Theoretical Analysis of Communication Networks in a Bipartite Setting

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***Abstract*- Many real-world network problems are modeled by digraphs. In this paper, we study orthogonal factorization for bipartite digraph, and show the following result: Let G be a bipartite (0, $mf-m+1$)-digraph. Let f be an integer-valued function defined on $V(G)$ such that $k\le f(x)$, and let $H_1, \cdots, H_k$ be an m-subdigraph of G. Then G has a (0, f)-factorization orthogonal to each H_i ($1\le i\le k$).**

***Keywords*- Computer Network; World Wide Web; Bipartite Digraph; Orthogonal Factorization**

I. INTRODUCTION

Many networks problems in the real-world can be modeled by digraphs (for instance, see [1, 2]). In such a network, an important example of is a communication network with vertices and arcs modeling cities and communication channels, respectively. Other examples are the railroad network with vertices and arcs representing railroad stations and railways between two stations, respectively, or the World Wide Web with vertices representing Web pages, and arcs corresponding to hyperlinks between Web pages. Orthogonal factorizations in digraphs are very important in network design, circuit layout, combinatorial design, and other applications, and attract a great deal of attentions from researchers. All digraphs considered in this paper are finite directed graphs with no loops or parallel arcs.

In recent years, the factorization orthogonal problem has gained attention in computer networks. Although there have been several recent advances in developing algorithms for computer networks problem, the study of base theoretic analysis of such algorithms has been largely limited. The bipartite setting of the computer networks problem is perhaps one of the simplest, and no result has been derived for it. For several results on bipartite settings, we refer to [3, 4, 5]. The contribution of this paper is to infer the necessary and sufficient condition for a bipartite digraph to admit a (g, f)-factor containing E_1 and excluding E_2. Then, obtained that bipartite (0, $mf-m+1$)-digraph has a (0, f)-factorization orthogonal.

The organization of this paper is as follows: we show the basic notations and give the necessary and sufficient condition for a bipartite digraph to admit a (g, f)-factor containing E_1 and excluding E_2 in Section II. Using these notions and lemma in Section II, we derive main result in Section III. At last, we pose some open problem in Section IV.

II. BASICS

Let G be a digraph with vertex set $V(G)$ and arc set $E(G)$. For any vertex $x\in V(G)$, the indegree and outdegree of x denoted by $\deg_G^-(x)$ and $\deg_G^+(x)$, respectively. We use uv to denote the arc with tail u and head v. Let $g=(g^-, g^+)$ and $f=(f^-, f^+)$ be pairs of positive integer-valued functions defined on $V(G)$ such that $g^-(x)\le f^-(x)$ and $g^+(x)\le f^+(x)$ for each $x\in V(G)$. If $g^-(x)\le \deg_G^-(x)\le f^-(x)$ and $g^+(x)\le \deg_H^+(x)\le f^+(x)$ for each $x\in V$, then a digraph G is called a (g, f)-digraph. A spanning subdigraph F of G is called a (g, f)-factor of G if F itself is an (g, f)-digraph. A subdigraph H of G is called an m-subdigraph if H has m arcs. Denote $g\le f$ if $g^-(x)\le f^-(x)$ and $g^+(x)\le f^+(x)$ for each $x\in V$, and $k\le g$ if $k\le \min\{g^-(x), g^+(x)\}$. A ($g$, f)-factorization $F=\{F_1, F_2, \cdots, F_m\}$ of G is a partition of E into arc-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. Let H be an m-subdigraph of G, and let $k\ge 1$ be a fixed integer. A factorization $F=\{F_1, F_2, \cdots, F_m\}$ of G is called k-orthogonal to H if $|E(H)\cap E(F_i)| = k$ for $i=1, \cdots, m$. Especially, 1-orthogonal is orthogonal.

In the following text, we always assume that $G=(X, Y)$ is a bipartite digraph. For any function f defined on $V(G)$ and $S\subseteq V(G)$, we write $f(S)$ for $\sum_{x\in S} f(x)$ and $f(\varnothing)=0$. For two subsets $S\subseteq X$ and $T\subseteq Y$, we write $E_G(S, T)$ for the set $\{uv : uv\in E, u\in S, v\in T\}$, and let $e_G(S,T) = |E_G(S, T)|$. Define

$$\gamma_{1G}(S,T;g,f) = f^+(S)-g^-(T)+e_G(X-S,T),$$
$$\gamma_{2G}(S,T;g,f) = f^-(T)-g^+(S)+e_G(S,Y-T),$$
$$\gamma_{3G}(S,T;g,f) = f^+(T)-g^-(S)+e_G(S,Y-T),$$
$$\gamma_{4G}(S,T;g,f) = f^-(S)-g^+(T)+e_G(X-S,T).$$

$\gamma_{1G}(S,T;g,f)$, $\gamma_{2G}(S,T;g,f)$, $\gamma_{3G}(S,T;g,f)$ and $\gamma_{4G}(S,T;g,f)$ are simply denoted as $\gamma_{1G}(S,T)$, $\gamma_{2G}(S,T)$, $\gamma_{3G}(S,T)$ and $\gamma_{4G}(S,T)$, respectively.

Let E_1 and E_2 be two disjoint subsets of $E(G)$, and let $S\subseteq X$ and $T\subseteq Y$ be two subsets of $V(G)$. Define, for i=1, 2.

$$E_{iS}=E_i\cap E(S,\ Y-T),$$

$$E_{iT}=E_i\cap E(X-S,\ T).$$
$$\alpha_S=|E_{1S}|,\quad \alpha_T=|E_{1T}|,$$
$$\beta_S=|E_{2S}|,\quad \beta_T=|E_{2T}|.$$

Gallai [6] obtained the necessary and sufficient condition for the existence of a (*g*, *f*)-factor in a digraph. Liu [7] gave a necessary and sufficient condition for a digraph to admit a (*g*, *f*)-factor containing E_1 and excluding E_2. Wang [8] obtained some results on orthogonal factorization for some special digraphs. Folkman and Fulkerson [9] obtained the necessary and sufficient condition for the existence of a (*g*, *f*)-factor in a bipartite graph. Liu [10] gave a necessary and sufficient condition for a bipartite graph to admit a (*g*, *f*)-factor containing E_1 and excluding E_2.

We first obtained the following necessary and sufficient condition for the existence of a (*g*, *f*)-factor in a bipartite digraph which follows by applying the technology used in [9].

Lemma 1. Let $G=(X, Y)$ be a bipartite digraph, and let $g=(g-, g+)$ and $f=(f^-, f^+)$ be pairs of positive integer-valued functions defined on $V(G)$ such that $g(x)\le f(x)$ for every $x\in V(G)$. Then G has a (*g*, *f*)-factor if and only if for all $S\subseteq X$, and $T\subseteq Y$, $\gamma_{1G}(S,T)\ge 0$, $\gamma_{2G}(S,T)\ge 0$, $\gamma_{3G}(S,T)\ge 0$, and $\gamma_{4G}(S,T)\ge 0$.

Let us now give a necessary and sufficient condition for a bipartite digraph to admit a (*g*,*f*)-factor containing E_1 and excluding E_2, which plays a crucial role in the proofs of our theorems.

Lemma 2. Let $G=(X, Y)$ be a bipartite digraph, and let $g=(g-, g+)$ and $f=(f^-, f^+)$ be pairs of positive integer-valued functions defined on $V(G)$ such that $g(x)\le f(x)$ for every $x\in V(G)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (*g*, *f*)-factor F such that $E_1\subseteq E(F)$ and $E_2\cap E(F)=\varnothing$ if and only if for all $S\subseteq X$, and $T\subseteq Y$, $\gamma_{1G}(S,T)\ge\alpha_S+\beta_T$, $\gamma_{2G}(S,T)\ge\alpha_T+\beta_S$, $\gamma_{3G}(S,T)\ge\alpha_T+\beta_S$, $\gamma_{4G}(S,T)\ge\alpha_S+\beta_T$.

Proof. First, we show that G has a (*g*, *f*)-factor with $E_2\cap E(F)=\varnothing$ if and only if

$$\gamma_{1G}(S,T)\ge\beta_T,\ \gamma_{2G}(S,T)\ge\beta_S,$$
$$\gamma_{3G}(S,T)\ge\beta_S,\ \gamma_{4G}(S,T)\ge\beta_T.$$

Let $G'=G-E_2$. Then the such desired (*g*, *f*)-factor exists if and only if G' has a (*g*, *f*)-factor if and only if, by Lemma 1, for any $S\subseteq X$, and $T\subseteq Y$,

$$\gamma_{1G'}(S,T)=f^+(-g^-(T)+e_{G'}(X'-S,T)\ge 0,$$
$$\gamma_{2G'}(S,T)=f^-(T)g^+(S)+e_{G'}(S,Y'-T)\ge 0,$$
$$\gamma_{3G'}(S,T)=f^+(T)g^-(S)+e_G(S,Y'-T)\ge 0,$$
$$\gamma_{4G'}(S,T)=f^-(S)g^+(T)+e_G(X'-S,Y)\ge 0.$$

It is easy to see that

$$\gamma_{1G'}(S,T)=\gamma_{1G}(S,T)-\beta_T,\tag{1}$$
$$\gamma_{2G'}(S,T)=\gamma_{2G}(S,T)-\beta_S,\tag{2}$$
$$\gamma_{3G'}(S,T)=\gamma_{3G}(S,T)-\beta_S,\tag{3}$$
$$\gamma_{4G'}(S,T)=\gamma_{4G}(S,T)-\beta_T.\tag{4}$$

Therefore, $\gamma_{1G'}(S,T)\ge 0$, $\gamma_{2G'}(S,T)\ge 0$, $\gamma_{3G'}(S,T)\ge 0$ and $\gamma_{4G'}(S,T)\ge 0$ if and only if $\gamma_{1G}(S,T)\ge\beta_T$, $\gamma_{2G}(S,T)\ge\beta_S$, $\gamma_{3G}(S,T)\ge\beta_S$ and $\gamma_{4G}(S,T)\ge\beta_T$.

Next, let us prove that there exists a (*g*, *f*)-factor in G containing all arcs of E_1 if and only if

$$\gamma_{1G}(S,T)\ge\alpha_S,\ \gamma_{2G}(S,T)\ge\alpha_T,$$
$$\gamma_{3G}(S,T)\ge\alpha_T,\ \gamma_{4G}(S,T)\ge\alpha_S.$$

For this purpose, let

$$g'^-(x)=\deg_G^-(x)-f^-(x),\quad f'^-(x)=\deg_G^-(x)-g^-(x),$$
$$g'^+(x)=\deg_G^+(x)-f^+(x),\quad f'^+(x)=\deg_G^+(x)-g^+(x),$$

and let

$$g'=(g'^-, g'^+),\ f'=(f'^-, f'^+).$$

Then such desired (*g*, *f*)-factor exists if and only if G has a (g', f')-factor excluding all arcs of E_1. According to the first statement, this is equivalent to

$$\gamma_{1G}(S,T;g',f')\ge\alpha_T,$$
$$\gamma_{2G}(S,T;g',f')\ge\alpha_S,$$
$$\gamma_{3G}(S,T;g',f')\ge\alpha_S,$$

and

$$\gamma_{4G}(S,T;g',f')\ge\alpha_T.$$

Note that

$$\begin{aligned}&\gamma_{1G}(S,T;g',f')\\&=f'^+(S)-g'^-(T)+e_G(X-S,T)\\&=\deg_G^+(S)-g^+(S)-\deg_G^-(T)\\&\quad+f^-(T)+e_G(X-S,T)\\&=f^-(T)+g^+(S)+e_G(S,Y-T)\\&=\gamma_{2G}(S,T;g,f).\end{aligned}$$

Similarly, we obtain

$$\gamma_{2G}(S,T;g',f')=\gamma_{1G}(S,T;g,f),$$
$$\gamma_{3G}(S,T;g',f')=\gamma_{4G}(S,T;g,f),$$
$$\gamma_{4G}(S,T;g',f')=\gamma_{3G}(S,T;g,f).$$

Hence, G has a (*g*, *f*)-factor containing all arcs of E_1 if and only if $\gamma_{1G}(S,T)\ge\alpha_S$, $\gamma_{2G}(S,T)\ge\alpha_T$, $\gamma_{3G}(S,T)\ge\alpha_T$, and $\gamma_{4G}(S,T)\ge\alpha_S$, as desired.

From the evidence offered above, we confirm that G has a (*g*, *f*)-factor F such that $E_1\subseteq E(F)$ and $E_2\cap E(F)=\varnothing$ if and only if G', as defined before, has a (*g*, *f*)-factor F with

$E_1 \subseteq E(F)$. By analyzing the preceding statement, this is equivalent to that $\gamma_{1G'}(S,T) \geq \alpha_S$, $\gamma_{2G'}(S,T) \geq \alpha_T$, $\gamma_{3G'}(S,T) \geq \alpha_T$ and $\gamma_{4G'}(S,T) \geq \alpha_S$. By (1)-(4), it is sure that G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$ if and only if for all $S \subseteq X$, and $T \subseteq Y$, $\gamma_{1G}(S,T) \geq \alpha_S + \beta_T$, $\gamma_{2G}(S,T) \geq \alpha_T + \beta_S$, $\gamma_{3G}(S,T) \geq \alpha_T + \beta_S$, $\gamma_{4G}(S,T) \geq \alpha_S + \beta_T$. □

In the present paper, we study the orthogonal factorizations in digraphs. The main result of this article is the following.

Theorem 1. Let G be a bipartite $(0, mf-m+1)$-digraph. let f be an integer-valued function defined on $V(G)$ such that $k \leq f(x)$, and let $H_1, \cdots, H_k$ be an m-subdigraph of G. Then G has a $(0, f)$-factorization orthogonal to each $H_i (1 \leq I \leq k)$.

III. PROOF OF MAIN RESULT

Let G be a bipartite $(0, mf-m+1)$-digraph where $m \geq 1$ is an integer.

Define

$$g^+(x)=\max\{0, \deg_G^+(x)-(m-1)f^+(x)+(m-1)-1\},$$

$$g^-(x)=\max\{0, \deg_G^-(x)-(m-1)f^-(x)+(m-1)-1\}.$$

$$\Delta_1(x)=\frac{1}{m}d_G^+(x)-g^+(x),$$

$$\Delta_2(x)=f^+(x)-\frac{1}{m}d_G^+(x),$$

$$\Delta_3(x)=\frac{1}{m}d_G^-(x)-g^-(x),$$

$$\Delta_4(x)=f^-(x)-\frac{1}{m}d_G^-(x).$$

By the above definitions, we have follow lemmas:

Lemma 3. For every $x \in V(G)$, we have

1) If $m \geq 2$, then $0 \leq g(x) < f(x)$;

2) If $g^+(x)=\deg_G^+(x)-(m-1)f^+(x)+(m-1)-1$, then $\Delta_1(x) \geq \frac{1}{m}$;

$g^-(x)=\deg_G^-(x)-(m-1)f^-(x)+(m-1)-1$, then $\Delta_3(x) \geq \frac{1}{m}$;

3) $\Delta_2(x) \geq \frac{m-1}{m}$ and $\Delta_4(x) \geq \frac{m-1}{m}$.

Proof. (1) Since G is a bipartite $(0, mf-m+1)$-digraph, and $m \geq 2$ is an integer. Thus, $0 \leq mf-m+1$. So, by the integer of function f, we have $f \geq 1$.

If $g^-(x)=0$, then $0 \leq g^-(x) < f^-(x)$ is obviously.

If $g^-(x)=\deg_G^-(x)-(m-1)f^-(x)+(m-1)-1$, then

$$\begin{aligned}
& f^-(x)-g^-(x) \\
&=f(x)-(\deg_G^-(x)-(m-1)f^-(x)+(m-1)-1) \\
&=m f^-(x)-m+2-\deg_G^-(x) \\
&\geq m f^-(x)-m+2-(m f^-(x)-m+1) \\
&=1.
\end{aligned}$$

Therefore, we get $0 \leq g^-(x) < f^-(x)$.

Similarity, we can get $0 \leq g^+(x) < f^+(x)$.

(2) In the terms of $g^+(x)=\deg_G^+(x)-(m-1)f^+(x)+(m-1)-1$, we obtain

$$\begin{aligned}
&\Delta_1(x) \\
&=\frac{1}{m}d_G^+(x)-g^+(x) \\
&=\frac{1}{m}d_G^+(x)-(\deg_G^+(x)-(m-1)f^+(x)+(m-1)-1) \\
&=\frac{m-1}{m}d_G^+(x)+(m-1)f^+(x)-(m-1)+1 \\
&\geq \frac{m-1}{m}(mf^+(x)-m+1)+(m-1)f^+(x)-(m-1)+1 \\
&=(1-m)f^+(x)+(m-1)-\frac{m-1}{m}+(m-1)f^+(x)-(m-1)+1 \\
&=\frac{1}{m}.
\end{aligned}$$

Similarity, we can show that $\Delta_3(x) \geq \frac{1}{m}$ if $g^-(x)=\deg_G^-(x)-(m-1)f^-(x)+(m-1)-1$.

(3) In fact,

$$\begin{aligned}
\Delta_2(x)&=f^+(x)-\frac{1}{m}d_G^+(x), \\
&\geq f^+(x)-\frac{1}{m}(mf^+(x)-m+1) \\
&=f^+(x)-f^+(x)+\frac{m-1}{m} \\
&=\frac{m-1}{m}.
\end{aligned}$$

Similarity, we can show that $\Delta_4(x) \geq \frac{m-1}{m}$. □

Lemma 4.

For every $x \in V(G)$, we have that

$$\gamma_{1G}(S,T;g,f)=\Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}e_G(X-S,T)+\frac{1}{m}e_G(S,Y-T),$$

$$\gamma_{2G}(S,T;g,f)=\Delta_1(S)+\Delta_2(T)+\frac{m-1}{m}e_G(S,Y-T)+\frac{1}{m}e_G(X-S,T),$$

$$\gamma_{3G}(S,T;g,f)=\Delta_3(T)+\Delta_4(S)+\frac{m-1}{m}e_G(S,Y-T)+\frac{1}{m}e_G(X-S,T),$$

and

$$\gamma_{4G}(S,T;g,f)=\Delta_3(S)+\Delta_4(T)+\frac{m-1}{m}e_G(X-S,T)+\frac{1}{m}e_G(S,Y-T).$$

Proof. We only proof the first inequality. The other can be veri1ed similarly. According to the definition of $\gamma_{1G}(S,T;g,f)$, we have

$$\begin{aligned}
&\gamma_{1G}(S,T;g,f)\\
&=e_G(X-S,T)-g^-(T)+f^+(S)\\
&=d_G^-(T)-e_G(S,T)-g^-(T)+f^+(S)\\
&=\frac{1}{m}d_G^-(T)-g^-(T)+(f^+(S)-\frac{1}{m}d_G^-(S))\\
&\quad+\frac{m-1}{m}e_G(X-S,T)+\frac{1}{m}e_G(X-S,T)\\
&=\Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}e_G(X-S,T)\\
&\quad+\frac{1}{m}e_G(S,Y-T).
\end{aligned}$$ □

Let $S\subseteq X$ and $T\subseteq Y$ be two subsets of $V(G)$. Let $S_0=\{x|\ x\in S,\ f(x)=1\}$, $S_1=S-S_0$; $S_0=\{x|\ x\in S,\ f(x)=1\}$, $S_1=S-S_0$. Then, we have

$S=S_0\cup S_1$, $S_0\cap S_1=\varnothing$,

$T=T_0\cup T_1$, $T_0\cap T_1=\varnothing$.

$\alpha_S=\alpha_{S_0}+\alpha_{S_1}$, $\alpha_T=\alpha_{T_0}+\alpha_{T_1}$,

$\beta_S=\beta_{S_0}+\beta_{S_1}$, $\beta_T=\beta_{T_0}+\beta_{T_1}$.

Lemma 5. Let E_1 and E_2 be two disjoint subsets of E.

(1)If $\gamma_{1G}(S_1,T_1;g,f)=e_G(G-S_1,T_1)-g^-(T_1)+f^+(S_1)\geq\alpha_{S_1}+\beta_{T_1}$, then $\gamma_{1G}(S,T;g,f)=e_G(G-S,T)-g^-(T)+f^+(S)\geq\alpha_S+\beta_T$.

(2)If $\gamma_{2G}(S_1,T_1;g,f)=e_G(S_1,G-T_1)-g^+(S_1)+f^-(T_1)\geq\alpha_{T_1}+\beta_{S_1}$, then $\gamma_{2G}(S,T;g,f)=e_G(S,G-T)-g^+(S)+f^-(T)\geq\alpha_T+\beta_S$.

(3)If $\gamma_{3G}(S_1,T_1;g,f)=e_G(S_1,G-T_1)-g^-(S_1)+f^+(T_1)\geq\alpha_{T_1}+\beta_{S_1}$, then $\gamma_{3G}(S,T;g,f)=e_G(S,G-T)-g^-(S)+f^+(T)\geq\alpha_T+\beta_S$.

(4)If $\gamma_{4G}(S_1,T_1;g,f)=e_G(G-S_1,T_1)-g^+(T_1)+f^-(S_1)\geq\alpha_{S_1}+\beta_{T_1}$, then $\gamma_{4G}(S,T;g,f)=e_G(G-S,T)-g^+(T)+f^-(S)\geq\alpha_S+\beta_T$.

Proof. We only proof the first inequality. The other can be veri1ed similarly. Since $e_G(G-S,T_0)-g^-(T_0)=e_G(G-S,T_0)\geq\alpha_{T_0}$, $0\leq d_G^+(x)\leq mf^+(x)-m+1$, and the vertex in S_0 have indegree 0 or 1 in G. We get

$$\begin{aligned}
|S_0|&\geq d_G^-(S_0)\\
&\geq e_G(S_0,T)+e_G(S_0,G-T)\\
&\geq\alpha_{S_0}+e_G(S_0,T_1).
\end{aligned}$$

Thus, when $\gamma_{1G}(S_1,T_1;g,f)\geq\alpha_{S_1}+\beta_{T_1}$, we obtain

$$\begin{aligned}
&\gamma_{1G}(S,T;g,f)\\
&=e_G(G-S,T)-g^-(T)+f^+(S)\\
&=f^+(S_1)+|S_0|+e_G(G-S,T_1)+\\
&\quad e_G(G-S,T_0)-g^-(T_1)\\
&\geq f^+(S_1)+\alpha_{S_0}+e_G(S_0,T_1)+\\
&\quad e_G(G-S,T_1)+\beta_{T_0}-g^-(T_1)\\
&=f^+(S_1)+\alpha_{S_0}+e_G(G-S_1,T_1)+\beta_{T_0}-g^-(T_1)\\
&=\gamma_{1G}(S_1,T_1;g,f)+\alpha_{S_0}+\beta_{T_0}\\
&\geq\alpha_{S_1}+\beta_{T_1}+\alpha_{S_0}+\beta_{T_0}\\
&=\alpha_S+\beta_T.
\end{aligned}$$ □

Lemma 6 [11]**.** Let G be an $(0, mf-m+1)$-digraph, $f(x)$ be a non-negative integer function defined on $V(G)$, H be an m-subdigraph of G, then G have a $(0, f)$-factorization orthogonal to each H.

Let G be a bipartite digraph, $f(x)$ be a non-negative integer function defined on $V(G)$ such that $f(x)\geq k$ for every $x\in V(G)$. Let $H_1,\cdots,H_k$ be an m-subdigraph of G, $g(x)$ as defined above. For $i=1,\cdots,k$, denote

$A_{i1}=\{xy\in E(H_i)|\ g(x)\geq 1$ and $g(y)\geq 1\}$,

$A_{i2}=\{xy\in E(H_i)|\ g(x)\geq 1$ or $g(y)\geq 1\}$,

$$A_i=\begin{cases}A_{i1}, & A_{i1}\neq\varnothing\\ A_{i2}, & A_{i1}=\varnothing\ \text{ and }\ A_{i2}\neq\varnothing\\ E(H_i), & \text{otherwise}\end{cases}$$

Choose $u_iv_i \in A_i$, i=1, …, k. Let E_1={ u_iv_i, i=1, …, k} and E_2= $(\bigcup_{i=1}^{k} E(H_i))$- E_1. Then $|E_1|=k$ and $|E_2|=(m-1)k$. For $S\subseteq X$ and $T\subseteq Y$, $E_{iS}=E_i \cap E(S, Y-T)$, E_{1T}, E_{2T}, E_{1S}, E_{2S}, $\alpha_S, \alpha_T, \beta_S, \beta_T$ as we defined above, then we have

$$\alpha_{S_1} \le \min\{k,|S_1|\},\ \ \alpha_{T_1} \le \min\{k,|T_1|\},$$
$$\beta_{S_1} \le \min\{(m-1)k,(m-1)|S_1|\},$$
$$\beta_{T_1} \le \min\{(m-1)k,(m-1)|T_1|\}.$$

In order to prove Theorem 1, Lemma 7 will be used later.

Lemma 7. Let $G=(X, Y)$ be a bipartite digraph and let f be a positive integer function defined on V with $f(x)\ge k$ for each $x\in V$, where $m\ge 2$ and $k\ge 2$ are two integers. If G is an $(0, mf-m+1)$-digraph, then G has a (g, f) -factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F)=\varnothing$.

Proof. By Lemma 2, we only to show that for all $S\subseteq X$, and $T\subseteq Y$,

$$\gamma_{1G}(S,T) \ge \alpha_S+\beta_T,\ \gamma_{2G}(S,T) \ge \alpha_T+\beta_S,$$
$$\gamma_{3G}(S,T) \ge \alpha_T+\beta_S,\ \gamma_{4G}(S,T) \ge \alpha_S+\beta_T.$$

Let S_1 and T_1 as defined above. By lemma 5, it is only need to show

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$=e_G(G-S_1,T_1)-g^-(T_1)+f^+(S_1)$$
$$\ge \alpha_{S_1}+\beta_{T_1},$$
$$\gamma_{2G}(S_1,T_1;g,f)$$
$$=e_G(S_1,G-T_1)-g^+(S_1)+f^-(T_1)$$
$$\ge \alpha_{T_1}+\beta_{S_1},$$
$$\gamma_{3G}(S_1,T_1;g,f)$$
$$=e_G(S_1,G-T_1)-g^-(S_1)+f^+(T_1)$$
$$\ge \alpha_{T_1}+\beta_{S_1},$$
$$\gamma_{4G}(S_1,T_1;g,f)$$
$$=e_G(G-S_1,T_1)-g^+(T_1)+f^-(S_1)$$
$$\ge \alpha_{S_1}+\beta_{T_1}.$$

We only to show the first inequality holds, the other inequality can be deal with in the similarity way. That is, we only to show

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$=e_G(G-S_1,T_1)-g^-(T_1)+f^+(S_1)$$
$$\ge \alpha_{S_1}+\beta_{T_1}.$$

By Lemma 3 and Lemma 4, we obtain

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$=\Delta_1(T_1)+\Delta_2(S_1)+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\ge \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$

If $T_1=\varnothing$, then $\beta_{T_1}=0$. We have

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$\ge e_G(G-S_1,T_1)-g^-(T_1)+f^+(S_1)$$
$$=f^+(S_1)\ge k|S_1| \ge |S_1|$$
$$\ge \alpha_{S_1}=\alpha_{S_1}+\beta_{T_1}.$$

Next, we assume $T_1=\varnothing$. Then there exist $x_1\in T_1$, such that

$$d_G^+(x_1)=\min\{d_G^+(x): x\in T_1\}.$$

By the definition of T_1, we have that for every $x\in T_1$,

$$g^-(x)=d_G^-(x)-((m-1)f^-(x)-(m-1)+1)\ge 1.$$

Thus,

$$g^-(x_1)=d_G^-(x_1)-((m-1)f^-(x_1)-(m-1)+1)\ge 1.$$

So, we get

$$d_G^-(x_1)\ge (m-1)f^-(x_1)-(m-1)+2$$
$$\ge (m-1)k-(m-1)+2$$
$$=(m-1)(k-1)+2.$$

Now, we consider following situations:

Case 1. $0\le |S_1| \le (m-1)(k-2)+1$.

In this case, we have

$$e_G(G-S_1,T_1)$$
$$\ge (d_G^+(x_1)-|S_1|)|T_1|$$
$$\ge [(m-1)(k-1)+2-(m-1)(k-2)-1]|T_1|$$
$$=m|T_1|.$$

Thus,

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$\ge \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\ge \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}m|T_1|+\frac{1}{m}e_G(S_1,Y-T_1)$$

$$\geq \frac{1}{m}+\frac{(m-1)}{m}\alpha_{S_1}+\beta_{T_1}+\frac{1}{m}\alpha_{S_1}$$
$$=\alpha_{S_1}+\beta_{T_1}.$$

Case 2. $|S_1|=(m-1)(k-2)+2$.

Subcase 2.1. $1\leq |T_1|\leq k-1$.

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$\geq \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$=\frac{|T_1|}{m}+\frac{(m-1)}{m}((k-2)(m-1)+2)+$$
$$\frac{m-1}{m}e_G(X-S_1,T_1)+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\geq \frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}k+\frac{m-1}{m}(k-2)(m-2)$$
$$+\frac{m-1}{m}\beta_{T_1}+\frac{1}{m}\alpha_{S_1}$$
$$\geq \frac{m-1}{m}\alpha_{S_1}+\frac{1}{m}\alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}$$
$$+\frac{m-1}{m}(k-2)(m-2)+\frac{m-1}{m}\beta_{T_1}$$
$$=\alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}$$
$$+\frac{m-2}{m(m-1)}(k-1)(m-1)+\frac{m-1}{m}(k-2)(m-2)$$
$$-\frac{m-2}{m(m-1)}(k-1)(m-1)$$
$$\geq \alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}+\frac{m-2}{m(m-1)}(m-1)|T_1|$$
$$+\frac{m-2}{m}[(k-2)(m-2)-1]$$
$$\geq \alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}+\frac{m-2}{m(m-1)}\beta_{T_1}$$
$$-\frac{m-2}{m}$$
$$>\alpha_{S_1}+\beta_{T_1}-1.$$

By $\gamma_{1G}(S_1,T_1;g,f)$ is an integer, we have $\gamma_{1G}(S_1,T_1;g,f)\geq \alpha_{S_1}+\beta_{T_1}$.

Subcase 2.2. $|T_1|=k$.

In this case, we have

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$=e_G(G-S_1,T_1)-g^-(T_1)+f^+(S_1)$$
$$=d_G^-(T_1)-|S_1|\,|T_1|-[\,d_G^-(T_1)-$$
$$((m-1)f(T_1)-(m-1)|T_1|+|T_1|)]+f^+(S_1)$$
$$=(m-1)f(T_1)-(m-1)|T_1|+|T_1|-|S_1|\,|T_1|+f^+(S_1)$$
$$\geq (m-1)k|T_1|-(m-1)|T_1|+|T_1|-k|T_1|+k|S_1|$$
$$=(m-1)(k-1)k+k$$
$$\geq k+(m-1)k$$
$$\geq \alpha_{S_1}+\beta_{T_1}.$$

Subcase 2.3. $|T_1|\geq k+1$.

In this case, we have

$$e_G(G-S_1,T_1)$$
$$\geq (d_G^+(S_1)-|S_1|)|T_1|$$
$$\geq [((k-1)(m-1)+2)-((k-2)(m-1)+2))]|T_1|$$
$$=(m-1)|T_1|.$$

Thus,

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$\geq \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\geq \frac{|T_1|}{m}+\frac{(m-1)}{m}((k-2)(m-1)+2)$$
$$+\frac{m-1}{m}(m-1)|T_1|+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\geq \frac{m-1}{m}\alpha_{S_1}+\frac{1}{m}\alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}\ \frac{m-1}{m}(m-1)k$$
$$+\frac{m-1}{m}(m-1)+\frac{m-1}{m}(k-2)(m-2)$$
$$\geq \alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}+\frac{m-2}{m(m-1)}k(m-1)$$
$$+\frac{m-1}{m}(k-2)(m-2)+\frac{m-1}{m}(m-1)$$
$$-\frac{m-2}{m(m-1)}(k-1)(m-1)$$
$$\geq \alpha_{S_1}+\beta_{T_1}+\frac{m-2}{m}[(k-1)(m-1)-k]+\frac{m-1}{m}$$
$$\geq \alpha_{S_1}+\beta_{T_1}.$$

Case 3. $|S_1|\geq (m-1)(k-2)+3$.

In this case, we get

$$\gamma_{1G}(S_1,T_1;g,f)$$
$$\geq \frac{|T_1|}{m}+\frac{(m-1)|S_1|}{m}+\frac{m-1}{m}e_G(X-S_1,T_1)$$
$$+\frac{1}{m}e_G(S_1,Y-T_1)$$
$$\geq \frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}[(k-2)(m-1)+3]$$
$$+\frac{m-1}{m}\beta_{T_1}+\frac{1}{m}\alpha_{S_1}$$
$$\geq \frac{1}{m}\alpha_{S_1}+\frac{m-1}{m}k+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}$$
$$+\frac{m-2}{m(m-1)}k(m-1)+\frac{m-1}{m}[(k-2)(m-1)+3]$$
$$-\frac{m-1}{m}k-\frac{m-2}{m(m-1)}k(m-1)$$
$$\geq \frac{1}{m}\alpha_{S_1}+\frac{m-1}{m}\alpha_{S_1}+\frac{1}{m(m-1)}\beta_{T_1}+\frac{m-1}{m}\beta_{T_1}$$
$$+\frac{m-2}{m(m-1)}\beta_{T_1}+\frac{m-2}{m}[(k-2)(m-2)-2]$$
$$+\frac{m-1}{m}$$
$$\geq \alpha_{S_1}+\beta_{T_1}+\frac{m-1}{m}-\frac{2(m-2)}{m}$$
$$=\alpha_{S_1}+\beta_{T_1}-\frac{m-3}{m}$$
$$>\alpha_{S_1}+\beta_{T_1}-1.$$

By $\gamma_{1G}(S_1,T_1;g,f)$ is an integer, we have $\gamma_{1G}(S_1,T_1;g,f) \geq \alpha_{S_1}+\beta_{T_1}$. □

Now, we begin to prove Theorem 1.

Proof. When k=1, we are down according to Lemma 6. Next, we assume $k\geq 2$. The result holds obviously for m=1. We assume that the result holds for $m-1$, where $m\geq 2$. By Lemma 7, we are sure that G has a (g, f) -factor F_1 such that $E_1\subseteq E(F_1)$ and $E_2\cap E(F_1)=\varnothing$. By the definition of $g(x)$, F_1 is a $(0, f)$-factor of G such that $E_1 \subseteq E(F_1)$ and $E_2\cap E(F_1)=\varnothing$.

Let $G'=G-E(F_1)$, then by the definition of $g(x)$, we have

$$0\leq d^+_{G'}(x)$$
$$=d^+_G(x)-d^+_{F_1}(x)$$
$$\leq d^+_G(x)-g(x)$$
$$\leq d^+_G(x)-[d^+_G(x)-((m-1)f^+(x)-(m-1)+1)]$$
$$=(m-1)f^+(x)-(m-1)+1.$$

Similarity, $0\leq d^-_{G'}(x)\leq (m-1)f^-(x)-(m-1)+1$ holds. Thus, G is a bipartite $(0, (m-1)f(x)-(m-1)+1)$-digraph. Let $H_i'=H_i-E_1$, where $1\leq i\leq k$. By the induction, G' has a $(0,f)$-factorization $F'=\{F_2, \cdots, F_m\}$ orthogonal to H_i' $(1\leq i\leq k)$. Thus, G has a $(0, f)$-factorization orthogonal to each $H_i (1\leq i\leq k)$.

IV. FURTHER WORK

It has showed by other author that the problems on (g, f)-factorizations, k-orthogonal or randomly k-orthogonal factorizations in undirected graph are in NP. From this point of view, the problems on (g, f)-factorizations, k-orthogonal or randomly k-orthogonal factorizations in digraph are also NP. Given a graph G (undirected graph or digraph), two integer functions $g=(g^-, g^+)$ and $f=(f^-, f^+)$ defined as the above and a positive integer k, is there a (g,f)-factorization $F=\{F_1,F_2,\cdots,F_m\}$ of G such that $m\leq k$? In views of the simple version of this problem can be regard as the edge-coloring problem, we can verify that the general version of this problem is NP-complete as well. In the terms of the above argument, clearly, the problem which asks whether a bipartite digraph has a (g, f)-factorization k-orthogonal (or factorizations randomly k-orthogonal) to a given km-subdigraph is also NP-complete since factorizations are the special case of orthogonal (or randomly orthogonal) factorizations. Note that, for undirected graph, there are polynomial algorithms for deciding whether a undirected graph G has a (g, f)-factor. When G is a bipartite undirected graph or $g(x)\neq f(x)$ for each $x\in V(G)$ holds, Heinrich et al. [12] gave a relatively simple existence criterion for a (g, f)-factor which leads to an (g, f)-factor algorithm with time complexity $O(g(V(G))|E(G)|)$; Hell and Kirkpatrick [13] gave $O(\sqrt{g(V(G))}\,|E(G)|)$ algorithms for this issue. Anstee [9] obtained a polynomial algorithm which either finds a (g, f)-factor or shows that one does not exist in $O(|V(G)|^3)$ operations for general (g, f)-factor problems.

The design of effective algorithm for giving a (g, f)-factorizations in bipartite digraph or checking whether there exists a factorization is a challenge work for us. However, these kinds of algorithms are much useful in computer network, such as transmission.

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