# Delay-dependent $H^\infty$ Control for Systems with Two Additive Time-vary Delays

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***Abstract*- This paper is concerned with the delay-dependent $H^\infty$ control problem for systems with two additive time-varying delays. We construct a new Lyapunov functional and give a tighter upper bound for the derivative of the Lyapunov functional, then employ a new method, the polyhedron method, to test the negative definiteness of the upper bound. New delay-dependent stability criteria are thus derived, which are less conservative than some existing ones. Based on the stability criteria a state feedback controller is constructed to guarantee that the closed-loop system is asymptotically stable with a prescribed $H^\infty$ disturbance attenuation level. Finally examples are given to show the advantages of the stability criteria and the effectiveness of the proposed control method.**

***Keywords*- *Additive Time-varying Delays; Polyhedron Method; Lyapunov Functional; Stability; H Control***

## I. INTRODUCTION

Over the last few decades systems with time delays have received considerable attention. The main reason is that they are often encountered in various practical systems, such as engineering systems, biology, economics, neural networks, networked control systems, and other areas [1-6]. Since time-delay is frequently the main cause of oscillation, divergence, or instability, considerable efforts have been devoted to stability for systems with time delays. According to whether stability criteria include the delay, they can be divided into two classes, that is, delay-independent ones and delay-dependent ones. Since delay-independent criteria tend to be more conservative especially for small size delays, more efforts have been put into delay-dependent stability. There are a number of delay-dependent stability results in the literature; we refer readers to the papers [7-23]. Among those papers, papers [17, 20-23] are related to systems with interval time-varying delay. It should be pointed out that all the stability results above are based on systems with one single delay in the state.

Recently Lam, Gao, and Wang [24] proposed a new model of system with two additive time-varying delay components; this model has a strong application background in remote control and networked control. Take a state-feedback networked control for example. Since the physical plant, controller, sensor, and actuator are located at different places, signals are transmitted from one device to another. Thus time delays will appear. Among these delays are two network-induced ones, one from sensor to controller and the other from controller to actuator. So the two delays will appear in the closed-loop system. Because of the network transmission conditions, the two delays are generally time-varying with different properties. It is not rational to lump the two delays into one delay. Therefore it is of significance to consider stability for systems with two additive time-varying delay components. Now we write this kind of system in the following:

$$\dot{x}(t) = Ax(t) + A_1 x(t - d_1(t) - d_2(t)) + Ew(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) + C_1 x(t - d_1(t) - d_2(t)) + Fw(t) + Du(t) \tag{2}$$

$$x(t) = \phi(t), \quad t \in [-h, 0]$$

where $x(t)$ is the state; $y(t)$ is the measurement; $u(t)$ is the control; $w(t) \in L_2[0,\infty]$ is the disturbance; $A$ , $A_1$ , $E$ , $B$ , $C$ , $C_1$ , $F$ , $D$ are known real constant matrices; $d_1(t)$ , $d_2(t)$ are two time-varying delays satisfying

$$0 \le d_1(t) \le h_1,\ 0 \le d_2(t) \le h_2 \tag{3}$$

and

$$\dot{d}_1(t) \le \mu_1,\ \dot{d}_2(t) \le \mu_2 \tag{4}$$

$\phi(t)$ is a real-valued initial function on $[-h, 0]$ with

$$h = h_1 + h_2 \tag{5}$$

Stability analysis was conducted in [24], and a delay-dependent stability criterion was obtained. An improved stability criterion was derived from [25] by constructing a Lyapunov functional to employ the information of the marginally delayed state $x(t-h)$ . However, another marginally delayed state $x(t-h_1)$ was not considered, which caused the integral $-\int_{t-h_1}^{t-d_1(t)} \dot{x}(\alpha)^T Z_1 \dot{x}(\alpha)\, d\alpha$ to be discarded in the estimating of the derivative of the Lyapunov functional. On the other hand, when estimating the derivative of the Lyapunov functional, there existed overly bounding for some integrals. Take $-\int_{t-d_1(t)}^{t} \dot{x}(s)^T Z_1 \dot{x}(s) ds$ as an example. By introducing

$$0 = 2\zeta(t)^T N\left[x(t) - x(t-d_1(t)) - \int_{t-d_1(t)}^{t} \dot{x}(\alpha)d\alpha\right]$$

with $\zeta(t)$ and $N$ an appropriate vector and a matrix respectively, it was estimated as

$$2\zeta(t)^T N[x(t) - x(t-d_1(t))] + \zeta(t)^T d_1(t)NZ_1^{-1}N^T\zeta(t)$$

where $d_1(t)NZ_1^{-1}N^T$ was further enlarged as $h_1NZ_1^{-1}N^T$ in [25]. This may also lead to conservatism.

In this paper we first revisit delay-dependent stability for systems for Systems (1) and (2). We will construct a new Lyapunov functional to employ the information of the marginally delayed state $x(t-h_1)$ as well as $x(t-h)$. Then we propose a so-called convex polyhedron method to avoid the overly bounding for the time derivative of the Lyapunov functional. The resulting stability criteria are less conservative. Then we apply the stability criteria to investigate $H^\infty$ control problem for the system, which is stated as: To design a state feedback controller $u(t) = Kx(t)$ for the system such that the closed-loop system is asymptotically stable with an $H^\infty$ disturbance attenuation level $\gamma > 0$, satisfying $\|y\|_2 < \gamma\|w\|_2$ for nonzero $w(t) \in L_2[0,\infty]$ under zero initial condition. A delay-dependent condition will be presented for the state feedback controller so that the closed-loop system is asymptotically stable with a prescribed $H^\infty$ disturbance attenuation level. Formulated in LMIs the condition is readily verified, and when it is feasible the controller can be constructed.

Notation: Throughout this paper the superscript '$T$' stands for matrix transposition. $I$ refers to an identity matrix with appropriate dimensions. For real symmetric matrices $X$ and $Y$, the notation $X > Y$ means that the matrix $X - Y$ is positive definite, and the $X \geq Y$ follows similarly. The symmetric term in a matrix is denoted by $*$. The space of square-integrable vector functions over $[0,\infty]$ is denoted by $L_2[0,\infty]$, and for $w = \{w(t)\} \in L_2[0,\infty]$ its norm by $\|w\|_2$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

## II. STABILITY ANALYSIS

Now we go to the stability analysis problem. Consider system (1) with $w(t) = u(t) = 0$, namely,

$$\dot{x}(t) = Ax(t) + A_1x(t-d_1(t)-d_2(t)) \tag{6}$$

Set

$$d(t) = d_1(t) + d_2(t) \tag{7}$$

$$\mu = \mu_1 + \mu_2 \tag{8}$$

Take $d_1(t)+d_2(t)$ as one delay $d(t)$ and then system (6) is changed into the following system:

$$\dot{x}(t) = Ax(t) + A_1x(t-d(t)) \tag{9}$$

with $0 \leq d(t) \leq h$, $\dot{d}(t) \leq \mu$.

For this system there are many delay-dependent stability criteria available, but when used to check the stability for (6), they are more conservative [24]. In the following part, we present a new stability result for System (6) by considering the two delays separately.

**Theorem 1**. The System (6) subject to (3) and (4) is asymptotically stable for given $h_1, h_2, \mu_1$ and $\mu_2$ if there exist matrices $P > 0$, $Q_i > 0, i = 1,2,3,4$, $Z_j > 0, j = 1,2$, $N = [N_1^T \quad N_2^T \quad N_3^T \quad 0 \quad 0]^T$, $S = [S_1^T \quad S_2^T \quad S_3^T \quad 0 \quad 0]^T$, $T = [T_1^T \quad T_2^T \quad T_3^T \quad 0 \quad 0]^T$ and $M = [M_1^T \quad M_2^T \quad M_3^T \quad 0 \quad 0]^T$ such that the following LMIs hold

$$\begin{bmatrix} \Phi & h_1N & h_2M \\ * & -h_1(Z_1+Z_2) & 0 \\ * & * & -h_2Z_2 \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} \Phi & h_1N & h_2S \\ * & -h_1(Z_1+Z_2) & 0 \\ * & * & -h_2Z_2 \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} \Phi & h_1S & h_1T & h_2M \\ * & -h_1Z_2 & 0 & 0 \\ * & * & -h_1Z_1 & 0 \\ * & * & * & -h_2Z_2 \end{bmatrix} < 0 \tag{12}$$

and

$$\begin{bmatrix} \Phi & h_1S & h_1T & h_2S \\ * & -h_1Z_2 & 0 & 0 \\ * & * & -h_1Z_1 & 0 \\ * & * & * & -h_2Z_2 \end{bmatrix} < 0 \tag{13}$$

where

$$\Phi = \begin{bmatrix} PA + A^TP + \sum_{i=1}^{4}Q_i & PA_1 & 0 & 0 & 0 \\ * & -(1-\mu)Q_3 & 0 & 0 & 0 \\ * & * & -(1-\mu_1)Q_1 & 0 & 0 \\ * & * & * & -Q_2 & 0 \\ * & * & * & * & -Q_4 \end{bmatrix}$$

$$+\begin{bmatrix} A^T \\ A_1^T \\ 0 \\ 0 \\ 0 \end{bmatrix}(h_1Z_1 + hZ_2)\begin{bmatrix} A^T \\ A_1^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$+\begin{bmatrix} N & S-M & M-N+T & -S & -T \end{bmatrix}$$

$$+\begin{bmatrix} N & S-M & M-N+T & -S & -T \end{bmatrix}^T \tag{14}$$

with $h$ and $\mu$ given in (5) and (8), respectively.

**Proof.** Define a Lyapunov functional for System (6):

$$V(t) = x(t)^TPx(t)$$

$$+\int_{t-d_1(t)}^{t} x(\alpha)^TQ_1x(\alpha)d\alpha + \int_{t-h}^{t} x(\alpha)^TQ_2x(\alpha)d\alpha$$

$$+\int_{t-d(t)}^{t} x(\alpha)^T Q_3 x(\alpha)d\alpha \; +\int_{t-h_1}^{t} x(\alpha)^T Q_4 x(\alpha)d\alpha$$
$$+\int_{-h_1}^{0}\int_{t+s}^{t} \dot{x}(\alpha)^T (Z_1+Z_2)\dot{x}(\alpha)d\alpha ds$$
$$+\int_{-h}^{-h_1}\int_{t+s}^{t} \dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha ds \qquad (15)$$

where $d(t)$ is given in (7). Then calculate the time derivative of the Lyapuonv functional along the trajectory of (6) yields

$$\dot{V}(t) \le 2x(t)^T P(Ax(t)+A_1x(t-d(t))) \; +\sum_{i=1}^{4} x(t)^T Q_i x(t)$$
$$-x(t-h_1)^T Q_4 x(t-h_1) - x(t-h)^T Q_2 x(t-h)$$
$$-(1-\mu)x(t-d(t))^T Q_3 x(t-d(t))$$
$$-(1-\mu_1)x(t-d_1(t))^T Q_1 x(t-d_1(t))$$
$$+[Ax(t)+A_1x(t-d_1(t)-d_2(t))]^T$$
$$\times[h_1Z_1+(h_1+h_2)Z_2][Ax(t)+A_1x(t-d_1(t)-d_2(t))]$$
$$-\int_{t-h_1}^{t} \dot{x}(\alpha)^T (Z_1+Z_2)\dot{x}(\alpha)d\alpha$$
$$-\int_{t-h}^{t-h_1} \dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha \qquad (16)$$

Note that

$$-\int_{t-h_1}^{t} \dot{x}(\alpha)^T (Z_1+Z_2)\dot{x}(\alpha)d\alpha \; -\int_{t-h}^{t-h_1} \dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha$$
$$=-\int_{t-d_1(t)}^{t} \dot{x}(\alpha)^T (Z_1+Z_2)\dot{x}(\alpha)d\alpha - \int_{t-h_1}^{t-d_1(t)} \dot{x}(\alpha)^T Z_1\dot{x}(\alpha)d\alpha$$
$$-\int_{t-d(t)}^{t-d_1(t)} \dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha - \int_{t-h}^{t-d(t)} \dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha \qquad (17)$$

By Leibniz-Newton formula, we have that

$$2\zeta(t)^T N\left[x(t)-x(t-d_1(t))-\int_{t-d_1(t)}^{t}\dot{x}(\alpha)d\alpha\right]=0 \qquad (18)$$

$$2\zeta(t)^T T\left[x(t-d_1(t))-x(t-h_1)-\int_{t-h_1}^{t-d_1(t)}\dot{x}(\alpha)d\alpha\right]=0 \qquad (19)$$

$$2\zeta(t)^T M\left[x(t-d_1(t))-x(t-d(t))-\int_{t-d(t)}^{t-d_1(t)}\dot{x}(\alpha)d\alpha\right]=0 \qquad (20)$$

$$2\zeta(t)^T S\left[x(t-d(t))-x(t-h)-\int_{t-h}^{t-d(t)}\dot{x}(\alpha)d\alpha\right]=0 \qquad (21)$$

where

$$\zeta(t)=\left[x(t)^T \; x(t-d_1(t)-d_2(t))^T \quad x(t-d_1(t))^T \quad x(t-h)^T \quad x(t-h_1)^T\right]^T \qquad (22)$$

From (16)-(21) it follows that

$$\dot{V}(t) \le \zeta(t)^T[\Phi+d_1(t)N(Z_1+Z_2)^{-1}N^T+(h_1-d_1(t))TZ_1^{-1}T^T$$
$$+d_2(t)MZ_2^{-1}M^T+(h-d(t))SZ_2^{-1}S^T]\zeta(t)$$
$$-\int_{t-d_1(t)}^{t}[\zeta(t)^T N+\dot{x}(\alpha)^T(Z_1+Z_2)]$$
$$\times(Z_1+Z_2)^{-1}[N^T\zeta(t)+(Z_1+Z_2)\dot{x}(\alpha)]d\alpha$$
$$-\int_{t-h_1}^{t-d_1(t)}[\zeta(t)^T T+\dot{x}(\alpha)^T Z_1]Z_1^{-1}[T^T\zeta(t)+Z_1\dot{x}(\alpha)]d\alpha$$
$$-\int_{t-d(t)}^{t-d_1(t)}[\zeta(t)^T M+\dot{x}(\alpha)^T Z_2]Z_2^{-1}[M^T\zeta(t)+Z_2\dot{x}(\alpha)]d\alpha$$
$$-\int_{t-h}^{t-d(t)}[\zeta(t)^T S+\dot{x}(\alpha)^T Z_2]Z_2^{-1}[S^T\zeta(t)+Z_2\dot{x}(\alpha)]d\alpha$$
$$\le \zeta(t)^T\varphi(d_1(t),d_2(t))\zeta(t) \qquad (23)$$

where

$$\varphi(d_1(t),d_2(t))$$
$$=\Phi+d_1(t)N(Z_1+Z_2)^{-1}N^T+(h_1-d_1(t))TZ_1^{-1}T^T$$
$$+d_2(t)MZ_2^{-1}M^T+(h-d(t))SZ_2^{-1}S^T.$$

Write $\alpha=d_1(t)/h_1$, $\beta=d_2(t)/h_2$, and then

$$\varphi(d_1(t),d_2(t))$$
$$=\Phi+d_1(t)N(Z_1+Z_2)^{-1}N^T$$
$$+(h_1-d_1(t))(TZ_1^{-1}T^T+SZ_2^{-1}S^T)$$
$$+d_2(t)MZ_2^{-1}M^T+(h_2-d_2(t))SZ_2^{-1}S^T$$
$$=\alpha[\Phi+h_1N(Z_1+Z_2)^{-1}N^T]$$
$$+(1-\alpha)[\Phi+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)]$$
$$+d_2(t)MZ_2^{-1}M^T+(h_2-d_2(t))SZ_2^{-1}S^T$$
$$=\alpha[\Phi+h_1N(Z_1+Z_2)^{-1}N^T$$
$$+d_2(t)MZ_2^{-1}M^T+(h_2-d_2(t))SZ_2^{-1}S^T]$$
$$+(1-\alpha)[\Phi+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)$$
$$+d_2(t)MZ_2^{-1}M^T+(h_2-d_2(t))SZ_2^{-1}S^T]$$
$$=\alpha[\beta(\Phi+h_1N(Z_1+Z_2)^{-1}N^T+h_2MZ_2^{-1}M^T)$$
$$+(1-\beta)(\Phi+h_1N(Z_1+Z_2)^{-1}N^T+h_2SZ_2^{-1}S^T)]$$
$$+(1-\alpha)[\beta(\Phi+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)+h_2MZ_2^{-1}M^T)$$
$$+(1-\beta)(\Phi+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)+h_2SZ_2^{-1}S^T)]. \qquad (24)$$

By Schur complement, (10)-(13) imply that $\varphi(d_1(t),d_2(t))<0$.

Therefore System (6) is asymptotically stable. This ends the proof.

**Remark 1.** Theorem 1 provides a new delay-dependent stability criterion for System (6) with two additive time-varying delay components. In a form of LMIs the criterion can be checked easily.

**Remark 2.** For the Lyapunov functional $V(t)$ in (15), the upper bound of $\dot{V}(t)$, which is given in (23), is tighter, due to that $\varphi(d_1(t),d_2(t))$ is not so enlarged as in [24, 25]. Noting $\varphi(d_1(t),d_2(t))$ dependent on the two time-varying delays rather than the upper bounds of the two time-varying delays, we have to adopt a new technique to test the negative definiteness for it. The basic idea is that a function matrix is negative definite over a convex polyhedron only if the matrix is negative definite at the vertexes. Note that

$$\varphi(h_1,h_2)=\Phi+h_1N(Z_1+Z_2)^{-1}N^T+h_2MZ_2^{-1}M^T$$
$$\varphi(h_1,0)=\Phi+h_1N(Z_1+Z_2)^{-1}N^T+h_2SZ_2^{-1}S^T$$
$$\varphi(0,h_2)=\Phi+h_1TZ_1^{-1}T^T+h_1SZ_2^{-1}S^T+h_2MZ_2^{-1}M^T$$

$$\varphi(0,0)=\Phi+h_1TZ_1^{-1}T^T+h_1SZ_2^{-1}S^T+h_2SZ_2^{-1}S^T.$$

From this we see that the negative definiteness of $\varphi(d_1(t),d_2(t))$ over the rectangle $0\le d_1(t)\le h_1$ $0\le d_2(t)\le h_2$ is determined by that of $\varphi(d_1(t),d_2(t))$ at the vertexes. We call this approach to negative definiteness a convex polyhedron method. Apparently the convex polyhedron method can be extended to more than two time-varying delays.

**Remark 3.** Gao et al. [25] took the advantage of $x(t-h)$ to derive a stability criterion, which improved over that in [24], but another marginally delayed state $x(t-h_1)$ was not used. In this paper, however, it is employed to define the Lyapunov functional $V(t)$ in (15), thus making $-\int_{t-h_1}^{t-d_1(t)}\dot{x}(\alpha)^T Z_1\dot{x}(\alpha)\,d\alpha$ retained in the estimate of $\dot{V}(t)$. Moreover, as $\dot{V}(t)$ is estimated like (23), $d_1(t)N(Z_1+Z_2)^{-1}N^T$ , $(h_1-d_1(t))TZ_1^{-1}T^T$ , $d_2(t)MZ_2^{-1}M^T$ and $(h-d(t))SZ_2^{-1}S^T$ are not so enlarged as in [25], but kept as they are. The resulting stability criterion Theorem 1 is expected to be less conservative than those in [25], as seen from the example in the following.

When $\mu_1$ and $\mu_2$ are unknown, setting $Q_1=Q_3=0$ we can obtain a delay-rate-independent stability criterion from Theorem 1 as follows.

**Corollary 1.** The system (6) subject to (3) is asymptotically stable for given $h_1$ and $h_2$ if there exist matrices $P>0$ , $Q_2>0$ , $Q_4>0$ $Z_j>0, j=1,2$ , $N=[N_1^T\quad N_2^T\quad N_3^T\quad 0\quad 0]^T$ , $S=[S_1^T\quad S_2^T\quad S_3^T\quad 0\quad 0]^T$ , $T=[T_1^T\quad T_2^T\quad T_3^T\quad 0\quad 0]^T$ and $M=[M_1^T\quad M_2^T\quad M_3^T\quad 0\quad 0]^T$ such that the following LMIs hold

$$\begin{bmatrix}\Phi_1 & h_1N & h_2M\\ * & -h_1(Z_1+Z_2) & 0\\ * & * & -h_2Z_2\end{bmatrix}<0$$

$$\begin{bmatrix}\Phi_1 & h_1N & h_2S\\ * & -h_1(Z_1+Z_2) & 0\\ * & * & -h_2Z_2\end{bmatrix}<0$$

$$\begin{bmatrix}\Phi_1 & h_1S & h_1T & h_2M\\ * & -h_1Z_2 & 0 & 0\\ * & * & -h_1Z_1 & 0\\ * & * & * & -h_2Z_2\end{bmatrix}<0$$

and

$$\begin{bmatrix}\Phi_1 & h_1S & h_1T & h_2S\\ * & -h_1Z_2 & 0 & 0\\ * & * & -h_1Z_1 & 0\\ * & * & * & -h_2Z_2\end{bmatrix}<0$$

Where

$$\Phi_1=\begin{bmatrix}PA+A^TP+Q_2+Q_4 & PA_1 & 0 & 0 & 0\\ * & 0 & 0 & 0 & 0\\ * & * & 0 & 0 & 0\\ * & * & * & -Q_2 & 0\\ * & * & * & * & -Q_4\end{bmatrix}$$

$$+\begin{bmatrix}A^T\\ A_1^T\\ 0\\ 0\\ 0\end{bmatrix}(h_1Z_1+hZ_2)\begin{bmatrix}A^T\\ A_1^T\\ 0\\ 0\\ 0\end{bmatrix}^T$$

$$+\begin{bmatrix}N & S-M & M-N & T-S & -T\end{bmatrix}$$

$$+\begin{bmatrix}N & S-M & M-N & T-S & -T\end{bmatrix}^T.$$

When $d_1(t)\equiv h_1$, that is, $d_1(t)$ is a constant delay, Theorem 1 reduces to the following corollary, which is useful for networked control.

**Corollary 2.** The system (6) with $d_1(t)\equiv h_1$ and $d_2(t)$ satisfying $0\le d_2(t)\le h_2$ and $\dot{d}(t)\le\mu_2$ is asymptotically stable for given $h_2>0$ , $h_1>0$ and $\mu_2$ if there exist $P>0$ , $Q_i>0$ , $i=1,2,3$ , $Z_j>0$ , $j=1,2$ , $N=[N_1^T\quad N_2^T\quad 0\quad 0]^T$ , $M=[M_1^T\quad M_2^T\quad 0\quad 0]^T$ and $S=[S_1^T\quad S_2^T\quad 0\quad 0]^T$ such that the following LMIs hold

$$\begin{bmatrix}\Phi_2 & h_1N & h_2M\\ * & -h_1Z_1 & 0\\ * & * & -h_2Z_2\end{bmatrix}<0,$$

$$\begin{bmatrix}\Phi_2 & h_1N & h_2S\\ * & -h_1Z_1 & 0\\ * & * & -h_2Z_2\end{bmatrix}<0$$

where

$$\Phi_2=\begin{bmatrix}PA+A^TP+\sum_{i=1}^{3}Q_i & PA_1 & 0 & 0\\ * & -(1-\mu_2)Q_3 & 0 & 0\\ * & * & -Q_1 & 0\\ * & * & * & -Q_2\end{bmatrix}$$

$$+\begin{bmatrix}A^T\\ A_1^T\\ 0\\ 0\end{bmatrix}(h_1Z_1+h_2Z_2)\begin{bmatrix}A^T\\ A_1^T\\ 0\\ 0\end{bmatrix}^T$$

$$+\begin{bmatrix}N & S-M & M-N & -S\end{bmatrix}+\begin{bmatrix}N & S-M & M-N & -S\end{bmatrix}^T.$$

**Proof.** Define the Lyapunov functional

$$V(t)=x(t)^TPx(t)+\int_{t-d(t)}^{t}x(\alpha)^TQ_3x(\alpha)d\alpha$$
$$+\int_{t-h_1}^{t}x(\alpha)^TQ_1x(\alpha)d\alpha+\int_{t-h}^{t}x(\alpha)^TQ_2x(\alpha)d\alpha$$

$$+\int_{-h_1}^{0}\int_{t+s}^{t}\dot{x}(\alpha)^T Z_1\dot{x}(\alpha)d\alpha ds+\int_{-h}^{-h_1}\int_{t+s}^{t}\dot{x}(\alpha)^T Z_2\dot{x}(\alpha)d\alpha ds\ .$$

Along a similar line as in the derivation of Theorem 1 the asymptotic stability can be established, and the proof is thus omitted.

**Remark 4.** Note that when $d_1(t)$ is a constant delay $h_1$, system (6) can be regarded as system (9) with interval time-varying delay: $h_1 \le d(t) \le h$ , $0 \le \dot{d}(t) \le \mu_2$ . The system can serve as a model for networked control systems with both network-induced delay and data dropout phenomenon [17, 25]. In the form of LMIs Corollary 2 can provide a delay-dependent stability criterion for this delayed system. Thanks to the convex polyhedron method Corollary 2 is less conservative than those recently reported in [20]; Refer to [26].

In the following part, we consider the example in [25] to show the reduced conservatism of our stability criteria.

**Example 1.** Consider the System (6) with

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix},\ A_1=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},\ \dot{d}_1(t)\le 0.1,\ \dot{d}_2(t)\le 0.8$$

For given upper bound $h_1$ of $d_1(t)$ , we intend to find the admissible upper bound $h_2$ of $d_2(t)$ , which guarantee the asymptotic stability of (6).

TABLE I ADMISSIBLE UPPER BOUND H2 FOR VARIOUS H1

| **Method** | $h_1$ | **1** | **1.2** | **1.5** |
|---|---|---|---|---|
| [24] | $h_2$ | 0.415 | 0.376 | 0.248 |
| [25] | $h_2$ | 0.512 | 0.406 | 0.283 |
| Theorem 1 | $h_2$ | 0.8731 | 0.6766 | 0.4529 |

When $h_2$ is given the admissible $h_1$ can be seen from Table 2.

TABLE II ADMISSIBLE UPPER BOUND H1 FOR VARIOUS H2

| **Method** | $h_2$ | **0.1** | **0.2** | **0.3** |
|---|---|---|---|---|
| [24] | $h_1$ | 2.263 | 1.696 | 1.324 |
| [25] | $h_1$ | 2.300 | 1.779 | 1.453 |
| Theorem 1 | $h_1$ | 2.5583 | 2.1003 | 1.8083 |

From the comparison between Table 1 and Table 2, it can be seen that Theorem 1 is less conservative than those in [24, 25].

When $d_1(t)$ is a constant delay $h_1$ , the system can be looked upon as those with interval time-varying delay. As indicated in Remark 4, the stability criterion Corollary 2 as well as that in [20] can be turned to for computing the admissible upper bound $h$ of $d(t)$ , which are shown in Table 3.

TABLE III ADMISSIBLE UPPER BOUND H FOR VARIOUS H1

| Method | $h_1$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| [20] | $h$ | 1.7423 | 2.4328 | 3.2234 | 4.0644 |
| Corollary 2 | $h$ | 2.0665 | 2.6181 | 3.3173 | 4.0905 |

Even as a delay-dependent criterion for systems with interval time-varying delay, Corollary 2 has advantages over some existing ones in the sense that the computed admissible upper bound of the time-varying delay is larger.

## III. $H^\infty$ CONTROL

Theorem 1 can be expected to be a useful tool for the $H^\infty$ control problem formulated above. We first write the closed-loop system formulated by Systems (1)-(2) and the controller $u(t)=Kx(t)$ .

$$\dot{x}(t)=(A+BK)x(t)+A_1x(t-d_1(t)-d_2(t))+Ew(t) \tag{25}$$

$$y(t)=(C+DK)x(t)+C_1x(t-d_1(t)-d_2(t))+Fw(t) \tag{26}$$

Now we present an $H^\infty$ performance analysis result in the following.

**Theorem 2.** The Systems (25)-(26) subject to (3) and (4) is asymptotically stable with an $H^\infty$ disturbance attenuation level $\gamma$ for given $h_1$ , $h_2$ , $\mu_1$ and $\mu_2$ , if there exist $P>0$ , $Q_i>0$ , $i=1,2,3,4$ , $Z_j>0$ , $j=1,2$ , $N=[N_1^T\ \ N_2^T\ \ N_3^T\ \ 0\ \ 0\ \ 0]^T$ ,

$$S=[S_1^T\ \ S_2^T\ \ S_3^T\ \ 0\ \ 0\ \ 0]^T,$$
$$T=[T_1^T\ \ T_2^T\ \ T_3^T\ \ 0\ \ 0\ \ 0]^T$$

and

$$M=[M_1^T\ \ M_2^T\ \ M_3^T\ \ 0\ \ 0\ \ 0]^T$$

so that the following LMIs hold

$$\begin{bmatrix}\Phi_c & h_1N & h_2M\\ * & -h_1(Z_1+Z_2) & 0\\ * & * & -h_2Z_2\end{bmatrix}<0 \tag{27}$$

$$\begin{bmatrix}\Phi_c & h_1N & h_2S\\ * & -h_1(Z_1+Z_2) & 0\\ * & * & -h_2Z_2\end{bmatrix}<0 \tag{28}$$

$$\begin{bmatrix}\Phi_c & h_1S & h_1T & h_2M\\ * & -h_1Z_2 & 0 & 0\\ * & * & -h_1Z_1 & 0\\ * & * & * & -h_2Z_2\end{bmatrix}<0 \tag{29}$$

and

$$\begin{bmatrix}\Phi_c & h_1S & h_1T & h_2S\\ * & -h_1Z_2 & 0 & 0\\ * & * & -h_1Z_1 & 0\\ * & * & * & -h_2Z_2\end{bmatrix}<0 \tag{30}$$

where

$$\Phi_c=\begin{bmatrix}\varphi_c & PA_1 & 0 & 0 & 0 & PE\\ * & -(1-\mu)Q_3 & 0 & 0 & 0 & 0\\ * & * & -(1-\mu_1)Q_1 & 0 & 0 & 0\\ * & * & * & -Q_2 & 0 & 0\\ * & * & * & * & -Q_4 & 0\\ * & * & * & * & * & -\gamma^2 I\end{bmatrix}$$

$$+\begin{bmatrix}(A+BK)^T\\ A_1^T\\ 0\\ 0\\ 0\\ E^T\end{bmatrix}[h_1Z_1+hZ_2]\begin{bmatrix}(A+BK)^T\\ A_1^T\\ 0\\ 0\\ 0\\ E^T\end{bmatrix}^T$$

$$+\begin{bmatrix}(C+DK)^T\\ C_1^T\\ 0\\ 0\\ 0\\ F^T\end{bmatrix}\begin{bmatrix}(C+DK)^T\\ C_1^T\\ 0\\ 0\\ 0\\ F^T\end{bmatrix}^T$$

$$+\begin{bmatrix}N & S-M & M-N+T & -S & -T & 0\end{bmatrix}$$
$$+\begin{bmatrix}N & S-M & M-N+T & -S & -T & 0\end{bmatrix}^T$$

with $\varphi_c=P(A+BK)+(A+BK)^TP+\sum_{i=1}^4 Q_i$ and $h$ given in Theorem1.

**Proof:** Suppose that (27)-(30) hold. By comparing $\Phi_c$ with $\Phi$ in (14), we can find that (27)-(30) imply (10)-(13). According to Theorem 1, the system is asymptotically stable.

Now using the same Lyapunov functional $V(t)$ in (15) and calculating $\dot{V}(t)$ similar to the derivation of Theorem 1 along the solution of Systems (25)-(26), we have

$$y(t)^T y(t)-\gamma^2 w(t)^T w(t)+\dot{V}(t)\le \bar{\zeta}(t)^T\bar{M}(\alpha,\beta)\bar{\zeta}(t) \qquad (31)$$

where

$$\begin{aligned}\bar{M}(\alpha,\beta)&=\alpha[\beta(\Phi_c+h_1N(Z_1+Z_2)^{-1}N^T+h_2MZ_2^{-1}M^T)\\&\quad+(1-\beta)(\Phi_c+h_1N(Z_1+Z_2)^{-1}N^T+h_2SZ_2^{-1}S^T)]\\&\quad+(1-\alpha)[\beta(\Phi_c+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)+h_2MZ_2^{-1}M^T)\\&\quad+(1-\beta)(\Phi_c+h_1(TZ_1^{-1}T^T+SZ_2^{-1}S^T)+h_2SZ_2^{-1}S^T)]\end{aligned}$$

where $\alpha$, $\beta$ are defined in the proof of Theorem 1, and

$$\bar{\zeta}(t)=\begin{bmatrix}\zeta(t)^T & w(t)^T\end{bmatrix}^T$$

with $\zeta(t)$ in (22). On the one hand, using the convex polyhedron method we can establish $\bar{M}(\alpha,\beta)<0$ by (27)-(30). On the other hand, under the zero condition we have $V(0)=0$ and $V(\infty)\ge 0$. So integrating both sides of (31) results in $\|y\|_2<\gamma\|w\|_2$ for all nonzero $w(t)\in L_2[0,\infty]$. This ends the proof.

Now we are in a position to deal with the $H^\infty$ control problem aforementioned.

**Theorem 3.** Consider Systems (1) and (2) with delays subject to (3) and (4). Given $\gamma$, $h_1,h_2,\mu_1$ and $\mu_2$, there exists a state-feedback controller $u(t)=Kx(t)$ ensuring that the closed-loop system is asymptotically stable with an $H^\infty$ disturbance attenuation level $\gamma$, if there exist matrices $\bar{K}$, $\bar{P}>0$, $\bar{Q}_i>0$, $i=1,2,3,4$, $\bar{Z}_j>0$, $j=1,2$,

$$\bar{N}=[\bar{N}_1^T\quad \bar{N}_2^T\quad \bar{N}_3^T\quad 0\quad 0\quad 0]^T,$$
$$\bar{S}=[\bar{S}_1^T\quad \bar{S}_2^T\quad \bar{S}_3^T\quad 0\quad 0\quad 0]^T,$$
$$\bar{T}=[\bar{T}_1^T\quad \bar{T}_2^T\quad \bar{T}_3^T\quad 0\quad 0\quad 0]^T$$

and

$$\bar{M}=[\bar{M}_1^T\quad \bar{M}_2^T\quad \bar{M}_3^T\quad 0\quad 0\quad 0]^T$$

so that the following LMIs hold

$$\begin{bmatrix}\Upsilon_i & \Gamma_i\\ \Gamma_i^T & \Lambda\end{bmatrix}<0,\ i=1,2,3,4 \qquad (32)$$

where

$$\Gamma_1=\Gamma_2=\begin{bmatrix}\bar{P}A^T+\bar{K}^TB^T & \bar{P}A^T+\bar{K}^TB^T & \bar{P}C^T+\bar{K}^TD^T\\ \bar{P}A_1^T & \bar{P}A_1^T & \bar{P}C_1^T\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ E^T & E^T & F^T\\ 0&0&0\\ 0&0&0\end{bmatrix}$$

$$\Gamma_3=\Gamma_4=\begin{bmatrix}\bar{P}A^T+\bar{K}^TB^T & \bar{P}A^T+\bar{K}^TB^T & \bar{P}C^T+\bar{K}^TD^T\\ \bar{P}A_1^T & \bar{P}A_1^T & \bar{P}C_1^T\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ E^T & E^T & F^T\\ 0&0&0\\ 0&0&0\\ 0&0&0\end{bmatrix} \qquad (33)$$

$$\Lambda=diag\{h_1^{-1}(\bar{Z}_1-2\bar{P}),h^{-1}(\bar{Z}_2-2\bar{P}),-I\} \qquad (34)$$

$$\Upsilon_1=\begin{bmatrix}\Psi & h_1\bar{N} & h_2\bar{M}\\ * & -h_1(\bar{Z}_1+\bar{Z}_2) & 0\\ * & * & -h_2\bar{Z}_2\end{bmatrix} \qquad (35)$$

$$\Upsilon_2=\begin{bmatrix}\Psi & h_1\bar{N} & h_2\bar{S}\\ * & -h_1(\bar{Z}_1+\bar{Z}_2) & 0\\ * & * & -h_2\bar{Z}_2\end{bmatrix} \quad (36)$$

$$\Upsilon_3=\begin{bmatrix}\Psi & h_1\bar{S} & h_1\bar{T} & h_2\bar{M}\\ * & -h_1\bar{Z}_2 & 0 & 0\\ * & * & -h_1\bar{Z}_1 & 0\\ * & * & * & -h_2\bar{Z}_2\end{bmatrix} \quad (37)$$

and

$$\Upsilon_4=\begin{bmatrix}\Psi & h_1\bar{S} & h_1\bar{T} & h_2\bar{S}\\ * & -h_1\bar{Z}_2 & 0 & 0\\ * & * & -h_1\bar{Z}_1 & 0\\ * & * & * & -h_2\bar{Z}_2\end{bmatrix} \quad (38)$$

with

$$\Psi=\begin{bmatrix}\psi & A_1\bar{P} & 0 & 0 & 0 & E\\ * & -(1-\mu)\bar{Q}_3 & 0 & 0 & 0 & 0\\ * & * & -(1-\mu_1)\bar{Q}_1 & 0 & 0 & 0\\ * & * & * & -\bar{Q}_2 & 0 & 0\\ * & * & * & * & -\bar{Q}_4 & 0\\ * & * & * & * & * & -\gamma^2 I\end{bmatrix}$$

$$+\begin{bmatrix}\bar{N} & \bar{S}-\bar{M} & \bar{M}-\bar{N}+\bar{T} & -\bar{S} & -\bar{T} & 0\end{bmatrix}$$

$$+\begin{bmatrix}\bar{N} & \bar{S}-\bar{M} & \bar{M}-\bar{N}+\bar{T} & -\bar{S} & -\bar{T} & 0\end{bmatrix}^T$$

with $\psi=A\bar{P}+\bar{P}A^T+\sum_{i=1}^{4}\bar{Q}_i$ and $h$ given in Theorem1. Moreover, if the foregoing condition is held, a desired controller gain matrix is given by

$$K=\bar{K}\bar{P}^{-1} \quad (39)$$

**Proof:** By Theorem 2 the closed-loop system is asymptotically stable with an H$^\infty$ disturbance attenuation level $\gamma$, if there exist $P>0$,

$$Q_i>0,\ i=1,2,3,4\ ,\ Z_j>0,\ j=1,2$$

$$N=[N_1^T\quad N_2^T\quad N_3^T\quad 0\quad 0\quad 0]^T,$$

$$S=[S_1^T\quad S_2^T\quad S_3^T\quad 0\quad 0\quad 0]^T,$$

$$T=[T_1^T\quad T_2^T\quad T_3^T\quad 0\quad 0\quad 0]^T$$

and

$$M=[M_1^T\quad M_2^T\quad M_3^T\quad 0\quad 0\quad 0]^T$$

so that the LMIs (27)-(30) hold.

Write

$$J_1=diag\{P^{-1},P^{-1},P^{-1},P^{-1},P^{-1},I,P^{-1},P^{-1}\},$$
$$J_2=diag\{P^{-1},P^{-1},P^{-1},P^{-1},P^{-1},I,P^{-1},P^{-1},P^{-1}\},$$
$\bar{P}=P^{-1}$, $\bar{Z}_i=P^{-1}Z_iP^{-1}$, $i=1,2$, $\bar{Q}_j=P^{-1}Q_jP^{-1}$, $j=1,2,3,4$,

$$\begin{bmatrix}\bar{M} & \bar{N} & \bar{S} & \bar{T}\end{bmatrix}=P^{-1}\begin{bmatrix}M & N & S & T\end{bmatrix}P^{-1}.$$

With these notations and (38) in mind, performing a congruence transformation to (27)-(28) and (29)-(30) by $J_1$ and $J_2$ respectively, by Schur complements we obtain

$$\begin{bmatrix}\Upsilon_i & \Gamma_i\\ \Gamma_i^T & \tilde{\Lambda}\end{bmatrix}<0,\ i=1,2,3,4 \quad (40)$$

where $\Gamma_i$ is given in (33), $\Upsilon_i(i=1,2,3,4)$ are defined in (35)-(38), and

$$\tilde{\Lambda}=diag\{-h_1^{-1}Z_1^{-1},-h^{-1}Z_2^{-1},-I\}.$$

Noting $(\bar{P}-\bar{Z}_i)\bar{Z}_i^{-1}(\bar{P}-\bar{Z}_i)\geq 0$ we have $\bar{P}\bar{Z}_i^{-1}\bar{P}\geq -\bar{Z}_i+2\bar{P}$. Therefore, $Z_i^{-1}\geq -\bar{Z}_i+2\bar{P}$. It follows immediately that $\tilde{\Lambda}\leq\Lambda$, which means that (40) holds if (32) do. The proof is thus completed.

**Remark 5.** Different from (40), conditions in (32) are linear in $\bar{P}$, $\bar{Q}_i$, $\bar{Z}_j$, $\bar{K}$, $\bar{N}$, $\bar{S}$, $\bar{T}$ and $\bar{M}$. As a result, for given $h_1,h_2,\mu_1$ and $\mu_2$ Theorem 3 provides an LMI approach to the H$^\infty$ control problem for systems with two additive time-varying delays. The feasibility of LMIs (32) guarantees the existence of H$^\infty$ state feedback controllers. Moreover, when LMIs (32) are feasible, the controller can be constructed with (39). However, the Condition (32) is slightly conservative compared with (40). Based on (40), one can obtain a less conservative controller at the cost of more complexity by Employing CCL method [27].

To illustrate the effectiveness of this control method we provide an example.

**Example 2.** Consider Systems (1) and (2) with parameters given as follows:

$$A=\begin{bmatrix}0.11 & 0\\ 0 & -0.9\end{bmatrix},\ A_1=\begin{bmatrix}-2 & 0\\ -1 & 1.1\end{bmatrix},\ E=\begin{bmatrix}0.56\\ 0.61\end{bmatrix},\ B=\begin{bmatrix}0.2\\ -2.5\end{bmatrix},$$
$$C=\begin{bmatrix}0.1 & 1.8\end{bmatrix},\ C_1=\begin{bmatrix}0.7 & -1\end{bmatrix},\ F=0.1,\ D=0.4.$$

For $h_1=0.1$, $h_2=0.4$, $\mu_1=0.1$, $\mu_2=0.2$ and $\gamma=1$ we can find LMIs in (32) are feasible with

$$\bar{P}=\begin{bmatrix}1.2730 & 0.4456\\ 0.4456 & 0.3246\end{bmatrix},\ \bar{K}=\begin{bmatrix}-0.0798 & 0.0583\end{bmatrix}.$$

By Theorem 3, there exists a state feedback controller

$$u(t)=\bar{K}\bar{P}^{-1}x(t)=\begin{bmatrix}-0.2417 & 0.5114\end{bmatrix}x(t)$$

so that the closed-loop system is asymptotically stable for $0\leq d_1(t)\leq 0.1$ $0\leq d_2(t)\leq 0.4$ with an H$^\infty$ disturbance attenuation level $\gamma=1$.

## IV. CONCLUSIONS

In this paper, delay-dependent H$^\infty$ control problem has been investigated for systems with two additive time-varying delay components. For one thing a new delay-dependent stability criterion was developed, which improves over existing ones in that it has less conservatism. A delay-rate-independent criterion was obtained as a by-product. When one of the delays is constant, a new stability criterion was given for systems with interval time-varying delay. Then examples were provided to illustrate the reduced conservatism of the criteria. Finally the $H^{\infty}$ control problem was solved via an LMI approach, which was demonstrated to be effective using another example.

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