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On the basisness in $L_2(0,1)$ of the root functions in not strongly regular boundary value problems

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Abstract. In the present article we consider the non-self adjoint Sturm-Liouville operators with periodic and anti-periodic boundary conditions which are not strongly regular. We obtain the asymptotic formulas for eigenvalues and eigenfunctions of these boundary value problems, when the potential $q(x)$ is a complex-valued function. Then using these asymptotic formulas, the Riesz basisness in $L_2(0,1)$ of the root functions are proved.

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Key words: Riesz basis, periodic and anti-periodic boundary conditions, not strongly regular boundary conditions, eigenvalue, eigenfunction, non-self adjoint Sturm-Liouville operator, Bari's theorem.

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1. Introduction

It is well known that the basisness of the root functions of a differential operator depends on regularity of boundary conditions generating the given differential operator. The basisness in the space $L_2(0,1)$ of the root functions of a linear differential operator of order n with regular (strongly regular, see. [1], p.71) boundary conditions is shown in [2, 3]. In [2, 4, 5] it is shown that the root functions of a boundary problem which is generated by not strongly regular boundary conditions may not be form a basis in $L_2(0,1)$. In [6], one non-classical heat conduction problem in homogeneous rod has been studied. This problem is reduced to the following boundary value problem

$$-y''(x) = \lambda y(x), \quad 0 < x < 1,$$
$$y(0) = 0, \; y'(0) = y'(1)$$

whose boundary conditions are regular, but not strongly regular. All the eigenvalues of this problem starting with the second one are double, the total number of associated functions is infinite. Nevertheless, in the paper it was established that the chosen specially system of the root functions forms an unconditional basis in $L_2(0,1)$.

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After this work, in [7], the boundary-value problem generated by the differential equation

$$y'' + q(x)y = \lambda y \tag{1.1}$$

and not strongly regular boundary conditions

$$y(0) - y(1) = 0, \quad y'(0) - y'(1) = 0 \tag{1.2}$$

or

$$y(0) + y(1) = 0, \quad y'(0) + y'(1) = 0 \tag{1.3}$$

was considered. Here, $q(x) \in C^{(4)}[0,1]$ was a complex valued function satisfying the condition $q(0) \neq q(1)$. In this paper, it was shown that the root functions of the boundary problems (1.1), (1.2) and (1.1), (1.3) formed Riesz basis in $L_2(0,1)$.

Let us present briefly the main definitions and fact which will be used in what follows.

Definition 1.1. *A system $\{\varphi_n\}_{n=1}^{\infty}$ forms a basis in a Banach space X if for any element $f \in X$ there exists a unique expansion of it in the elements of the system, i.e. the series $\sum\limits_{j=1}^{\infty} c_j \varphi_j$ convergent to f in the norm of the space X.*

Definition 1.2. *[8,9] A system $\{\varphi_n\}_{n=1}^{\infty}$ is called a Riesz basis of the Hilbert space H if there exists a bounded linear invertible operator A such that the system $\{A\varphi_n\}_{n=1}^{\infty}$ forms an orthonormal basis in H.*

Theorem 1.1. *[8,9] If the sequence $\{\varphi_j\}_{j=1}^{\infty}$ is complete in the Hilbert space H, there corresponds to it a complete biorthogonal sequence $\{\psi_j\}_{j=1}^{\infty}$, and for any $f \in H$ one has $\sum\limits_{j=1}^{\infty} |(f,\varphi_j)| < \infty$, $\sum\limits_{j=1}^{\infty} |(f,\psi_j)|^2 < \infty$, then the sequence $\{\psi_j\}_{j=1}^{\infty}$ forms a Riesz basis in H.*

We consider the boundary-value problems (1.1), (1.2) and (1.1), (1.3), where $q(x) \in C^{(4)}[0,1]$ is a complex-valued function. Without loss of generality, we can assume that $\int\limits_0^1 q(x)dx = 0$.

In the present paper, in Section 2 we obtain the asymptotic formulas of eigenvalues and eigenfunctions of the boundary problems (1.1), (1.2). In Section 3, using these asymptotic formulas and Theorem 1.1, we prove the basisness in $L_2(0,1)$ of the root functions of the boundary problem (1.1), (1.2). In Section 4, similar results are obtained for the boundary problem (1.1), (1.3).

2. The asymptotic formulas for eigenvalues and eigenfunctions of the periodic problem

First we shall prove the following lemma.

Lemma 2.1. *All eigenvalues of the boundary-value problem (1.1), (1.2), starting from some number, are simple and form two infinite sequences $\lambda_{k,1}$, $\lambda_{k,2}$, $k = N, N+1, \cdots$, where N is a positive integer and*

$$\lambda_{k,1} = -(2k\pi)^2 - \frac{q'(1) - q'(0) + \int_0^1 q^2(t)dt}{(4k\pi)^2} + O(\frac{1}{k^3}), \tag{2.1}$$

$$\lambda_{k,2} = -(2k\pi)^2 + \frac{q'(1) - q'(0) - \int_0^1 q^2(t)dt}{(4k\pi)^2} + O(\frac{1}{k^3}), \tag{2.2}$$

and the corresponding eigenfunctions are of the form

$$y_{k,1}(x) = \sin 2k\pi x + O(\frac{1}{k}), \tag{2.3}$$

$$y_{k,2}(x) = \cos 2k\pi x + O(\frac{1}{k}). \tag{2.4}$$

Proof. We assume that $q(0) = q(1)$. The case $q(0) \neq q(1)$ was investigated in [7]. Consider the equation (1.3) or

$$y'' + q(x)y + \mu^2 y = 0, \tag{2.5}$$

where $\mu = \sqrt{-\lambda}$ and $\sqrt{r}e^{i\varphi/2}$ for $-\pi < \varphi \leq \pi$. From [1,10], it is well known that the eigenvalues of the boundary problem (1.1), (1.2) are asymptotically located in pairs, i.e.

$$\lambda_{k,1} = \lambda_{k,2} + O(k^{1/2}) = -(2k\pi)^2 \left\{ 1 + \frac{\xi_0}{k} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \ (k = N, N+1, \cdots).$$

It follows from the last relation that

$$\begin{aligned} \mu_{k,1} &= \sqrt{-\lambda_{k,1}} = 2k\pi \left\{ 1 + \frac{\xi_0}{2k} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \ (k = N, N+1, \cdots) \\ \mu_{k,2} &= \sqrt{-\lambda_{k,2}} = 2k\pi \left\{ 1 + \frac{\xi_0}{2k} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \ (k = N, N+1, \cdots). \end{aligned}$$

Hence, there exists a positive number c_o such that $|\Im(\mu_{k,1})| \leq c_o$ and $|\Im(\mu_{k,2})| \leq c_o$. Thus, the relation

$$\mu_{k,1}, \mu_{k,2} \in Q = \{\mu : \Re(\mu) \geq 0, |\Im(\mu)| \leq c_o\}$$

holds for all $k = N, N+1, \cdots$. It is easy to verify that $Q \subset S_0 - ic_o \equiv T$, where $S_0 = \left\{\mu : 0 \leq \arg \mu \leq \frac{\pi}{2}\right\}$.

From [1,10], it is well known that in a region T of the complex plane μ the equation (2.5) has two linear independent solutions $\varphi_1(x,\mu)$, $\varphi_2(x,\mu)$ satisfying the relations

$$\varphi_j(x,\mu) = e^{\mu\omega_j x} \left\{ \sum_{m=0}^{6} \frac{u_m(x)}{(2\omega_j \mu)^m} + O(\frac{1}{\mu^7}) \right\}, \ (j = 1, 2),$$

$$\varphi_j'(x,\mu) = \mu\omega_j e^{\mu\omega_j x}\left\{u_0(x) + \sum_{m=1}^{6}\frac{u_m(x)+2u_{m-1}'(x)}{(2\omega_j\mu)^m} + O(\frac{1}{\mu^7})\right\},\ (j=1,2),$$

where

$$\omega_1 = -\omega_2 = i,\ u_0(x)\equiv 1,\ \ u_m(x) = -\int_0^x l\left(u_{m-1}(t)\right)dt,\ m=1,2,3,4,5,6.$$

It follows that

$$\varphi_j(0,\mu) = 1 + O(\frac{1}{\mu^7}),$$

$$\begin{aligned}
\varphi_j(1,\mu) \;=\; & e^{\mu\omega_j}\Bigg\{1 - \frac{1}{(2\omega_j\mu)^3}[q'(1)-q'(0)+\int_0^1 q^2(t)dt] + \frac{1}{(2\omega_j\mu)^4}[q''(1)-q''(0) \\
& +\frac{5}{2}q^2(1) - \frac{3}{2}q^2(0) - q(0)q(1)] - \frac{1}{(2\omega_j\mu)^5}[q'''(1)-q'''(0)+7q(1)q'(1) \\
& -5q(0)q'(0) - q(0)q'(1) - q(1)q'(0) + (q(1)-q(0))\int_0^1 q^2(t)dt \\
& +2\int_0^1 q^3(t)dt - \int_0^1 q'^2(t)dt] + \frac{1}{(2\omega_j\mu)^6}[q^{(4)}(1) - q^{(4)}(0) + 9q(1)q''(1) \\
& -7q(0)q''(0) - q(0)q''(1) - q(1)q''(0) + \frac{11}{2}q'^2(1) - \frac{9}{2}q'^2(0) - q'(0)q'(1) \\
& +\frac{15}{2}q^3(1) - \frac{7}{2}q^3(0) - \frac{5}{2}q(0)q^2(1) - \frac{3}{2}q(1)q^2(0) \\
& +(q'(1)-q'(0))\int_0^1 q^2(t)dt + \frac{1}{2}(\int_0^1 q^2(t)dt)^2] + O(\frac{1}{\mu^7})\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\varphi_j'(0,\mu) \;=\; & \mu\omega_j\Bigg\{1 - \frac{2q(0)}{(2\omega_j\mu)^2} + \frac{2q'(0)}{(2\omega_j\mu)^3} - \frac{1}{(2\omega_j\mu)^4}[2q''(0)+2q^2(0)] \\
& +\frac{1}{(2\omega_j\mu)^5}[2q'''(0)+8q(0)q'(0)] - \frac{1}{(2\omega_j\mu)^6}[2q^{(4)}(0) \\
& +10q'^2(0) + 12q(0)q''(0) + 4q^3(0)] + O(\frac{1}{\mu^7})\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\varphi_j'(1,\mu) \;=\; & \mu\omega_j e^{\mu\omega_j}\Bigg\{1 - \frac{q(0)+q(1)}{(2\omega_j\mu)^2} + \frac{1}{(2\omega_j\mu)^3}[q'(1)+q'(0) - \int_0^1 q^2(t)dt] \\
& -\frac{1}{(2\omega_j\mu)^4}[q''(1)+q''(0) + \frac{3}{2}q^2(1) + \frac{3}{2}q^2(0) - q(0)q(1)]
\end{aligned}$$

$$+\frac{1}{(2\omega_j\mu)^5}[q'''(1)+q'''(0)+5q(1)q'(1)+5q(0)q'(0)-q(0)q'(1)$$
$$-q(1)q'(0)+(q(1)+q(0))\int_0^1 q^2(t)dt-2\int_0^1 q^3(t)dt+\int_0^1 q'^2(t)dt]$$
$$-\frac{1}{(2\omega_j\mu)^6}[q^{(4)}(1)+q^{(4)}(0)+\frac{13}{2}q'^2(1)+\frac{9}{2}q'^2(0)-q'(0)q'(1)$$
$$+7q(1)q''(1)+7q(0)q''(0)-q(0)q''(1)-q(1)q''(0)+\frac{7}{2}q^3(1)$$
$$+\frac{7}{2}q^3(0)-\frac{3}{2}q(0)q^2(1)-\frac{3}{2}q(1)q^2(0)+(q'(1)+q'(0))\int_0^1 q^2(t)dt$$
$$\left.-\frac{1}{2}(\int_0^1 q^2(t)dt)^2]+O(\frac{1}{\mu^7})\right\}.$$

Let us substitute all these expressions into the characteristic determinant

$$\Delta(\mu)=\begin{vmatrix} U_1(\varphi_1) & U_1(\varphi_2) \\ U_2(\varphi_1) & U_2(\varphi_2) \end{vmatrix},$$

where $U_1(y)=y(1)-y(0)$, $U_2(y)=y'(1)-y'(0)$.

By elementary transformations, we obtain the relation

$$\begin{aligned}(i\mu)^{-1}\Delta(\mu) \quad = \quad & e^{2i\mu}\left\{1-\frac{2q(0)}{(2i\mu)^2}-\frac{1}{(2i\mu)^3}\int_0^1 q^2(t)dt-\frac{1}{(2i\mu)^4}[2q''(0)-\frac{1}{2}q^2(1)\right.\\
&+\frac{3}{2}q^2(0)+q(0)q(1)]-\frac{1}{(2i\mu)^5}[q(1)q'(1)-q(0)q'(1)-q(0)q'(0)\\
&+q(1)q'(0)-2q(0)\int_0^1 q^2(t)dt+2\int_0^1 q^3(t)dt-\int_0^1 q'^2(t)dt]\\
&-\frac{1}{(2i\mu)^6}[2q^{(4)}(0)+\frac{1}{2}q'^2(1)+\frac{21}{2}q'^2(0)-q'(0)q'(1)-q(1)q''(1)\\
&+11q(0)q''(0)+q(0)q''(1)+q(1)q''(0)-2q^3(1)+3q^3(0)\\
&\left.+3q(0)q^2(1)-(\int_0^1 q^2(t)dt)^2]+O(\frac{1}{\mu^7})\right\}\end{aligned}$$

$$-2e^{i\mu}\left\{1-\frac{2q(0)}{(2i\mu)^2}-\frac{1}{(2i\mu)^4}[2q''(0)+2q^2(0)]-\frac{1}{(2i\mu)^6}[2q^{(4)}(0)\right.$$
$$\left.+12q(0)q''(0)+10q'^2(0)+4q^3(0)]+O(\frac{1}{\mu^7})\right\}$$
$$+\left\{1-\frac{2q(0)}{(2i\mu)^2}+\frac{1}{(2i\mu)^3}\int_0^1 q^2(t)dt-\frac{1}{(2i\mu)^4}[2q''(0)-\frac{1}{2}q^2(1)\right.$$
$$+\frac{3}{2}q^2(0)+q(0)q(1)]+\frac{1}{(2i\mu)^5}[q(1)q'(1)-q(0)q'(0)-q(0)q'(1)$$
$$+q(1)q'(0)-2q(0)\int_0^1 q^2(t)d+2\int_0^1 q^3(t)dt-\int_0^1 q'^2(t)dt]$$
$$-\frac{1}{(2i\mu)^6}[2q^{(4)}(0)+\frac{1}{2}q'^2(1)+\frac{21}{2}q'^2(0)-q'(0)q'(1)-q(1)q''(1)$$
$$+11q(0)q''(0)+q(0)q''(1)+q(1)q''(0)-2q^3(1)+3q^3(0)$$
$$\left.+3q(0)q^2(1)-(\int_0^1 q^2(t)dt)^2]+O(\frac{1}{\mu^7})\right\}, \tag{2.6}$$

for $\mu\in T$ sufficiently large in absolute value.

Let $b(\mu)$ be the coefficient of $e^{2i\mu}$ in (2.6). Using the expansion

$$\frac{1}{1-x}=1+x+x^2+x^3+O(x^4),\ \ x\to 0,$$

it can be easily seen that the relation

$$b^{-1}(\mu) = 1+\frac{2q(0)}{(2i\mu)^2}+\frac{1}{(2i\mu)^3}\int_0^1 q^2(t)dt+\frac{1}{(2\omega_j\mu)^4}[2q''(0)-\frac{1}{2}q^2(1)+\frac{11}{2}q^2(0)$$
$$+q(0)q(1)]+\frac{1}{(2i\mu)^5}[q(1)q'(1)-q(0)q'(0)-q(0)q'(1)+q(1)q'(0)$$
$$+2q(0)\int_0^1 q^2(t)dt+2\int_0^1 q^3(t)dt-\int_0^1 q'^2(t)dt]+\frac{1}{(2i\mu)^6}[2q^{(4)}(0)$$
$$+\frac{1}{2}q'^2(1)+\frac{21}{2}q'^2(0)-q'(0)q'(1)-q(1)q''(1)+19q(0)q''(0)$$
$$+q(0)q''(1)+q(1)q''(0)-2q^3(1)+17q^3(0)+q(0)q^2(1)$$
$$\left.+4q(1)q^2(0)+\frac{1}{2}(\int_0^1 q^2(t)dt)^2]+O(\frac{1}{\mu^7})\right\} \tag{2.7}$$

holds for $\mu \in T$ sufficiently large in absolute value.

Thus, for $\mu \in T$ sufficiently large in absolute value, the equation $\Delta(\mu) = 0$ is equivalent to the equation

$$(i\mu)^{-1} b^{-1}(\mu) \Delta(\mu) e^{i\mu} = 0. \tag{2.8}$$

Using (2.6), (2.7) and the relations $q(1) = q(0)$ and $q'(1) \neq q'(0)$, from the equation (2.8), we obtain two equations

$$\mu_{k,1} = 2k\pi + \frac{q'(1) - q'(0) + \int_0^1 q^2(t)dt}{(4k\pi)^3} + O(\frac{1}{k^4}), \tag{2.9}$$

$$\mu_{k,2} = 2k\pi - \frac{q'(1) - q'(0) - \int_0^1 q^2(t)dt}{(4k\pi)^3} + O(\frac{1}{k^4}). \tag{2.10}$$

By Rouche's theorem, we have asymptotic expressions for the roots $\mu_{k,1}$ and $\mu_{k,2}$, $k = N, N+1, \cdots$, of the equations (2.9) and (2.10), respectively, where N is a positive integer

$$\mu_{k,1} = 2k\pi + \frac{q'(1) - q'(0)}{(4k\pi)^3} + O(\frac{1}{k^4}), \tag{2.11}$$

$$\mu_{k,2} = 2k\pi - \frac{q'(1) - q'(0)}{(4k\pi)^3} + O(\frac{1}{k^4}). \tag{2.12}$$

Note that $\mu_{k,1}$ and $\mu_{k,2}$ are simple roots of the equations (2.9) and (2.10), respectively. From the relations (2.11), (2.12) and the relations $\lambda_{k,1} = -\mu_{k,1}^2$, $\lambda_{k,2} = -\mu_{k,2}^2$, we obtain the formula (2.1) and observe that these eigenvalues are simple.

Let us calculate $U_2(\varphi_1(x, \mu_{k,1}))$ and $U_2(\varphi_2(x, \mu_{k,1}))$. Since

$$e^{i\mu_{k,1}} - 1 = \frac{q'(1) - q'(0) + \int_0^1 q^2(t)dt}{(2i\mu_{k,1})^3} + O(\frac{1}{\mu_{k,1}^4}),$$

we have

$$\begin{aligned} U_2(\varphi_1(x, \mu_{k,1})) &= \varphi_1'(1, \mu_{k,1}) - \varphi_1'(0, \mu_{k,1}) \\ &= i\mu_{k,1} e^{i\mu_{k,1}} [1 - \frac{q(1) + q(0)}{(2i\mu_{k,1})^2} + \frac{q'(1) + q'(0) - \int_0^1 q^2(t)dt}{(2i\mu_{k,1})^3} + O(\frac{1}{\mu_{k,1}^4})] \\ &\quad - i\mu_{k,1}[1 - \frac{2q(0)}{(2i\mu_{k,1})^2} + \frac{2q'(0)}{(2i\mu_{k,1})^3} + O(\frac{1}{\mu_{k,1}^4})] \\ &= \frac{q'(1) - q'(0)}{(2i\mu_{k,1})^2} + O(\frac{1}{\mu_{k,1}^3}). \end{aligned} \tag{2.13}$$

In a similar way, we obtain

$$U_2(\varphi_2(x,\mu_{k,1})) = \frac{q'(1)-q'(0)}{(2i\mu_{k,1})^2} + O(\frac{1}{\mu_{k,1}^3}).$$

Without loss of generality, we can assume that $q'(1)-q'(0)\neq 0$. Since $U_2(\varphi_j(x,\mu_{k,1}))\neq 0$, $j=1,2$, and $q'(1)-q'(0)\neq 0$, we seek the eigenfunction $y_{k,1}(x)$ corresponding to the eigenvalue $\lambda_{k,1}$ in the form

$$y_{k,1}(x) = \frac{(2i\mu_{k,1})^2}{2i\left[q'(1)-q'(0)\right]}\begin{vmatrix} \varphi_1(x,\mu_{k,1}) & \varphi_2(x,\mu_{k,1}) \\ U_2(\varphi_1(x,\mu_{k,1})) & U_2(\varphi_2(x,\mu_{k,1})) \end{vmatrix}. \tag{2.14}$$

From the equalities

$$\varphi_j(x,\mu_{k,1}) = e^{\mu_{k,1}\omega_j x}\left[1+\frac{u_1(x)}{(2w_j\mu_{k,1})}+\frac{u_2(x)}{(2w_j\mu_{k,1})^2}+O(\frac{1}{\mu_{k,1}^3})\right],\ \ j=1,2$$

and the formulas (2.13), (2.14) we obtain

$$y_{k,1}(x) = \sin\mu_{k,1}x + O(\frac{1}{\mu_{k,1}}).$$

Therefore, the eigenfunction $y_{k,1}(x)$ satisfies the asymptotic formula (2.5).

In a similar way, since $U_1(\varphi_j(x,\mu_{k,1}))\neq 0$, $j=1,2$, and $q'(1)-q'(0)\neq 0$, we can seek the eigenfunctions $y_{k,2}(x)$ corresponding to the eigenvalues $\lambda_{k,2}$ in the form

$$y_{k,2}(x) = -\frac{(2i\mu_{k,2})^3}{4\left[q'(1)-q'(0)\right]}\begin{vmatrix} \varphi_1(x,\mu_{k,2}) & \varphi_2(x,\mu_{k,2}) \\ U_1(\varphi_1(x,\mu_{k,2})) & U_1(\varphi_2(x,\mu_{k,2})) \end{vmatrix}.$$

Thus, we obtain

$$y_{k,2}(x) = \cos\mu_{k,2}x + O(\frac{1}{\mu_{k,2}}).$$

This completes the proof of the lemma.

3. The Riesz basisness in $L_2(0,1)$ of the root functions for the periodic problem

Theorem 3.1. *The root functions of the boundary problem (1.1), (1.2) form a Riesz basis in* $L_2(0,1)$.

Proof. The system of the root functions of the boundary problem (1.1), (1.2) is complete and minimal in $L_2(0,1)$. The minimality of this system follows from the fact that this system has a biorthogonal system consisting of the root functions of the adjoint operator

$$\begin{aligned} l^*(v) &= v'' + \overline{q(x)}v, \\ v(1) &= v(0),\ \ v'(1) = v'(0). \end{aligned}$$

For any $f \in L_2(0,1)$, with a direct computation we have that $\sum\limits_{n=N}^{\infty} |(f, y_{k,1})|^2 < \infty$, $\sum\limits_{n=N}^{\infty} |(f, y_{k,2})|^2 < \infty$. On the other hand, the eigenfunctions of the adjoint operator have of the form

$$\upsilon_{k,1}(x) = 2\sin 2k\pi x + O(\frac{1}{k}), \tag{3.1}$$

$$\upsilon_{k,2}(x) = 2\cos 2k\pi x + O(\frac{1}{k}), \tag{3.2}$$

and the inequalities $\sum\limits_{n=N}^{\infty} |(f, \upsilon_{k,1})|^2 < \infty$ and $\sum\limits_{n=N}^{\infty} |(f, \upsilon_{k,2})|^2 < \infty$ hold. According to Theorem 1.1, the root functions of the boundary problem (1.1), (1.2) form a Riesz basis in $L_2(0,1)$. This completes the proof.

4. The basisness in $L_2(0,1)$ of the root functions for the anti-periodic boundary-value problem

Similarly, the following results are obtained for the boundary problem (1.1), (1.3).

Lemma 4.1. *All eigenvalues of the boundary value problem (1.1), (1.3), starting from some number, are simple and form two infinite sequence* $\lambda_{k,1}$, $\lambda_{k,2}$, $k = N, N+1, \cdots$, *where* N *is a positive integer and*

$$\lambda_{k,1} = -\left[(2k+1)\pi\right]^2 + \frac{q'(1) - q'(0) - \int\limits_0^1 q^2(x)dx}{[2(2k+1)\pi]^2} + O(\frac{1}{k^3}), \tag{4.1}$$

$$\lambda_{k,2} = -\left[(2k+1)\pi\right]^2 - \frac{q'(1) - q'(0) + \int\limits_0^1 q^2(x)dx}{[2(2k+1)\pi]^2} + O(\frac{1}{k^3}), \tag{4.2}$$

and the corresponding eigenfunctions are of the form

$$y_{k,1}(x) = \sin(2k+1)\pi x + O(\frac{1}{k}), \tag{4.3}$$

$$y_{k,2}(x) = \cos(2k+1)\pi x + O(\frac{1}{k}). \tag{4.4}$$

Proof. In the anti-periodic case, in a similar way to the proof Lemma 2.1, we have the relations

$$e^{i\mu} + 1 = \frac{q'(1) - q'(0) - \int\limits_0^1 q^2(x)dx}{(2i\mu)^3} + O(\frac{1}{\mu^4}),$$

$$e^{i\mu} + 1 = -\frac{q'(1) - q'(0) - \int\limits_0^1 q^2(x)dx}{(2i\mu)^3} + O(\frac{1}{\mu^4}).$$

From these relations we can obtain (4.1) and (4.2). Again in a similar way to the proof Lemma 2.1, we obtain

$$
\begin{aligned}
U_1(\varphi_1(x,\mu_{k,1})) &= \varphi_1(1,\mu_{k,1})+\varphi_1(0,\mu_{k,1}) \\
&= \frac{2[q'(1)-q'(0)]}{(2i\mu_{k,1})^3}+O(\frac{1}{\mu_{k,1}^4}),
\end{aligned}
$$

$$
\begin{aligned}
U_1(\varphi_2(x,\mu_{k,1})) &= \varphi_2(1,\mu_{k,1})+\varphi_2(0,\mu_{k,1}) \\
&= -\frac{2[q'(1)-q'(0)]}{(2i\mu_{k,1})^3}+O(\frac{1}{\mu_{k,1}^4}).
\end{aligned}
$$

Since $U_1(\varphi_j(x,\mu_{k,1}))\neq 0$,$j=1,2$, we can seek the eigenfunction $y_{k,1}(x)$ corresponding to the eigenvalue $\lambda_{k,1}$ in the form

$$
y_{k,1}(x)=-\frac{(2i\mu_{k,1})^3}{[q'(1)-q'(0)]}\begin{vmatrix} \varphi_1(x,\mu_{k,1}) & \varphi_2(x,\mu_{k,1}) \\ U_1(\varphi_1(x,\mu_{k,1})) & U_1(\varphi_2(x,\mu_{k,1})) \end{vmatrix}.
$$

Hence, we have

$$
y_{k,1}(x)=\sin(2k+1)\pi x+O(\frac{1}{k}),
$$

i.e., the formula (4.3) satisfies. In similar way we can obtain the formula (4.4).

Theorem 4.1. *The root functions of the boundary problem (1.1), (1.3) form a Riesz basis in* $L_2(0,1)$.

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