ when $t \to \infty$. A much stronger notion *uniform stability* refers to the property that the operator norm $\|S(t)\|_H \leq f(t) \to 0, t \to \infty$ where$f(t)$ is a (continuous) one-variable function. PDE systems with stronger dissipation/damping often yield uniform stability, while PDE systems with weaker dissipation yield strong stability only. It is not surprised that with the full strength of the boundary damping $(\alpha, \beta > 0, \rho \geq 0)$, a uniform exponential decay rate of the full statecould be obtained. Reducing the strength of the boundary damping$(\alpha, \beta = 0 \; or \; \alpha = 0, \beta > 0)$brings about complexity of stability analysis. Indeed, for the undamped model (with $\alpha, \beta = 0, \rho \geq 0$), the dissipation propagated from the Navier-Stokes equation to the wave equation is “ too weak” and it does not affect the boundary normal displacement of the solid, which is known to be critical for uniform stability of wave dynamics [25]. Uniform stability of the entire coupled system is thus impossible. In such configuration strong stability result is the best one can hope for. However, boundary damping alone does not suffice to achieve strong stability of the full state. It is here where we have to encompass the geometry of the interface Γ_s. Appropriate geometric conditions on Γ_s have to be identified in order to deal with lack of strength of the dissipation.

C. Applications, Motivation and Challenges

Fluid structure interaction, as described above, has been an active area of research in mathematics, physics and engineering, with applications ranging from cell biology, biomedical engineering and naval and space engineering [19] [12] [20]. Examples include a submarine submerged in the water or cells in the human body fluid:

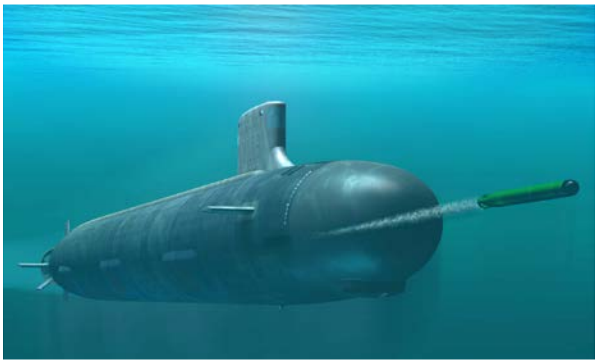

Fig. 2 Submarine – An example of Fluid-structure interaction

The major mathematical difficulty stems from the mismatch between the boundary regularity of the hyperbolic wave equation and the parabolic Navier-Stokes equation, which does not provide sufficient regularity for the boundary traces. In dealing with this particular difficulty, several strategies have been developed in earlier mathematical literatures where either a structural damping is added to the wave equation or a very smooth local-in-time solution were considered. Only recently the existence, uniqueness (in two dimension), of the solutions in the *natural energy level* were shown to hold [8]. This was accomplished by taking advantage of recently discovered hyperbolic trace theory [27] applied on the interface of the structure. Regularity of weak solutions was subsequently developed in [9], and also in [16], [17] [18] for a slightly different topological setting. Smooth solutions with moving interface have been analyzed in [13].

In the context of stability, stability results are available for the*linearized* model with the presence of pressure: strong stability in [1] [2] [4] where geometric dependency is first discovered; exponential decay rate with additional boundary damping in [2] [3]. The main tool used to establish the strong stability results for linear models is spectral theory, which has no extension to nonlinear models. Our main challenge is to develop new approaches and new tools adequate for the treatment of stability analysis in the *nonlinear models*.

Thus, *nonlinearity* and the presence of the *pressure* term in the fluid equation are two main new aspects and challenges of the analysis. First stability results obtained for a *nonlinear* version of the structure in (1) are in [28] [29] [30]. The present paper extends and generalizes these stability results by accounting for *nonlinear effects in the modeling of the structure*. The effectiveness of the overall damping mechanism is analyzed in the context of geometric configurations of the solid.

II. PRELIMINARIES

Before introducing the main results, we will review some definitions and preliminary facts pertinent to the subsequent analysis.

A. Phase Space and Energy Functional

As in [8], we define the following key spaces:

$$H \equiv \left\{u \in \left[L_2(\Omega_f)\right]^2 : \operatorname{div} u = 0\right\}, V \equiv H \cap [H^1_{\partial\Omega}(\Omega_f)]^2$$

and the finite energy space for state variables $[u; w; w_t]^T$:

$$\mathcal{H} \equiv H \times [H^1(\Omega_s)]^2 \times [L_2(\Omega_s)]^2$$

where $H^1_{\partial\Omega}(\Omega_f)$ denotes $H^1(\Omega_f)$, Sobolev space with zero boundary conditions imposed on the boundary $\partial\Omega$.

The following (standard) notations will be used:

$$(u,v)_f = \int_{\Omega_f} uv\, d\Omega_f, (u,v)_s = \int_{\Omega_s} uv\, d\Omega_s,$$

$$\langle u,v\rangle_s = \int_{\Gamma_s} uv\, d\Gamma_s,\ (u,v)_{1,f} = \int_{\Omega_f} \nabla u \cdot \nabla v\, d\Omega_f,$$

$$|u|_{\propto,D} = |u|_{H^{\propto}(D)}, |u|_{0,f} = |u|_{0,\Omega_f},$$

$$Q_s \equiv (0,T] \times \Omega_s; Q_f \equiv (0,T] \times \Omega_f;$$

$$\Sigma_s \equiv (0,T] \times \Gamma_s; \Sigma_f \equiv (0,T] \times \Gamma_f;$$

B. Existence, Uniqueness and Regularity of Finite Energy Solutions

Motivated by feedback stabilization results for the pure wave equation [24] [39] [25] [23], a natural feedback to consider is in a form of a porous force acting on the interface Γ_s and given by:

$$F(w) \equiv \frac{\partial w}{\partial n} + \alpha w, \alpha \geq 0 \tag{2}$$

Projecting the equations on H and utilizing the boundary conditions allows us to define weak solutions of the PDE system.

Definition II.1. (Weak Solution.) Let $(u_0; w_0; w_1) \in \mathcal{H}$ and $T > 0$. We say that a triple $(u; w; w_t) \in C_w([0; T]; \mathcal{H})$ is a weak solution of (1) with the feedback control given by (2) iff $(u; w; w_t)$ satisfies variational form of the original PDEequation a.e. in $t \in (0; T)$

$$(u_t,\phi)_f + \langle\frac{\partial w}{\partial n} + \alpha w, \phi\rangle + (\nabla u, \nabla\phi)_f + \big((u\cdot\nabla)u,\phi\big)_f - \langle\frac{1}{2}(u\cdot\vec{n})u,\phi\rangle = 0, \qquad \forall\phi \in V \tag{3}$$

$$(w_{tt},\psi)_s - \langle\frac{\partial w}{\partial n},\psi\rangle + (\nabla w,\nabla\psi)_s + (\rho g(w_t),\psi)_s = 0, \qquad \forall\psi \in [H^1(\Omega_s)]^n \tag{4}$$

We recall some results describing well-posedness and regularity of finite energy solutions. Global-in-time existence of the weak solutions is obtained in [8] for $\rho = 0$. When $\rho \geq 0$, the argument in [8] could be supplemented by taking advantage of monotone operator theory in line with [32]. The Assumption 1 plays an essential role for this generalization.

Theorem II.2. *(Existence and uniqueness of weak solutions [8]) Assume that Assumption 1 is in force. Given any initial condition $(u_0, w_0, w_1) \in \mathcal{H}$ and any $T > 0$, there exists unique weak solution $(u, w, w_t) \in C_w([0,T],\mathcal{H})$ to the System (1) with the following additional properties:*

1) $u \in L_2(0,T;V), u_t \in L_2(0,T;V')$;
$$w_{tt} \in L_2(0,T;[H^1(\Omega_s)^n]'),$$
$$u\mid_{\Gamma_s} = w_t\mid_{\Gamma_s} + \beta\left(\frac{\partial w}{\partial n} + \alpha w\right)\mid_{\Gamma_s}$$
2) *When* $\beta = 0$,
$$\frac{\partial w}{\partial n} \in L_2\left((0,T);\left[H^{-\frac{1}{2}}(\Gamma_s)\right]^n\right),$$
$$w_t\mid_{\Gamma_s} \in L_2\left((0,T);\left[H^{\frac{1}{2}}(\Gamma_s)\right]^n\right),$$
3) *When* $\beta > 0$,
$$\frac{\partial w}{\partial n} \in L_2\big((0,T);[L_2(\Gamma_s)]^n\big)$$
$$w_t\mid_{\Gamma_s} \in L_2\big((0,T);[L_2(\Gamma_s)]^n\big),$$

Moreover, the said solution depends continuously on the initial data (with respect to the topology induced by $\mathcal{H}$)

Remark 2. We note that the definition of weak solutions does not require test functions ψ to satisfy typical [14] compatibility conditions on the interface Γ_s . This is possible due to established in [8] sharp regularity of the normal derivatives of the displacement w (see the third bullet in the definition). As a consequence, the variational form of the equation is amenable to *numerical approximations by using Finite Element Methods*.

Additional regularity including differentiability of weak solutions is asserted in [9] (see also [16] [17] for different topological configuration)

Theorem II.3. *(Regularity [9]) Assume that Assumption 1 is inforce and that $g \in C^1(R)$. Let $(u_0, w_0, w_1) \in \left[H^2(\Omega_f)\right]^n \cap V \times [H^2(\Omega_s)]^n \times [H^1(\Omega_s)]^n$ satisfy the usual boundary compatibility conditions imposed on the boundary.*

Then, for any T > 0 we have:

1) $(u,p) \in L_2\left((0,T);\ \left[H^2(\Omega_f)\right]^n \times H^1(\Omega_f)\right)$
2) $(u_t, w_t, w_{tt}) \in L_\infty\left((0,T);\mathcal{H}\right)$,
   $w \in L_\infty\left((0,T);[H^2(\Omega_s)]^n\right)$.

Theorem II.2 and Theorem II.3 were proved in [8] [9] with the parameter $\alpha,\beta = 0$. However, the same proof can be carried out when $\alpha > 0, \beta > 0, \rho \geq 0$ or $\alpha = 0, \beta > 0$, $\rho \geq 0$. In this process, Assumption 1 along with differentiability of g play critical role.

C. Energy Functional and Energy Identity

Let $u; w$ be regular solutions obtained in Theorem II.3. Choose test functions $\phi = u$ and $\psi = w_t$ in the weak Formulations (3)-(4). Noticing cancellation occurring in nonlinear term and utilizing the transmission condition $u = w_t + \beta\left(\frac{\partial w}{\partial n} + \alpha w\right)$ on Γ_s , one obtains the following energy identity for $0 \leq s \leq t$,

$$E_\alpha(t) + \int_s^t \left[|\nabla u|^2_{0,\Omega_f} + \beta \left| \frac{\partial w}{\partial n} + \alpha w \right|^2_{0,\Gamma_s} + (\rho g(w_t), w_t)_{0,\Omega_s} \right] d\tau = E_\alpha(s) \tag{5}$$

where $E_\alpha(t)$ is the energy functional defined as

$$E_\alpha(t) \equiv \frac{1}{2}\left[|u_t(t)|^2_{0,\Omega_f} + |\nabla w(t)|^2_{0,\Omega_s} + |w_t(t)|^2_{0,\Omega_s} + \alpha\left|w(t)\right|^2_{0,\Gamma_s}\right] \tag{6}$$

The energy identity (5) reveals that there are three potential sources of dissipation: one propagated from the Navier-Stokes equation, one from the interior damping $\rho g(w_t)$ (since g is assumed to be monotone, $(\rho g(w_t), w_t) \geq 0$) and the last from the boundary dynamic damping$\beta F(w)$. With the presence of the dynamic damping, dissipation also has impact on the boundary normal displacement of the solid. Hence, a (uniform) exponential decay rate could be expected if $\beta > 0$.

When $\alpha = 0$, the energy functional $E_{\alpha=0}(t) = E_0(t)$ provides the total *energy* of the system, and is only a *semi-norm* on the phase space $\mathcal{H}$. Thus, there could be zero energy solutions which might have non zero displacement of the solid. On the other hand, with the presence of the static damping ($\alpha > 0$), the energy functional determines a full norm on the phase space $\mathcal{H}$, a result of Poincare's in equality and trace theory. How to eliminate nonzero steady states will be a major issue we have to deal with when $\alpha = 0$.

Denote the dissipation terms in (5) as

$$D(t) = |\nabla u|^2_{0,\Omega_f} + \beta\left|\frac{\partial w}{\partial n} + \alpha w\right|^2_{0,\Gamma_s} + (\rho g(w_t), w_t)_{0,\Omega_s}$$

Then, the energy identity can be rewritten as

$$E_\alpha(t) + \int_s^t D(\tau)d\tau = E_\alpha(s),\ \ 0 \leq s \leq t \tag{7}$$

D. Main Results

Since the interior damping might not provide dissipation of energy, the stability results obtained in this paper depend on the strength of the boundary controls and the geometry of the domain and can be summarized as follows:

- If $\alpha, \beta = 0$, that is when the dissipation is the weakest, by imposing geometric conditions such as partially flatness on the interface Γ_s, the energy $E_0(t)$ will decay to zero. The System (1) is strongly stable but there is no rate of decay. In addition, there is no information on stability of elastic displacement. ($E_0(t)$ accounts only for gradients of the displacement).
- If $\alpha > 0, \beta = 0$, (i.e. static damping $\alpha > 0$ is added to the model),the result in point above strengthens up to yield strong stability of the *full elastic state* which includes also the displacement.
- If $\alpha, \beta > 0$, that is when the dissipation is the strongest, without assuming any geometric conditions on the interface Γ_s , the full state $|Y(t)|_{\mathcal{H}}$ for any $Y(t) \equiv (u(t), w(t), w_t(t)) \in \mathcal{H}$ decays to zero at a uniform exponential rate. The System (1) is uniformly exponentially stable;
- If $\beta > 0, \alpha = 0$, assuming the star-shaped geometric condition on Γ_s, the energy $E_0(t)$ decays to zero at a uniform exponential rate. The energy of the System (1) achieves uniform exponential stability by relaying on added boundary friction and suitable geometry of the domain. However, there is no information on the decay of the displacement of elastic body. For the latter, static damping $\alpha > 0$ needs to be added.

Detailed statements of our main results are given in the following theorems. We first formulate the following Geometric Assumption.

Assumption 2. (a) Γ_s contains a flat portion Γ_0 with positive measure;

Theorem II.4. (Strong Stability of Energy) *Let $\alpha = 0, \beta > 0$,*

n = 2. In addition to Assumption 1, impose geometric Assumption 2. Then, for any initial data $(u_0, w_0, w_1) \in \mathcal{H}$, one has that the energy functional for the system (1) tends to 0 as $t \to \infty$. This is tosay:

$$E_0(t) \to 0,\ as\ t \to \infty \tag{8}$$

Theorem II.5. (Strong Stability of Full State) *Let $\alpha > 0, \beta > 0$, n = 2. In addition to Assumption 1, impose geometric Assumption 2. Then, for any initial data $(u_0, w_0, w_1) \in \mathcal{H}$, one has that the full state for the system (1) tends to 0 as $t \to \infty$. This is to say:*

$$|u(t), w(t), w_t(t)|_{\mathcal{H}} = |Y(t)|_{\mathcal{H}} \to 0, when\ t \to \infty \tag{9}$$

Theorem II.6. (Exponential Decay Rates of Full State).

With reference to the model introduced in (1) under Assumption 1 and dynamic damping $F(w)$ specified in (2), let dim Ω=2 and let $\beta > 0, \alpha > 0$. For any initial data $(u_0, w_0, w_1) \in \mathcal{H}$, we have that weak solutions to (1) satisfy:

$$|u(t), w(t), w_t(t)|_{\mathcal{H}} = |Y(t)|_{\mathcal{H}} \leq C\left(E(0)\right)e^{-\omega t} \tag{10}$$

for some constant $\omega > 0$ and all $t \geq 0$.

Theorem II.7. (Exponential Decay Rates for the Energy).

Let $\beta > 0$and $\alpha = 0$, n = 2. In addition to Assumption 1, assume the additional geometric condition:

Assumption 3. $h \cdot \vec{n} > 0$ with $h(x) \equiv x - x_0$, where $\vec{n}$is the unit outward normal vector of Γ_s;

Then, there exist constants $M \geq 1$ and δ>0, such that

$$E_0(t) \leq Me^{-\delta t} E_0(0), \forall t \geq 0$$

Remark 3. The dimensionality of the domain is restricted to two for different reasons. For strong stability, the argument depends on the locally Lipschitz property of the flow on the phase space $\mathcal{H}$, which does not hold true in three dimensional space. For uniform stability, the method used does not depend on the dimensionality of the domain. However, when n = 3, weak solutions are not known to be unique, thus the decay rates obtained for strong solutions only can not be extended to all weak solutions. In that case the result remains valid for smooth solutions which are global (e.g. corresponding to small initial data -as shown in [9]).

E. Discussions of the stability results

- **Geometry dependence.** An interesting aspect of the stability results is geometry dependence. Geometric conditions listed in Assumption 2 are critically used to handle the buildup of pressure on the normal direction of the interface. Condition (a) in Assumption 2, the partially flatness condition, is compatible with what discovered for the linearized model [1]. Condition (b) in Assumption 2, though generically true, is new and essential to nonlinear aspects of the model. Both conditions are violated for a highly symmetric domain, for example, a two dimensional disc. We should point out that partially flatness condition is only one of the effective geometric conditions dealing with the buildup of the pressure. The star shaped geometric condition (Assumption 3) is crucial to uniform stability when only the dynamic damping is present on the boundary. It allows to get rid of tangential propagation of the energy near the boundary. This is reflected by critical boundary inequalities in the key PDE estimate.

- **Blow-up of solutions generated from steady states - the role of the static damping** Nonlinearity of the system introduces the most substantial difficulty into the problem. For the linearized model, spectral theory allows to decompose the phase space into a one dimensional unstable subspace containing the steady states solutions and a stable subspace. The stability analysis could thus be done on the 'mode-out' part of the phase space invariant under the dynamics. However, for nonlinear model, such route is inapplicable. In fact, due to the mixing of nonlinear terms [22], initial steady states might blow up in time eventually. Therefore, stability results for energy only with the absence of the static damping is the best under the given circumstances. This phenomenon could be illustrated by considering the following elementary example:

Example.

$$\begin{cases} w_{tt} - \Delta w + g(w_t) = 0, & \text{in}\Omega \\ \dfrac{\partial w}{\partial n} = 0, & \text{on}\Gamma \equiv \partial\Omega \end{cases} \tag{11}$$

where Ω is a bounded domain in an Eclidean space and g is a continuous and monotone increasing function.

The energy identity is given by

$$\tilde{E}(t) + \int_0^t \int_\Omega g(w_t) w_t d\Omega ds = \tilde{E}(0)$$

with the energy functional $\tilde{E}(t)$ defined as

$$\tilde{E}(t) = \frac{1}{2}[|\nabla w(t)|^2 + |w_t(t)|^2]$$

Clearly, $w(t) \equiv C, C$: a constant, is a zero energy solution. However, it might generate solutions that will eventually blow up.

Let $g(s) = s^p (p \neq 1)$. Suppose $w(t,x) \equiv Cf(t)$ is a solution of (11). Then, $f(t)$ satisfies the equation

$$Cf_{tt} + g(Cf_t) = 0$$

Solving the equation yields

$$f(t) = C^{-1}\left[C_p(t + c_1)^{\frac{p-2}{p-1}} + c_2\right], p \neq 1$$

Thus,

$$w(t,x) = C_p(t + c_1)^{\frac{p-2}{p-1}} + c_2, p \neq 1$$

$w(t,x)$is a finite energy solution if $p > 1$, since $w_t(t,x) \sim t^{-\frac{1}{p-1}} \to 0$ as $t \to \infty$, thus $\tilde{E}(t) \to 0$ as $t \to \infty$. However, if $p > 2$, the solution will eventually blow up: $u(t,x) \sim t^{\frac{p-2}{p-1}} \to \infty$, as $t \to \infty$.

It is at this point when static damping $\alpha > 0$ becomes critical. It "stabilizes" asymptotically steady states which may blow up at infinite time due to the nonlinearity. This is confirmed by the results in Theorem II.6 and Theorem II.5.

- **Our approach and the strategy.** To cope with the instability of solutions generated from steady states, we will apply methods used for nonlinear wave propagation. More specifically, transformation of dynamics for strong stability exploited in [31] and use of a special multiplier for uniform stability [21] are critical ingredients. For strong stability of the undamped model, following the procedure introduced in [31], we will first transform the original system into equivalent first order system where the energy functional will then become a full norm on the transformed phase space. For uniform stability when $\alpha = 0, \beta > 0$, we will apply a special multiplier partitioning potential and kinetic energy which enable us to dispense with the lower order terms that appear in an intermediate estimate. This step is inspired by the method presented in [22].

III. STRONG STABILITY WITHOUT ANY FRICTIONAL DAMPING

In this section, we will establish strong stability for theundamped model $(\alpha \geq 0, \beta = 0)$.We will show that the total energy, kinetic and potential energy $E_0(t)$ decays to zero when timegoes to infinity. We should point out that since we do not assume $\rho > 0$ in general, we do not account for dissipative effect of mechanical damping due to the nonlinear term $g(w_t)$. The effective source of energy dissipation which drives the entire system to stability is the one propagated from the Navier-Stokes equation.

A. Overview

The model without the frictional damping possesses a few distinct features in the context of strong stability: (a) the dissipation is *weak*; (b) the resolvent operator is *not* compact; (c) the dynamics is partially hyperbolic. These features render the standard tools used for the study of strong stability of nonlinear systems not applicable in the present situation. Indeed, a classical tool is LaSalle's Invariance Principle [35]. A key hypothesis assumed by this principle(and its variants) is the compactness of the orbits, often secured by the compactness of the resolvent of the semigroup generated by the flow. However, this latter property, while typical in parabolic flows, does not hold in hyperbolic-like dynamics, e.g. the wave equation component in the system we consider. Some known nonlinear methods [6] [7] [10] for studying asymptotic stability require one of the following conditions to be satisfied: (i) semigroup associated with the linearization be "smoothing" (parabolic like situation), or (ii) the nonlinear generator be m-monotone, or (iii) linearization be exponentially stable , or (iv) linear generator be monotone and nonlinear perturbation *weakly sequentially compact*. In the case of the model under consideration neither of these options is available.

The approach we develop is motivated by a *relaxed* version of LaSalle's Invariance Principle [37], based on the concept of 'relaxed' ω -limit set, which yields strong stability in a suitable weakened topology. In order to follow this route, as mentioned earlier, we will first transform the system following method introduced in [31]. Once the correct dynamical system is identified, we shall show that this system admits a "relaxed" ω -limit set containing only the trivial solution. The main technical difficulty that need to be addressed are: (1) to improve weak into strong convergence - a challenging endeavor in the absence of compactness and (2) to identify ω -limit sets with suitable equilibria of coupled dynamics. The first task will be handled by exploiting suitable multipliers that are harmonic extensions of Stokes operator. We will rely on the geometric conditions and a micro-local analysis to fulfill the second task.

B. Transformation of the Dynamics.

Since the energy relation provides information only on the gradient of the displacement (without controlling the entire L_2 norm, where the latter may increase in time), we will construct a new dynamical system which accounts for the "degeneracy" of the energy. To achieve this we shall proceed as in [31]. We consider the space defined as

$$\mathcal{H}_0 \equiv H \times U \times [L_2(\Omega_s)]^2$$

where

$$U \equiv L^2_\nabla(\Omega_s) \equiv \{\nabla h, h \in [H^1(\Omega_s)]^2\}$$

Note, that $L^2_\nabla(\Omega_s)$ is the space of vector tensors of order four, i.e.: $\nabla h = \begin{pmatrix}\nabla h_1\\ \nabla h_2\end{pmatrix}$. As shown in [31], $L^2_\nabla(\Omega_s)$ is a closed subspace of $[L_2(\Omega_s)]^2 \times [L_2(\Omega_s)]^2$ and so is a Hilbert space. With the above notation (in the sequel we shall omit explicit writing of multiple copies of the vector spaces), we shall rewrite the original system as a dynamical system governed by the variables $\big(u(t), \xi(t), v(t)\big) \in \mathcal{H}_0$ which satisfy:

fluid equation in the variable $u \in H$:

$$(u_t, \phi)_f + \langle \xi \cdot \vec{n}, \phi\rangle + (\nabla u, \nabla\phi)_f + \big((u\cdot\nabla)u, \phi\big)_f - \langle \frac{1}{2}(u\cdot\vec{n})u, \phi\rangle = 0, \qquad \forall \phi \in V \tag{12}$$

and solid equation in the variable $(\xi, v) \in L^2_\nabla(\Omega_s) \times L_2(\Omega_s)$:

$$\begin{cases} \xi_t = \nabla v, & \text{in}\Omega_s \times (0,\infty)\\ v_t = \nabla\cdot\xi - \rho g(v), & \text{in}\Omega_s \times (0,\infty)\\ v\mid_{\Gamma_s} = u\mid_{\Gamma_s}, & \text{on}\Gamma_s \times (0,\infty)\end{cases} \tag{13}$$

The equivalent variational form is the following:

$$\begin{aligned} &(u_t, \phi)_f - \langle \xi \cdot \vec{n}, \phi\rangle + (\nabla u, \nabla\phi)_f + \big((u\cdot\nabla)u, \phi\big)_f - \langle \frac{1}{2}(u\cdot\vec{n})u, \phi\rangle = 0, \qquad \forall \phi \in V\\ &(\xi_t, \Psi)_s = (\nabla v, \Psi)_s, \qquad \forall \Psi \in L^2_\nabla(\Omega_s)\\ &(v_t, \Psi)_s = \langle \xi\cdot\vec{n}, \psi\rangle - (\xi, \nabla\psi)_s - (\rho g(v), \psi)_s, \qquad \forall \psi \in H^1(\Omega_s)\end{aligned}$$

with the transmission condition

$$v\mid_{\Gamma_s} = u\mid_{\Gamma_s}, \tag{14}$$

(14) is supplied with the initial conditions:

$$u(0) = u_0 \in H, \xi(0) = \xi_0 \in L^2_\nabla(\Omega_s), v(0) = v_0 \in L_2(\Omega_s)$$

Time derivatives are defined distributionally. It is clear that every solution (u, w, w_t) of the original problem corresponds to (u, ξ, v) with the identification: $\xi = \nabla w, v = w_t$. Also, having given variable (u, ξ, v), we can easily reconstruct w from $w(t) = w(0) + \int_0^t v(s)ds$. Of course, the latter quantity may not be bounded in time when $t \to \infty$.

- **Energy identity and energy functional for the transformed dynamics.** Energy method applied to strong solutions of (14), i.e. taking $\phi = u, \Psi = \xi, \psi = v$ gives

$$E_0(t) + \int_0^t [|\nabla u|^2_{0,f} + (\rho g(v), v)_{0,s}]ds = E_0(0)$$

where

$$E_0(t) \equiv \frac{1}{2}[|u|^2_{0,f} + |\xi|^2_{0,s} + |v|^2_{0,s}]$$

Thus, the energy function defines a norm on $\mathcal{H}_0$ defined above. In fact, "both" energies for the original system and the transformed one are the same. The original system defined in the variables $(u; w; w_t)$ corresponds in a one to one manner to a "new" system $(u; \xi;\ v)$ where $u = u, \xi = \nabla w, v = w_t$.

By Theorem II.6 we can construct a nonlinear semigroup $S_0(t): \mathcal{H}_0 \to \mathcal{H}_0$ such that for all $(u_0, \xi_0, v_0) \in \mathcal{H}_0, S_0(t)(u_0, \xi_0, v_0) = \big(u(t), \xi(t), v(t)\big)$. Then, $(S_0(t), \mathcal{H}_0)$ defines a dynamical system [11].

We define the following weak ω -limit set and the set D – smooth data.

Definition III.1. (weak ω -limit set) Let $(u(t); w(t); w_t(t))$ be weak solution of (1) specified in Theorem II.6 corresponding to the initial data $(u_0, \xi_0, v_0) \in \mathcal{H}_0$. We say a point $(\bar{u}_0, \bar{\xi}_0, \bar{v}_0) \in \mathcal{H}_0$ is in the weak ω-limit set $\omega(u_0, \xi_0, v_0)$ if there exists a sequence $t_n \to \infty$ such that $\big(u(t_n), v(t_n)\big) \to (\bar{u}_0, \bar{v}_0)$ strongly in $L_2(\Omega_f) \times L_2(\Omega_s)$ and $\xi(t_n) \rightharpoonup \bar{\xi}_0$ in $L^2_\nabla(\Omega_s)$.

Definition III.2. (Smooth data) We say the data $(u_0, \xi_0, v_0) \in \mathcal{H}_0$ is *smooth*, if it is contained in the following set D:

$$D = \{(u_0, \xi_0, v_0) \in V \times L^2_\nabla(\Omega_s) \times H^1(\Omega_s)\ such\ that$$
$$P_H \Delta u_0 \in L_2(\Omega_f), \mathrm{div}\xi_0 \in L_2(\Omega_s), \xi_0 \cdot \vec{n} \in H^{-\frac{1}{2}}(\Omega_s)$$
$$u_0 \mid_{\Gamma_s} = v_0 \mid_{\Gamma_s}, \langle \xi_0 \cdot \vec{n} - \frac{\partial u_0}{\partial n} + \frac{1}{2}(u_0 \cdot \vec{n}) u_0, \phi \rangle = 0, \phi \in V\}$$

C. Weak ω –Limit Set is $\{0\}$ *for Smooth Initial Data in D*

In this section, we will show that the dynamical system (14) admits a weak ω -limit set which is zero in the topology of $\mathcal{H}_0$. We should first point out that the weak ω -limit set is not empty since $V \times H^1(\Omega_s)$ is strongly compact in $H \times L_2(\Omega_s)$ and boundedness of ξ_n in $L^2_\nabla(\Omega_s)$ implies existence of weakly convergent subsequence $\xi_{n,k}$ in $L^2_\nabla(\Omega_s)$.

Let $(\bar{u}_0, \bar{\xi}_0, \bar{v}_0)$ be an element in $\omega(u_0, \xi_0, v_0)$ for $(u_0, \xi_0, v_0) \in D$. By definition, there exists a sequence $t_n \to \infty$ such that $(u(t_n), v(t_n)) \to (\bar{u}_0, \bar{v}_0)$ strongly in $L_2(\Omega_f) \times L_2(\Omega_s)$ and $\xi(t_n) \rightharpoonup \bar{\xi}_0$ weakly in $L^2_\nabla(\Omega_s)$, where $X(t; X_0) \coloneqq (u(t), \xi(t), v(t))$ is a solution with the initial data $X_0 = (u_0, \xi_0, v_0)$. For this sequence t_n, consider the translate $X_n(t) \coloneqq X(t + t_n; X_0)$. Since by energy identity (5),

$$\|X_n(t)\|_{\mathcal{H}_0} \le \|X_0\|_{\mathcal{H}_0}, \forall t \in \mathbb{R}^+$$

X_n is a bounded sequence in $L_\infty\big((0,\infty); \mathcal{H}_0\big)$. Thus, X_n has a subsequence, which we will denote by $X_n \coloneqq (u_n, \xi_n, v_n)$ such that X_n converges to $\bar{X} \coloneqq \big(\bar{u}, \bar{\xi}, \bar{v}\big)$ weakly in $L_2((0,T); \mathcal{H}_0)$ and weak* in $L_\infty((0,\infty); \mathcal{H}_0)$. We will first show that $\bar{u} = 0$. In fact, we actually have a much stronger result:

Lemma III.3. $u_n \to 0$ *in* $C((0,T], V)$ *for each* $T \ge 0$.

The key result we used to show this lemma is Lemma 3.1 in [28], which establishes the positive invariance of the set D under the dynamics on $\mathcal{H}_0$ and boundedness of time derivatives in $L_\infty((0,\infty); \mathcal{H}_0)$ for trajectories originating in D.

To see full details of the proof of this lemma, please consult the proof of Lemma 4.1 in [28]. The convergence of u_n obtained in Lemma III.3 allows us to pass the limit in the weak Formulation (14) and it turns out that $(\bar{\xi}, \bar{v})$ satisfies a specialDirichlet -Stokes problem stated in the following lemma whoseproof is technical (see [28]).

Lemma III.4. $[\bar{\xi}, \bar{v}]$ *is a weak solution of the following problem:*

$$\begin{cases} \bar{\xi}_t = \nabla \bar{v}, & \text{in}\ \Omega_s \times (0, T_1) \\ \bar{v}_t = \nabla \cdot \bar{\xi} - \rho g(\bar{v}) \text{in}\ \Omega_s \times (0, T_1) & \\ \bar{v} = 0, \bar{\xi} \cdot \vec{n} = p(t)\vec{n} \text{on}\ \Gamma_s \times (0, T_1) & \\ \rho(x) g(v) v = 0 & \text{in}\ \Omega_s \times (0, T_1) \end{cases} \tag{15}$$

with initial condition $\big(\bar{u}(0), \bar{\xi}(0), \bar{v}(0)\big) = \big(\bar{u}_0, \bar{\xi}_0, \bar{v}_0\big) \in D$ *and* $p \in L_\infty(0, T_1)$ *where* T_1 *is arbitrary.*

Our next step is to analyze the overdetermined problem (15) and show that the solution to (15) is stationary. We have the following lemma.

Lemma III.5. *With reference to the overdetermined boundary system (15) the following hold:*

- *Under part (a) of the Assumption 2 the energy* $E_0(t)$ *is a strict Lyapunov function on* $\mathcal{H}_0$. *Solutions to (15) are stationary.*
- *Under the full strength of Assumption 2 the only solution of (15) is the trivial one.*

Proof: Let (ξ, v) be a solution to the overdetermined problem specified in (15). Let D_τ denotes tangential derivative applied to the flat portion of the boundary $\Gamma_0 \subset \Gamma_s$. D_τ is orthogonal to $\vec{n}$ and commutes with $\vec{n}$ on Γ_0 (flatness assumption). D_τ can be naturally extended into a small collar near Γ_s – denoted by $\Omega_0 \subset \Omega_s$. We denote

$$\xi_\tau \equiv D_\tau \xi, \quad v_\tau \equiv D_\tau v, \quad in\ \Omega_0$$

Exploiting the flatness of the boundary Γ_0, we obtain the following system satisfied for the new variables (ξ_τ, v_τ) in Ω_0.

$$\begin{cases} \xi_{\tau,t} = \nabla v_\tau, & \text{in}\Omega_0 \\ v_{\tau,t} = \nabla \cdot \xi_\tau - \rho g'(v) v_\tau - \rho_\tau g(v), & \text{in}\Omega_0 \\ \xi_\tau \cdot \vec{n} = 0, v_\tau = 0, & \text{on}\ \Gamma_0 \end{cases} \tag{16}$$

The above system can be reduced to the following system:

$$\begin{cases} v_{\tau,tt} = \Delta v_\tau - \rho g'(v) v_\tau - \rho_\tau g(v), & \text{in}\Omega_0 \\ v_\tau = 0, \dfrac{\partial v_\tau}{\partial n} = 0, & \text{on}\Gamma_0 \end{cases} \tag{17}$$

The distribution law gives that $(\rho g(v), v) = 0$. Therefore, on $supp\ \rho(x)$, $(g(v), v) = 0$. But g is monotone, thus, on $supp\ \rho(x)$, $v \equiv 0$. This implies that $\rho g'(v) v_\tau = 0$ on $supp\ \rho(x)$ and since further $g(0) = 0$, $\rho_\tau g(v) = 0$ on $supp\ \rho(x)$. Thus, (17) could be reduced to the wave equation with the overdetermined boundary data on Γ_0:

$$\begin{cases} v_{\tau,tt} = \Delta v_\tau, & \text{in}\Omega_0 \\ v_\tau = 0, \dfrac{\partial v_\tau}{\partial n} = 0, & \text{on}\Gamma_0 \end{cases} \tag{18}$$

By the unique continuation property [26] [34], we conclude that $v_\tau = 0$, in Ω_0. Applying now the classical Holmgren's Uniqueness Theorem we extend local uniqueness to the global, claiming

$$v_\tau \equiv 0, \ \ in \ \Omega_s$$

The above condition implies that v is constant in y. Therefore, for any fixed $x \in \Omega_s$, $v(x, y, t) = v(x, y^*, t)$ for any $y^* \in \Gamma_s$, $y \in \Omega_s$and $t \in \mathbb{R}^+$. But on the boundary Γ_s, v is identically zero for all t . Thus,

$$v \equiv 0, \ \ in \ \Omega_s \times \mathbb{R}^+$$

Going back to the original system we obtain that $\xi_t = \nabla \cdot v \equiv 0$, which then implies that $E_0(t)$ is a strict Lyapunov's function on $\mathcal{H}_0$. This proves the first part of the Lemma.

Since we assume $g(0) = 0$ and $v \equiv 0$ as proved above, $g(v) \equiv 0$. Hence, for the second part of the Lemma, we are led to considerthe stationary problem:

$$\begin{cases} \text{div}\xi = 0, & \text{in}\Omega_s \\ \xi \cdot \vec{n} = p\vec{n}, & \text{on}\Gamma_s \end{cases} \tag{19}$$

with$p$ being now just a constant. We shall show that p must be zero. Indeed, compatibility on the boundary enforces $p \int_{\Gamma_s} \vec{n} ds = 0$, which is impossible (due to geometric condition) unless $p = 0$. So we have div$\xi = 0$ and $\xi \cdot \vec{n} = 0$ on the boundary Γ_s. Since $\xi \in L^2_\nabla(\Omega_s)$ we have that $\xi = \nabla h$ for some $h \in H^1(\Omega)$. This implies

$$\begin{cases} \Delta h = 0, & \text{in}\Omega_s \\ \dfrac{\partial h}{\partial n} = 0, & \text{on}\Gamma_s \end{cases} \tag{20}$$

The above can happy only if $h =$ *constant*. But then $\xi \equiv 0$, proving that both $v \equiv 0, \xi \equiv 0$. This completes the proof of the second part of the Lemma.

Lemma III.5 and Lemma III.4 imply the following important Corollary:

Corollary 1. *Under the geometric Assumption 2 we have that weak ω limit set for the dynamical system* $(S_0(t), \mathcal{H}_0)$ *consists of zero element only. This is to say* $(\bar{u}, \bar{\xi}, \bar{v}) \equiv 0$, *in* $\mathcal{H}_0$ Strong ω –limit set is {0} for smooth initial data in D Our goal in this section is to improve weak convergence of $\xi_n(s)$in$L^2_\nabla(\Omega_s)$ to the strong convergence.

Lemma III.6. *Assume the geometric conditions imposed by Assumption 2. Then for all initial data in D, we have that* $(u_n(t), \xi_n(t), v_n(t)) \to 0$ *strongly for all* $t \in [0, T]$.

To prove this lemma, it suffices to show that the convergence of $\xi_n$to zero is strong in $L^2_\nabla(\Omega_s)$. Here the idea is to utilize certain harmonic extensions associated with Stokes operator. To this aim, we define the following Stokes extension of the Dirichlet map D

$$z = Dg \Leftrightarrow \begin{cases} \Delta z = 0, \quad \text{div}\, z = 0, & \text{in}\Omega_f \\ z \mid_{\Gamma_s} = g, & \text{on}\Gamma_s \end{cases} \tag{21}$$

where we assume the compatibility $\int_{\Gamma_s} g\vec{n}\, d\Gamma_s = 0$. Stokes theory [38] gives that $D: H^\alpha(\Gamma_s) \to H^{\alpha+\frac{1}{2}}(\Omega_f)$ is well definedand continuous.In particular, D is continuous from $H^{\frac{1}{2}}(\Gamma_s)$ to V.

The task left is to construct a sequence $w_n(t) \to 0$ strongly in $H^{1-\epsilon}(\Omega_f)$ from $\xi_n(t)$ and then choose test functions $\phi = Dw_n \mid_{\Gamma_s}(t)$ and $\psi = w_n \mid_{\Gamma_s}(t)$ in (14) and carefully verify that all terms in the resulting identity eventually vanish as $t \to \infty$. The geometric condition $\int_{\Gamma_s} \vec{n}\, d\Gamma_s \neq 0$ in conjunction with a compactness-uniqueness argument is invoked to establish the convergent of $w_n(t)$ in the desired space. (see [28] for details)

D. Strong ω –limit set is {0} *for any initial data in D*

The final step in the proof is to show strong stability for arbitrary initial data in $\mathcal{H}_0$. Theorem II.6 defines semigroup $S_0(t): \mathcal{H}_0 \to \mathcal{H}_0$ so that for any data $x \in \mathcal{H}_0$, $S_0(t)x$ is weak solution of (14). Lemma III.6 asserts that this semigroup, when restricted to D, is strongly stable. Thus the proof of strong stability on $\mathcal{H}_0$ entails proving that the nonlinear semigroup $S_0(t)$ describing the flow is locally Lipschitz on $\mathcal{H}_0$. This last property depends critically on two-dimensionality of the domain. The required argument calls for appropriate estimates applied to the difference of two solutions (see [28]).

IV. UNIFORM STABILITY FOR THE MODEL WITH A FRICTIONAL DAMPING – THE STRATEGY

We present first the strategy used for the proof of Theorem II.6. The proof is based on the multiplier's method. As usual, the critical step in proving Theorem II.6 is the following estimate:

Theorem IV.1. *Under the conditions of Theorem II.6, there exists a time T > 0 and a constant* $C_T > 0$ *, such that the energy at t = T is dominated by the dissipation for all initial condition*$(u_0, w_0, w_1) \in \mathcal{H}$*:*

$$E(T) \leq C_T \int_0^T D(t)\, dt \tag{22}$$

Once Theorem IV.1 is established, using the energy identity (5) and following the nonlinear version of an inductive argument in [32], one is able to show Theorem II.6. Thus, the main task is to establish the validity of Theorem IV.1.

This task will require different arguments when $\alpha > 0$ and when $\alpha = 0$. The reason for this is that when $\alpha, \beta > 0$, there are no steady states in the dynamics. Thus we can easily replace gradient norms by $H^1(\Omega_s)$ norms. However, when $\alpha = 0$, the steady states need to be accounted for and the analysis become more delicate. We shall begin with the case when $\alpha > 0$.

In order to achieve (22) with $\alpha > 0$, we shall split our proof into the following two steps: first, we show the following "suboptimal" estimate containing tangential on the boundary derivatives. And then using micro-local estimate along with compactness/uniqueness argument we will be able to absorb this term into the dissipation.

Theorem IV.2. *There exists a time T > 0 and a constant* $C_T > 0$, *C > 0, such that the following estimate for the energy at t = T holds true for all initial condition*$(u_0, w_0, w_1) \in \mathcal{H}$*:*

$$E(T) \le C_T \int_0^T D(t)dt + C\int_0^T |D_\tau w|^2_{L_2(\Gamma_s)}dt + C_\alpha \int_0^T |w|^2_{L_2(\Omega_s)}dt \tag{23}$$

where$D_\tau w$ *is the tangential derivative of* w *on* Γ_s.

In the second step we shall eliminate the last two terms on the right hand side of (23).

In the case $\alpha = 0$, Inequality (22) will be obtained directly. This will be possible due to the construction of a special multiplier partitioning potential and kinetic energy. The presence of additional geometric condition will allow to dispense with the tangential derivatives in the inequality.

V. PROOF OF THEOREM II.6 WITH DYNAMIC AND STATIC DAMPING.

Without loss of generality, we assume $\beta = 1$ throughout the rest of the paper and α positive in this section. We consider the transformed problem.

A. Proof of Theorem IV.2

Let $h(x) \coloneqq x - x^0$ where x is an arbitrary vector in $\overline{\Omega}_s$ and x^0 is a fixed vector in $\mathbb{R}^n$. We multiply the equations in (13) with the three conventional multipliers:

- Multiplying $v_t = \operatorname{div}\xi - \rho g(v)$ by $h\cdot\xi$;
- Multiplying $\xi_t = \nabla v$by hv;
- Multiplying $v_t = \operatorname{div}\xi - \rho g(v)$by w,

where w is reconstructed from ξ via $w(t) = w_0 + \int_0^T v(s)\,ds$ for given initial condition w_0. And then integrating by parts will yield some useful intermediate inequalities. The total energy could be recovered by applying the transmission condition $v = u - \beta(\xi\cdot\vec{n} + \alpha w)$ on Γ_s on the boundary term of the intermediate inequalities. By carefully estimating boundary terms in the inequalities, we shall follow a similar argument as in [28] to prove Theorem IV.2. In order to estimate the nonlinear term involving function $g(v)$ we rely on the monotonicity and growth restriction imposed by Assumption 1.

B. Proof of Theorem IV.1

Equipped with the estimate in Theorem IV.2, we shall continue with the proof of Theorem IV.1.

- **Estimation of the tangential derivative**

We revoke a critical result from [33] on the tangential derivative $|D_\tau w|$ for the solution w of the wave equation in (1). Let $\epsilon > 0$ and $0 < \alpha < \frac{T}{2}$ be arbitrary,

$$\begin{aligned}\int_\alpha^{T-\alpha} |\xi\cdot\tau|^2_{L_2(\Gamma_s)}dt &\le C_{T,\epsilon}\left\{\int_0^T \left[|\xi\cdot\nu|^2_{L_2(\Gamma_s)} + |w_t|^2_{L_2(\Gamma_s)}\right]dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\epsilon}(\Omega_s)}\,dt\right\}\\ &\le C_{T,\epsilon}\left[\int_0^T D(t)dt + (\alpha+1)\int_0^T |w|^2_{H^{\frac{1}{2}+\epsilon}(\Omega_s)}\,dt\right]\end{aligned} \tag{24}$$

where in the last step we have used

$$|\xi\cdot\vec{n}|^2_{L_2(\Gamma_s)} \le |D(t)| + C_\epsilon \alpha^2 |w|^2_{\frac{1}{2}+\epsilon,\Gamma_s}$$

Combining (23) with (24), applying interpolation scale between $L_2$and H^1, which then yields the estimate for $H^{\frac{1}{2}+\epsilon}$, we conclude that (see [30])

$$E(T) \le C_{T,\epsilon}\left[\int_0^T D(t)dt + \int_0^T |w|^2_{0,\Omega_s}dt\right] \tag{25}$$

- **Absorption of l.o.t. in (25)**

In this step, we absorb the lower order terms in (25). We shall apply standard, by now, nonlinear version of the compactness/ uniqueness argument [32], where the uniqueness comes from the fact that $\alpha > 0$. (see [30] for details)

Lemma V.1. *With reference to the damped system (1) with*$\alpha, \beta > 0$, *there exists a constant* $C_T(E(0)) > 0$, *such that the following inequality holds:*

$$\int_0^T |w|^2_{0,\Omega_s}dt \le C_T\int_0^T D(t)\,dt \tag{26}$$

By combining the estimate in Theorem IV.2, Lemma V.1 and (25) leads to the result stated in Theorem IV.1.

VI. PROOF OF THEOREM II.6 WITH DYNMAIC DAMPING ONLY.

In this step, we will show the uniform stability result for the model without the static damping but subject to the additional geometry condition placed on Ω: $h\cdot\vec{n} > 0$. We are thus in the framework when $\beta > 0$ and α = 0. The key point of the proof is that rather than using *w* as the multiplier in equipartition of energy, inspired by [22] we will use a different multiplier that is linked to projection on unstable manifold.

We shall use projector operator which allows to separate steady states from the solution -the idea employed for the damped wave equation in [22]. Let $\{\phi_i\}$ be the orthonormal basis of $L_2(\Omega_s)$ formed by the eigen functions of the eigenvalue problem $-\Delta\phi = \lambda\phi$ with Neumann boundary condition $\frac{\partial\phi}{\partial n} = 0$ satisfying the condition that $0 \le \lambda_1 \le \cdots \lambda_i \le \cdots$. where λ_i is the corresponding eigenvalue of ϕ_i. Recall that ϕ_1 is constant, thus let P be the projection from $L_2(\Omega_s)$ to the subspace expanded by ϕ_1, then,

$$Pw = \frac{1}{|\Omega_s|}\int_{\Omega_s} w(x)dx,\ \forall w \in L_2(\Omega_s) \tag{27}$$

And a classical version of the Poincare's inequality states that there exists a constant $C>0$ depending only on Ω such that $|w - Pw|_{L_2} \leq C|\nabla w|_{L_2}, \forall\, w \in L_2(\Omega)$.

Applying the multiplier $w - Pw$ to the equation $v_t = \operatorname{div} \xi - \rho g(v)$ enables us to achieve equipartition of the energy, thus avoid lower order terms in the estimates and establish the following inequality

$$E_0(T) \leq C_T \int_0^T D(t)dt \qquad (28)$$

which is the statement in Theorem IV.1. Thus, using an inductive argument along with evolution property, the energy functional decays to zero uniformly at an exponential rate.

VII. CONCLUSIONS AND OPEN PROBLEMS

The following conclusions are derived from the analysis presented.

- Full stability analysis of a fluid structure interaction model has been carried out. This accounts for a passive damping (no active feedback controls) and active damping (both static and dynamic) actuated by boundary feedback controls applied to the interface.
- The results obtained depend on the interaction of geometry of the solid with the control mechanism.
- In the case of passive damping (controls un-active) strong stability of the energy is achieved for bodies which contained a flat portion. This flat segment could be arbitrary small but of a positive measure.
- In the case of active dynamic feedback control applied to the interface, the energy decays exponentially to zero (uniform stability) provided the geometry of the body satisfies the "star shaped condition".
- In the case of active feedback controls - *both dynamic and static* - not only the energy but the full $|Y(t)|_{\mathcal{H}}$ norm of the solution converges exponentially to zero. In addition, the domain can be arbitrary and it is not required to satisfy any geometric restrictions.
- The notion of weak-finite energy solution allows one for a direct implementation of Finite Element Method FEM.

Regarding future research, several open problems are natural to state.

1) Extend the analysis to the three dimensional case. For strong stability, the main obstacle is the lack of Lipschitz estimate for the solutions. However, partial results relying on a weaker concept of a solution may be possible to obtain. For uniform stability, the aim will be to construct a good approximation of each specified solution and then to derive decay rates ,for this approximation.

2) Another interesting problem is to investigate the stability results when the nonlinear damping $g(s)$ is polynomially bounded at infinity. This might affect the decay rates, which amy no longer be exponential but polynomial only.

3) Weaker forms of static damping (sublinear) could be also considered.

ACKNOWLEDGMENT

Research partially supported by the National Science Foundation (NSF) Grant DMS 0606682 and Air Force Office of Scientific Research (AFOSR) Grant FA9550-09-1-0459.

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