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β-Closed spaces and β-θ-subclosed graphs

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Abstract. The purpose of this paper is to achieve various characterizations of β-closed spaces [2], specially, in terms of new types of graphs under the terminology β-θ-subclosed graphs of functions and in terms of a generalized complete accumulation point. Apart from several properties, a sufficient condition for common fixed points of a family of functions having β-θ-subclosed graphs is also given.

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1. Introduction

Motivated by the various usefulness of compactness many mathematicians have tried to generalize this notion. In the course of their attempts, several weaker and stronger versions of compactness have been studied in detail. It is seen from the literature that certain open-like sets have been employed for such investigations. In [1] Monsef et. al introduced the notion of β-open sets (semi-preopen sets [4]) and since its introduction such sets along with some of their relevant concepts have been investigated by many. Mention may be made [2, 3, 4, 5, 6, 9, 10, 16]. Monsef et. al [2] have taken up an investigation of a sort of covering property, known as β-closedness with the help of the notion of β-open sets. A topological space X is said to be β-closed [2] if every β-open cover of X admits a finite subfamily whose β-closures cover X. In this paper we intend to undertake a further study of such concept. Joseph and Kwack [11] have characterized S-closed spaces in various ways adopting the techniques which have been found useful for compact spaces and some of its generalizations like H-closed spaces and minimal Hausdorff spaces. Analogue of such characterizations for β-closed spaces are given here.

In section § 2, we state some existing definitions and results as a prerequisite for the development of subsequent sections. In section § 3, we derive various characterizations of β-closed spaces, specially, in terms of filter bases, in terms of nets with well ordered directed sets and in terms of a generalized complete accumulation point. Section § 4 concerns from

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several points of view. First, in introducing and characterizing the notions of (θ, β)-continuity and β-θ-subclosedness of graphs of functions, second, to obtain several relevant properties of such functions along with a theorem that concerns on common fixed points of a family of functions having β-θ-subclosed graphs and finally, to exploit these ideas in achieving some characterizations of β-closed spaces.

Throughout this paper, spaces always mean a topological space without any separation axioms and $\psi : X \to Y$ denotes a single valued function of a space (X, τ) into a space (Y, τ_1). The closure and the interior of a subset S of a space X are denoted by $cl(S)$ and $int(S)$ respectively. We recall the following well known definitions:

A subset S of a space (X, τ) or X is said to be α-open [15] (resp. semi-open [12], preopen [14], β-open [1] or semi-preopen [4]) if $S \subset int(cl(int(S)))$ (resp. $S \subset cl(int(S)$, $S \subset int(cl(S))$, $S \subset cl(int(cl(S))))$. We denote the classes of all open (resp. α-open, semi-open, preopen, β-open) sets in a space (X, τ) by $O(X)$ (resp. $\tau_\alpha = \alpha(X)$, $SO(X)$, $PO(X)$, $\beta O(X) = SPO(X)$)) and that containing a point x of (X, τ) by $O(X, x)$ (resp. $\alpha(X, x)$, $SO(X, x)$, $PO(X, x)$, $\beta O(X, x)$). Moreover it is well known that $\tau \subset \tau_\alpha = PO(X) \cap SO(X) \subset PO(X) \cup SO(X) \subset \beta O(X)$. The complement of a β-open set is called β-closed. Preclosed and semi-closed sets are similarly defined. The β-closure (resp. preclosure, semi closure) of S denoted by $\beta cl(S)$ (resp. $pcl(S)$, $scl(S)$) is the intersection of all β-closed (resp. pre closed, semi-closed) subsets of X containing S. β-interior of S, denoted by $\beta int(S)$ is defined as usual. A space X is called QHC [7] (resp. S-closed [17]), s-closed [13], P-closed [8]) if every open (resp. semi-open, semi-open, preopen) cover of X has a finite subfamily, whose closures (resp. closures, semi-closures, pre-closures) cover X. For any filter base $\mathscr{F}$, adherence of $\mathscr{F}$ is written as $ad\mathscr{F}$. A filter base $\mathscr{F}$ is said to (a) β-θ-adhere at x (written as $x \in \beta$-θ-$ad\mathscr{F}$) if for each $F \in \mathscr{F}$ and each $V \in \beta O(X, x)$, $F \cap \beta cl(V) \neq \emptyset$. (b) β-θ-converge to x if for each $V \in \beta O(X, x)$, there is an $F \in \mathscr{F}$ such that $F \subset \beta cl(V)$. The corresponding definitions of nets are obvious.

2. Prerequisites

The following definitions and results which already have been found in literature [16] in the language of semipre-open sets are being restated in the language of β-open sets which will be frequently used in the subsequent sections.

Definition 2.1. A subset S of a space (X, τ) is said to be β-regular (=semipre-regular [16]) if it is both β-open as well as β-closed.

The family of all β-regular sets of a space X and that containing a point x of X are respectively denoted by $\beta R(X)$ and $\beta R(X, x)$.

Lemma 2.2 [16]. For a subset A of a space X, $A \in \beta O(X)$ if and only if $\beta cl(A) \in \beta R(X)$.

Definition 2.3. A point $x \in X$ is said to be in the β-θ-closure (=sp-θ-closure [16]) of A, denoted by β-θ-$cl(A)$, if $A \cap \beta cl(V) \neq \emptyset$ for every $V \in \beta O(X, x)$. If β-θ-$cl(A) = A$, then A is said to be β-θ-closed (=sp-θ-closed [16]). The complement of a β-θ-closed set is said to be

β-θ-open (=sp-θ-open [16]).

Lemma 2.4 [16]. For a subset A of a space X, β-θ-$cl(A) = \cap\{R : A \subset R$ and $R \in \beta R(X)\}$.

Lemma 2.5 [16]. Let A and B be any subsets of a space X. Then the following properties hold:
(i) $x \in \beta$-θ-$cl(A)$ if and only if $A \cap V \neq \emptyset$ for each $V \in \beta R(X, x)$.
(ii) if $A \subset B$ then β-θ-$cl(A) \subset \beta$-θ-$cl(B)$.
(iii) β-θ-$cl(\beta$-θ-$cl(A)) = \beta$-θ-$cl(A)$.
(iv) intersection of an arbitrary family of β-θ-closed sets in X is β-θ-closed in X.
(v) A is β-θ-open if and only if for each $x \in A$, there exists $V \in \beta R(X, x)$ such that $x \in V \subset A$.
(vi) If $A \in \beta O(X)$ then $\beta cl(A) = \beta$-θ-$cl(A)$.
(vii) If $A \in \beta R(X)$ then A is β-θ-closed.

Remark 2.6 [16]. T. Noiri [16] has shown that β-regular $\Rightarrow$ β-θ-open $\Rightarrow$ β-open. But the converses are not necessarily true.

3. β-Closed Spaces

Definition 3.1. A space X is said to be β-closed [2] if every cover of X by β-open sets has a finite subfamily whose β-closures cover X.

The following characterizations of β-closed spaces are quite obvious.

Theorem 3.2. For a space X, the following are equivalent:
(a) X is β-closed.
(b) every cover of X by β-regular sets has a finite subcover.
(c) for every family $\{U_\alpha \in \beta R(X) : \alpha \in I\}$ such that $\cap\{U_\alpha : \alpha \in I\} = \emptyset$, there exists a finite subset I_0 of I such that $\cap\{U_\alpha : \alpha \in I_0\} = \emptyset$.
(d) every cover of X by β-θ-open sets has a finite subcover.

Definition 3.3. A point x in a space X is called a β-θ-complete accumulation point (β-θ-c.a.p., for short) of a subset S of X if $|S| = |S \cap V|$ for each $V \in \beta R(X, x)$, where $|S|$ denotes the cardinality of the subset S.

Theorem 3.4. The following are equivalent for a space X
(a) X is β-closed.
(b) every infinite subset X has a β-θ-c.a.p. in X.
(c) each net with a well ordered directed set as its domain β-θ-adheres to a point in X.

Proof. $(a) \Rightarrow (b)$: Let I be an infinite subset in a β-closed space X and also let $N = \{x \in X : X$ is not a β-θ-c.a.p. of $I\}$. So for each $x \in N$, there exists a $B_x \in \beta R(X, x)$ such that $|I \cap B_x| < |I|$. If N is the whole space, then it follows from the theorem 3.2 that the cover $\{B_x : x \in N\}$ has a finite subcover, say, $\{B_{x_1}, B_{x_2}, ...,B_{x_k}\}$. Now $I \subset \cup\{B_{x_i} \cap I : i = 1, 2, ...k\}$ and $|I| = max\{|B_{x_i} \cap I| : i = 1, 2, ...k\}$ — a contradiction. So, I has a β-θ-c.a.p. in X.

$(b) \Rightarrow (a)$: Conversely, let X be not β-closed. Then by theorem 3.2 there exists a cover $\mathscr{U}$ of X by β-regular sets with no finite subcover. Consider $\beta = min\{|\mathscr{U}^\star| : \mathscr{U}^\star \subset \mathscr{U}$ and $\mathscr{U}^\star$ is cover of X $\}$ where $|.|$ denotes the cardinality. Let $\mathscr{U}_0 \subset \mathscr{U}$ be a cover of X for which $|\mathscr{U}_0| = \beta$. Clearly $\beta \geq \aleph_0$. By well ordering of $\mathscr{U}_0$ by some minimal well-ordering $\prec$, we have $|\{U : U \in \mathscr{U}_0$ and $U \prec U_0\}| < |\{U; U \in \mathscr{U}_0\}|$, for each $U_0 \in \mathscr{U}_0$. Clearly X cannot have any subcover with cardinality less than β and hence for each $U \in \mathscr{U}_0$, there exists a point $x_U \in X - \cup\{U_0 \cup \{x_{U_0}\} : U_0 \in \mathscr{U}_0$ and $U_0 \prec U\}$. This can always be done otherwise one can choose from $\mathscr{U}_0$ a cover of smaller cardinality. Let $S = \{x_U : U \in \mathscr{U}_0\}$ and x be any point of X. Since $\mathscr{U}$ is a cover of X, $x \in U^\star$ for some $U^\star \in \mathscr{U}_0$. But by the choice of x_U, $x_U \in U^\star$ implies $U \prec U^\star$. Therefore, $W = \{U \in \mathscr{U}_0$ and $x_U \in U^\star\} \subset \{U \in \mathscr{U}_0 : U \prec U^\star\}$. But $|W| < \beta$, by the minimality of $\prec$. So, $|S \cap U^\star| < \beta$. Since for $U_1, U_2 \in \mathscr{U}_0$ with $U_1 \neq U_2$, we have $x_{U_1} \neq x_{U_2}$, then $|S| = \beta \geq \aleph_0$. Therefore the infinite set S has no β-θ-c.a.p. in X — a contradiction. So, X is β-closed.

$(c) \Rightarrow (b)$: Let I be an infinite subset of X. By Zorn's lemma, I can be assumed to be net with a well ordered directed set as its domain. So, it has a β-θ-adherent point say, x and clearly x is an β-θ-c.a.p. of I.

$(a) \Rightarrow (c)$: Let $\{x_\lambda\}_{\lambda \in D}$ be a net with well ordered directed set D, having no β-θ-adherent point in X, So, for each $x \in X$, there is a β-regular set $U_x \in \beta R(X, x)$ and a $\lambda_x \in D$ such that $x_\lambda \in X - U_x, \forall\ \lambda \geq \lambda_x$. Since X is β-closed, the cover $\{U_x : x \in X\}$ has a finite subcover, say, $\{U_{x_1}, ..., U_{x_k}\}$. Let $\{\lambda_{x_1}, ..., \lambda_{x_k}\}$ be the corresponding elements in D which is finite and hence by the well orderedness of D, there exists a largest element say λ_{x_k} in D. Then $x_\lambda \in \cap_{i=1}^{k}(X - U_{x_i}) = X - \cup_{i=1}^{k} U_{x_i} = \emptyset$, for $\lambda > \lambda_{x_k}$ — a contradiction. Therefore the net $\{x_\lambda\}_{\lambda \in D}$ has a β-θ-adherent point in X.

Theorem 3.5. The following are equivalent for a space X

(a) X is β-closed.

(b) each family of β-θ-closed sets with the finite intersection property has nonempty intersection.

(c) each filter base on X has at least one β-θ-adherent point.

(d) each filter base on X with atmost one β-θ-adherent point is β-θ-convergent.

(e) every maximal filter base β-θ-converges to some point in X.

Proof. $(a) \Leftrightarrow (b)$: Obvious.

$(b) \Rightarrow (c)$: Let $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ be a filter base on X. Then $\mathscr{F}^\star = \{\beta\text{-}\theta\text{-}cl(F_\alpha) : \alpha \in I\}$ is a family of β-θ-closed sets with the finite intersection property. Then by (b) $\beta\text{-}\theta\text{-}ad\mathscr{F} = \cap\mathscr{F}^\star \neq \emptyset$.

$(c) \Rightarrow (b)$: Let $\Omega = \{F_\alpha : \alpha \in I\}$ be a family of β-θ-closed sets having finite intersection property. Let $\Omega^\star$ be the family of all sets of Ω together with their all finite intersections. Clearly, $\Omega^\star$ is a filter base on X and hence by (c), $\Omega^\star$ β-θ-adheres to some point say x in X. So, $x \in \cap\Omega^\star \subset \cap\Omega$.

$(c) \Rightarrow (d)$: Let $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ be a filter base on X with $\beta\text{-}\theta\text{-}ad\mathscr{F} \subset \{x\}$ for some $x \in X$. Then by (c), $\beta\text{-}\theta\text{-}ad\mathscr{F} = \{x\}$. Suppose that there exists an $U \in \beta R(X, x)$ such that $F_\alpha \cap (X - U) \neq \emptyset$, for all $\alpha \in I$. Then $\mathscr{F}^\star = \{F_\alpha - U : \alpha \in I\}$ is a filter base on X. But by (c), $\mathscr{F}^\star$ has at least one β-θ-adherent point. Now, $\cap_{\alpha \in I}\beta\text{-}\theta\text{-}cl(F_\alpha - U) \subset (\cap_{\alpha \in I}\beta\text{-}\theta\text{-}cl(F_\alpha)) \cap (X - U) = \{x\} \cap (X - U) = \emptyset$ — a contradiction. So for each $U \in \beta R(X, x)$ there

exists an $F_\alpha \in \mathscr{F}$ with $F_\alpha \subset U$. Therefore $\mathscr{F}$ β-θ-converges to x.
$(d) \Rightarrow (c)$: Suppose that $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ is a filter base on X with no β-θ-adherent point in X. By hypothesis (d), $\mathscr{F}$ β-θ-converges to a point say x in X. Let $F_\alpha \in \mathscr{F}$ and $U \in \beta R(X,x)$. Then there exists an $F_\alpha^{'} \in \mathscr{F}$ such that $F_\alpha^{'} \subset U$. Since $\mathscr{F}$ is a filter base on X, there exists an $F_{\alpha^\star} \in \mathscr{F}$ such that $F_{\alpha^\star} \subset F_\alpha \cap F_\alpha^{'} \subset F_\alpha \cap U$. Since $F_\alpha^\star$ is non-empty, $F_\alpha \cap U \neq \emptyset$. So, $x \in \beta$-θ-$cl(F_\alpha)$ and this holds for every $F_\alpha \in \mathscr{F}$. Therefore, x is a β-θ-adherent point of $\mathscr{F}$ — a contradiction.
$(e) \Rightarrow (c)$: Let $\mathscr{F}$ be a filter base on X and $\mathscr{F}^\star$ be a maximal filter base such that $\mathscr{F} \subset \mathscr{F}^\star$. By (e), $\mathscr{F}^\star$ β-θ-converges to some point x in X. For each $U \in \beta R(X,x)$, there exists an $F^\star \in \mathscr{F}^\star$ such that $F^\star \subset U$. So for each $F \in \mathscr{F}$, $\emptyset \neq F \cap F^\star \subset F \cap U$. Therefore x is a β-θ-adherent point of $\mathscr{F}$.
$(c) \Rightarrow (e)$: Obvious.

Remark 3.6. Equivalent formulations of the characterizations of β-closed spaces in terms of nets and ultranets are quite similar to the above theorem and are omitted.

Theorem 3.7. (X,τ) is β-closed if and only if (X,τ_α) is β-closed.

Proof. The result follows from the well known fact that in any space (X,τ), $\beta O(X,\tau) = \beta O(X,\tau_\alpha)$.

Since every open set is β-open, the following theorem is quite obvious.

Theorem 3.8. (a) Every β-closed space is quasi H-closed.
(b) Every β-compact space [2] (a space is β-compact if every β-open cover of has a finite subcover) is β-closed.

Remark 3.9. The converse of the results (a) and (b) in theorem 3.8 are not true in general. Furthermore the concepts of compactness and β-closedness are independent.

Example 3.10. Example of a compact (and hence quasi H-closed) space which is not β-closed.

Let $X = N$ be the set of all naturals with the co-finite topology τ. Here $SO(X) = \tau$ and $PO(X) = \beta O(X) = \{S \subset X : S \text{ is infinite }\} \cup \{\emptyset\}$. Since for a subset S, $\beta cl(S) = S \cup int(cl(int(S)))$, so the $\beta cl(A_i) = A_i$ when $A_i = N_e \cup \{i\}$, $i \in N$ and N_e be the set of all even positive integers. If we take the β-open cover $\mathscr{U} = \{A_i : i = 1,3,5,7,...\}$ of X, where $A_i = N_e \cup \{i\}$ then it has no finite subfamily whose β-closures cover X. So (X,τ) is not β-closed but (X,τ) is obviously compact and hence it is quasi H-closed.

Example 3.11. Example of an infinite β-closed space which is neither β-compact nor compact.

Let X be the set of reals with the topology τ in which non-void open sets are those subsets of X which contain the point 1. Clearly the space (X,τ) is not compact and hence not β-compact (as every β-compact space is obviously compact). We claim that in this space (X,τ) every non-void β-open set must contains the point 1. Indeed, let S be a non-void subset of X

such that $1 \notin S$. Since a subset A is β-open if $A \subset cl(int(cl(A)))$, then S can not be β-open. Hence X is the only β-closed set containing any non-void β-open set. Thus the β-closure of a single non-void β-open set is X and therefore X is β-closed.

We recall that a space (X, τ) is said to be submaximal [7] if every dense subset of X is open and extremally disconnected [15] if the closure of each open set is open in X.

Theorem 3.12. Let (X, τ) be a extremally disconnected space. Then (X, τ) is β-closed if and only if (X, τ) is P-closed.

Proof. As a space (X, τ) is extremally disconnected if and only $PO(X) = \beta O(X)$, the result follows immediately.

Theorem 3.13. If (X, τ) is submaximal and extremally disconnected then the following are equivalent:
(a) (X, τ) is β-closed.
(b) (X, τ) is P-closed.
(c) (X, τ) is s-closed.
(d) (X, τ_α) is β-closed.
(e) (X, τ) is QHC.
(f) (X, τ_α) is S-closed.

Proof. The proof follows from the fact that if (X, τ) is a submaximal extremally disconnected space then $\tau = \tau_\alpha = SO(X) = PO(X) = \beta O(X)$ [6].

§ 4. (θ,β)-Continuity and β-θ-Subclosed Graph

Definition 4.1. A function $\psi : X \to Y$ is (θ, β)-continuous if each filter base $\mathscr{F}$ on X, satisfies $\psi(ad\mathscr{F}) \subset \beta\text{-}\theta\text{-}ad\psi(\mathscr{F})$.

Theorem 4.2. For a function $\psi : X \to Y$, the following are equivalent:
(a) ψ is (θ, β)-continuous.
(b) for each $A \subset X$, $\psi(cl(A)) \subset \beta\text{-}\theta\text{-}cl(\psi(A))$.
(c) for each $x \in X$ and each $V \in \beta O(Y, \psi(x))$, there exists an open set U containing x such that $\psi(U) \subset \beta\text{-}cl(V)$.
(d) for each $W \in \beta R(Y, \psi(x))$, there is an open set U containing x such that $\psi(U) \subset W$.
(e) for each β-θ-closed set B of Y,$\psi^{-1}(B)$ is closed in X.
(f) for each $B \subset Y$ $cl(\psi^{-1}(B)) \subset \psi^{-1}(\beta\text{-}\theta\text{-}cl(B))$.
(g) for each $x \in X$ and each filter base $\mathscr{F}$ on X with $\mathscr{F} \to x$, the filter base $\psi(\mathscr{F})$ β-θ-converges to $\psi(x)$.
(h) for each $x \in X$ and every net (x_λ) in X with $(x_\lambda) \to x$, $\psi(x_\lambda)$ β-θ-converges to $\psi(x)$.

Theorem 4.3. If $\psi : X \to Y$ is (θ, β)-continuous and Y is Hausdorff then the graph $G(\psi)$ of ψ is closed in $X \times Y$.

Proof. Let $(x, y) \notin G(\psi)$. Then $y \neq \psi(x)$. As Y is being Hausdorff, there are disjoint open sets U and V in Y containing y and $\psi(x)$ respectively such that $U \cap \beta\text{-}cl(V) = \emptyset$. By (θ, β)-continuity of ψ, there is a $W \in O(X, x)$ such that $\psi(W) \subset \beta\text{-}cl(V)$. Then $W \times U$ is an open set in $X \times Y$ containing (x, y) such that $G(\psi) \cap (W \times U) = \emptyset$. Therefore $G(\psi)$ is closed.

Theorem 4.4. Let $G(\psi) : X \to X \times Y$ be the graph function of the function $\psi : X \to Y$. Then ψ is (θ, β)-continuous if $G(\psi)$ is so.

Proof. Let $x \in X$ and W be any β-open set containing $\psi(x)$ in Y. If $U \in \beta O(X)$ and $W \in \beta O(Y)$ then we claim that $U \times W \in \beta O(X \times Y)$. Indeed, since U and W are β-open sets, there exists $V \in PO(X)$ and $K \in PO(Y)$ such that $V \subset U \subset cl(V)$ and $K \subset W \subset cl(K)$. Clearly $V \times K \subset U \times W \subset cl(V) \times cl(K) = cl(V \times K)$, and $V \times K \in PO(X \times Y)$. So $U \times W$ is β-open in $X \times Y$. Thus $X \times W \in \beta O(X \times Y)$ containing $G(\psi)(x)$. Since $G(\psi)$ is (θ, β)-continuous, there exists an open set O containing x such that $G(\psi)(O) \subset \beta cl(X \times W) \subset X \times \beta cl(W)$. Therefore we have $\psi(O) \subset \beta cl(W)$ and hence ψ is (θ, β)-continuous.

Definition 4.5. A function $\psi : X \to Y$ has a β-θ-subclosed graph if $\beta\text{-}\theta\text{-}ad\psi(\Omega) \subset \{\psi(x)\}$ for each $x \in X$ and each filter base Ω on $X - \{x\}$ with $\Omega \to x$.

Equivalently, ψ has a β-θ-subclosed graph if and only if for each $x \in X$ and each net (x_λ) in $X - \{x\}$ with $x_\lambda \to x$, $\psi(x_\lambda)$ β-θ-adheres to atmost $\psi(x)$.

Theorem 4.6. The following are equivalent for spaces X, Y and for the function $\psi : X \to Y$
(a) ψ has a β-θ-subclosed graph.
(b) for each $(x, y) \notin G(\psi)$, there are open sets W containing x in X and some β-open set V containing y in Y satisfying $G(\psi) \cap (W - \{x\}) \times \beta cl(V)) = \emptyset$.
(c) for each $(x, y) \notin G(\psi)$, there exist an open set W containing x in X and some β-open set V containing y in Ysuch that $G(\psi) \cap (W \times (\beta cl(V) - \{\psi(x)\})) = \emptyset$.
(d) for each $(x, y) \notin G(\psi)$, there exist an open set W containing x in X and β-open set V containing y in Y such that $\psi(W) \cap (\beta cl(V) - \{\psi(x)\}) = \emptyset$.

Proof. $(a) \Rightarrow (b)$: Let $\psi : X \to Y$ be a function having β-θ-subclosed graph and $(x, y) \notin G(\psi)$. Consider $\mathscr{F} = \{W - \{x\} : W \in O(X, x)\}$. If $\mathscr{F}$ is a filter base then $\mathscr{F} \to x$ and since ψ has a β-θ-subclosed graph, $y \notin \beta\text{-}\theta\text{-}ad\psi(\mathscr{F})$. Hence there exist an $F(= W - \{x\}$, for some $W \in O(X, x)) \in \mathscr{F}$ and a β-open set V containing y in Y such that $\psi(F) \cap \beta cl(V) = \emptyset$ i.e. $\psi(W - \{x\}) \cap \beta cl(V) = \emptyset$. Therefore $G(\psi) \cap (W - \{x\} \times \beta cl(V)) = \emptyset$. If $\mathscr{F}$ is not a filter base then $W = \{x\}$ for some $W \in O(X, x)$ and the rest is obvious.
$(b) \Rightarrow (c)$: If possible, let $(z, \psi(z)) \in G(\psi) \cap (W \times \beta cl(V) - \{\psi(x)\}) = \emptyset$ where W and V are sets as in the hypothesis (b). Then $z \in W$ and $\psi(z) \in \beta cl(V) - \{\psi(x)\}$. Clearly $\psi(z) \neq \psi(x)$ and hence $z \neq x$. Since $z \in W - \{x\}$ and $\psi(z) \in \beta cl(V)$ then $\psi(z) \in \psi(W - \{x\}) \cap \beta cl(V) = \emptyset = G(\psi) \cap ((W - \{x\}) \times \beta cl(V))$ — a contradiction.
$(c) \Rightarrow (d)$: Obvious.
$(d) \Rightarrow (a)$: Suppose $\mathscr{F}$ is filter base in $X - \{x\}$ such that $\mathscr{F} \to x$ and also suppose that $y \neq \psi(x)$. Then $(x, y) \notin G(\psi)$. So by hypothesis (d), there is an open set W containing x in X and a β-open set V containing y in Y such that $\psi(W) \cap (\beta cl(V) - \{\psi(x)\}) = \emptyset$. Since $\mathscr{F} \to x$ then $F \subset W$ for some $F \in \mathscr{F}$. Therefore $\psi(F) \cap (\beta cl(V) - \{\psi(x)\}) = \emptyset$. Now as $\mathscr{F}$ is a filter base in $X - \{x\}$, $\psi(F) \cap \beta cl(V) = \emptyset$. So, $y \notin \beta\text{-}\theta\text{-}ad\psi(\mathscr{F})$. Therefore β-θ-

$ad\psi(\mathscr{F}) \subset \{\psi(x)\}$.

Theorem 4.7. If $\phi : X \to Y$ is (θ, β)-continuous and if $\psi : X \to Y$ has a β-θ-subclosed graph then the set $\Delta_X(\phi, \psi) = \{x \in X : \phi(x) = \psi(x)\}$ is a closed subset of X.

Proof. Suppose $x_0 \in cl(\Delta_X(\phi, \psi)) - \Delta_X(\phi, \psi)$. Then there is a filter base $\mathscr{F}$ on $\Delta_X(\phi, \psi)$ such that $\mathscr{F} \to x_0$. Since $\psi(F) = \phi(F)$, for each $F \in \mathscr{F}$ and since ψ has a β-θ-subclosed graph, we have β-θ-$ad\phi(\mathscr{F}) = \beta$-$\theta$-$ad\psi(\mathscr{F}) \subset \{\psi(x)\}$. As ϕ is (θ, β)-continuous then for each $F \in \mathscr{F}$, we have $x_0 \in cl(F) \subset cl(\phi^{-1}(\phi(F))) \subset \phi^{-1}(\beta$-$\theta$-$cl(\phi(F))$ (last inclusion follows from theorem 4.2). So, $\phi(x_0) \in \beta$-θ-$cl(\phi(\mathscr{F}))$ for each $F \in \mathscr{F}$. Therefore, $\phi(x_0) \in \beta$-θ-$ad\phi(\mathscr{F})$ and hence $\phi(x_0) = \psi(x_0)$ — a contradiction. So $\Delta_X(\phi, \psi)$ is closed in X.

Definition 4.8. A topological space X is said to be β-connected [3] if X can not be expressed as the union of two non-empty disjoint β-open sets.

Corollary 4.9. If X is β-connected and if $\psi : X \to X$ has a β-θ-subclosed graph then the set of fixed points of ψ is a closed subset of X.

Proof. Since X is β-connected and also since $\beta cl(V) \in \beta O(X)$ for $V \in \beta O(X)$ (by lemma 2.2), for a nonempty β-open set W of X, $\beta cl(W) = X$. So, the identity function $\phi : X \to X$ is always (θ, β)-continuous. Hence by the above theorem 4.7, the result is being followed.

The following theorem establishes on common fixed points of a family of functions having β-θ-subclosed graphs.

Theorem 4.10. Let Ω be a family of functions from a β-connected β-closed space X into itself with β-θ-subclosed graphs. If for each finite $\Omega_0 \subset \Omega$ there is an $x \in X$ such that $\psi(x) = x$ for all $\psi \in \Omega_0$ then there exists an $x \in X$ such that $\psi(x) = x$ for all $\psi \in \Omega$.

Proof. Since X is β-connected, the identity function $\phi : X \to X$ is (θ, β)-continuous. Now by theorem 4.7, $\mathscr{F} = \{\Delta_X(\phi, \psi) : \psi \in \Omega\}$ is a family of closed subsets of X. By hypothesis, Ω has finite the intersection property. Let $\mathscr{F}_0$ be the filter base generated by $\mathscr{F}$. Since X is β-closed, by theorem 3.2, β-θ-$ad\mathscr{F}_0 \neq \emptyset$. Hence $\emptyset \neq \beta$-θ-$ad\mathscr{F}_0 \subset \cap_{\psi \in \Omega} \Delta_X(\phi, \psi)$. Therefore, there is at least one $x \in X$ satisfying $\psi(x) = \phi(x) = x$ for all $\psi \in \Omega$.

Theorem 4.11. If $A \subset X$ and $\psi : X \to Y$ has a β-θ-subclosed graph then the restriction $\psi_A : A \to Y$ has a β-θ-subclosed graph.

Proof. Straightforward.

It is well known that inverse the image of a compact set of a function with closed graph is closed. The following theorem shows a analogous result for functions having β-θ-subclosed graph (A subset B of X is called β-closed with respect to X written as β-set if every cover of B by β-open sets of X has a finite subfamily whose β-closures cover B).

Theorem 4.12. If $\psi : X \to Y$ is a function with a β-θ-subclosed graph then $\psi^{-1}(B)$ is closed in X for each β-set B in Y.

Proof. Let $x \in cl(\psi^{-1}(B)) - \psi^{-1}(B)$. Then there is a filter base $\mathscr{F}$ on $\psi^{-1}(B)$ such that $\mathscr{F} \to x$. Since ψ has a β-θ-subclosed graph, β-θ-$ad\psi(\mathscr{F}) \subset \{\psi(x)\}$. Now as B is being a β-set, it can be easily verified that $B \cap \beta$-θ-$ad\psi(\mathscr{F}) \neq \emptyset$. Therefore $\psi(x) \in B$ and hence $x \in \psi^{-1}(B)$ —- a contradiction.

Theorem 4.13. The following are equivalent for a β-T_2 space (X,τ):
(a) (X,τ) is β-closed.
(b) for any space Y, every functions $f : Y \to X$ with β-θ-subclosed graph is (θ,β)-continuous.
(c) for all spaces Y, Z and all functions $\phi : Y \to X$ and $\psi : Z \to X$ with β-θ-subclosed graphs, the set $D(\phi,\psi) = \{(y,z) \in Y \times Z : \phi(y) = \psi(z)\}$ is closed in $Y \times Z$.
(d) for any space Y and every function $\psi : Y \to X$ having β-θ-subclosed graph, the set $D(\psi) = \{(y_1,y_2) \in Y \times Y : \psi(y_1) = \psi(y_2)\}$ is closed in $Y \times Y$.

Proof. $(a) \Rightarrow (b)$: Let $f : Y \to X$ be a function which has a β-θ-subclosed graph. To show f is (θ,β)-continuous, we will have to show that $f(ad\mathscr{F}) \subset \beta$-$\theta$-$adf(\mathscr{F})$, for any filter base $\mathscr{F}$ on Y. Let $x \in f(ad\mathscr{F})$. Then $x = f(y)$ for some $y \in ad\mathscr{F}$. Let $\mathscr{F}_0 = \{(U \cap F) - \{y\} : F \in \mathscr{F}$ and $U \in O(Y,y)\}$.
Case-I: Let $\mathscr{F}_0$ be a filter base on $Y - \{y\}$. Then clearly $\mathscr{F} \to y$ in Y. Since f has a β-θ-subclosed graph, β-θ-$adf(\mathscr{F}_0) \subset \{f(y)\}$. Also, as X is β-closed, by theorem 3.5, we get β-θ-$adf(\mathscr{F}_0) = \{f(y)\}$. So $x = f(y) \in \beta$-θ-$adf(\mathscr{F}_0) \subseteq \beta$-$\theta adf(\mathscr{F})$.
Case-II: Let $\mathscr{F}_0$ be not a filter base on $Y - \{y\}$. Then $U_0 \cap F_0 = \{y\}$ for some $U_0 \in O(Y,y)$ and $F_0 \in \mathscr{F}_0$. We claim that $y \in F$ for each $F \in \mathscr{F}$. Indeed, if it is not true, then for some $F' \in \mathscr{F}$, $y \notin F'$. Select an $F'' \in \mathscr{F}$ such that $F'' \subseteq F_0 \cap F'$. So, $(U_0 \cap F'') - \{y\} \subseteq (U_0 \cap F_0) - \{y\} = \emptyset$. Therefore, $U_0 \cap F'' = \{y\}$ and hence $y \in F'' \subseteq F_0 \cap F'$. This shows $y \in F'$ — a contradiction. So, $x = f(y) \in f(F)$ for each $F \in \mathscr{F}$ and hence $x \in \beta$-θ-$adf(\mathscr{F})$. Therefore, in any case, f is (θ,β)-continuous.
$(b) \Rightarrow (a)$: If possible let (X,τ) be not β-closed. Then by theorem 3.5, there exists a filter base $\mathscr{F}$ on X with β-θ-$ad\mathscr{F} = \emptyset$. Choose $x_o \in X$ and let $\tau_0 = \{B \subset X : x_0 \notin B\} \cup \{B \subset X : x_0 \in B$ and $F \subset B$ for some $F \in \mathscr{F}\}$. In [11] it has been shown that τ_0 is a topology on X. We shall show that the identity function $f : (X,\tau_0) \to (X,\tau)$ has a β-θ-subclosed graph but f is not (θ,β)-continuous. For this let $\mathscr{G}$ be a filter base on $X - \{x\}$ such that $\mathscr{G} \to x$ in (X,τ_0). We claim that $x = x_0$. If not then $\{x\}$ is an open set in (X,τ_0) and hence the filter base $\mathscr{G}$ on $X - \{x\}$ can not converge to x in (X,τ_0) — a contradiction. Also we claim that $\mathscr{F} \subset \mathscr{G}$. Indeed, for each $F \in \mathscr{F}$, we have $F \cup \{x_0\} \in \tau_0$ and since $\mathscr{G} \to x = x_0$ in (X,τ_0), there exists a $G \in \mathscr{G}$ such that $G \subset F \cup \{x_0\}$. So, $G \subset F$ and hence $F \in \mathscr{G}$. Hence, β-θ-$adf(\mathscr{G}) = \beta$-θ-$ad\mathscr{G} \subseteq \beta$-$\theta$-$ad\mathscr{F} = \emptyset$. Therefore, f has a β-θ-subclosed graph. But this f is not (θ,β)-continuous. In fact, $x_0 \in ad\mathscr{F}$ in (X,τ_0) but $f(x_0) = x_0 \notin \beta$-$\theta$-$adf(\mathscr{F})$. This contradicts the hypothesis (b). So (X,τ) is β-closed.
$(b) \Rightarrow (c)$: Let (y,z) be a limit point of $D(\phi,\psi)$. Then there exists a net $\{(y_\lambda,z_\lambda) : \lambda \in I\}$ in $D(\phi,\psi) - \{(y,z)\}$ with $\{(y_\lambda,z_\lambda) \to (y,z)$. So, either (y_λ) is in $Y - \{y\}$ or (z_λ) is in $Z - \{z\}$, say (y_λ) is in $Y - \{y\}$; since ϕ has a β-θ-subclosed graph, $\phi(y_\lambda)$ has atmost one β-θ-adherent point say, $\phi(y)$. Now as by hypothesis (b), ϕ is (θ,β)-continuous, hence $\phi(y_\lambda)$ β-θ-converges to $\phi(y)$ only. But as $\phi(y_\lambda) = \psi(z_\lambda)$ for each $\lambda \in I$, the net $\psi(z_\lambda)$ is also β-θ-converging to $\phi(y)$ only. Since ψ is (θ,β)-continuous (by hypothesis (b)) $\psi(z_\lambda)$ β-θ-

converges to $\psi(z)$. Since X is β-T_2, we have $\phi(y) = \psi(z)$ and so $(y,z) \in D$. Therefore, D is closed in $Y \times Z$.

$(c) \Rightarrow (d)$: Obvious.

$(d) \Rightarrow (a)$: Suppose (X,τ) is not β-closed. So by remark 3.6, there exists a net $(x_\lambda)_{\lambda\in I}$ in X which has no β-θ-adherent point. We may choose $x_0, x_1 \in X$ with $x_0 \neq x_1$ and assume without loss of generality that $(x_\lambda)_{\lambda\in I}$ is a net in $X - \{x_0, x_1\}$. Let $Z = X$ and $\tau^\star = \{U \subset Z : U \cap \{x_0, x_1\} = \emptyset\}$ or $\{U \subset Z : U \cap \{x_0, x_1\} \neq \emptyset$ and $S_\lambda = \{x_\lambda : \lambda \geq \lambda_0\} \subset U$ for some $\lambda_0 \in I\}$. Clearly $\tau^\star$ is a topology on Z. Let $\psi : (Z,\tau^\star) \to (X,\tau)$ be define by $\psi(x_0) = x_1$, $\psi(x_1) = \psi(x_0)$ and $\psi(x) = x$ if $x \notin \{x_0, x_1\}$. We arrive at a contradiction by showing that $D(\psi)$ is not closed but this function ψ has a β-θ-subclosed graph. From the definition, it is clear that $(x_0, x_1) \notin D(\psi)$. But (x_0, x_1) is a limit point of $D(\psi)$. Indeed, let W be an open set containing (x_0, x_1) in $Z \times Z$. The definition of $\tau^\star$ ensures that $(S_{\lambda_1} \cup \{x_0\}) \times (S_{\lambda_2} \cup \{x_1\}) \subseteq W$ for some $\lambda_1, \lambda_2 \in I$. Now $(x_\lambda, x_\lambda) \in W \cap D(\psi)$ for $\lambda > \lambda_1, \lambda_2$ provides (x_0, x_1) is a limit point of $D(\psi)$. So $D(\psi)$ is not closed. To show ψ has a β-θ-subclosed graph, let $N = (z_\beta)_{\beta\in J}$ be a net in $Z - \{z\}$ converging to z. Obviously $z = x_0$ or x_1, otherwise $\{z\}$ would be an open set containing z and hence the net $(z_\beta)_{\beta\in J}$ could not converge to z. Suppose $z = x_0$ (say). If possible, let $\psi(N)$ β-θ-adheres to some point $x \in X$. But as the net $\psi((x_\lambda)_{\lambda\in I}) = (x_\lambda)_{\lambda\in I}$ has no β-θ-adherent point, there exists $V \in \beta R(X,x)$ and a $\lambda_0 \in I$ such that $S_\lambda = \{x_\lambda : \lambda \geq \lambda_0\} \subset X - V$, for all $\lambda \geq \lambda_0$. Since the net $N = (z_\beta)_{\beta\in J}$ is converging to x_0, $\{z_\beta : \beta \geq \beta_0\} \subset S_\lambda \cup \{x_0\}$ for some $\beta_0 \in J$. Obviously, no z_β can be x_0 and x_1 as well for $\beta \geq \beta_0$. So, $\{\psi(z_\beta) : \beta \geq \beta_0\} = \{z_\beta : \beta \geq \beta_0\} \subseteq S_\lambda \subset X - V$. Hence $\psi(N)$ can not β-θ-adhere to x. So ψ has a β-θ-subclosed graph. Therefore X is β-closed.

Theorem 4.14. A space (X,τ) is β-closed if and only if for any space Z and any functions $\phi, \psi : Z \to X$ with β-θ-subclosed graphs, $\Delta = \{z \in Z : \phi(z) = \psi(z)\}$ is closed in Z.

Proof. Suppose X is not β-closed. Then by remark 3.6, there exists a net $S = (x_\lambda)_{\lambda\in I}$ in X having no β-θ-adherent point in X. Consider two points x_o, x_1 in X with $x_o \neq x_1$ and put $Z = X$ and assume without loss of generality that $S = (x_\lambda)_{\lambda\in I}$ is a net $X - \{x_1\}$. Let $\tau^\star = \{U \subset Z : x_1 \notin U\} \cup \{U \subset Z : T_{\lambda_0} = \{x_\lambda : \lambda \geq \lambda_0\} \subset U$ for some $\lambda_0 \in I\}$. Then $\tau^\star$ is a topology on Z. We now define two functions $\phi, \psi : (Z,\tau^\star) \to (X,\tau)$ as follows:

$\phi(z) = z$ for $z \in Z$ and $\psi(z) = z$ for $z \in Z - \{x_1\}$ and $\psi(x_1) = x_0$. We now claim that ϕ and ψ has β-θ-subclosed graphs. Let $N = (z_\mu)_{\mu\in J}$ be a net on $Z - \{z\}$ converging z. If $z \neq x_1$ then $N = (z_\mu)_{\mu\in J}$ cannot converge to z as $\{x_1\}$ is an open set in $(Z,\tau^\star)$ — a contradiction. So $z = x_1$. If possible, let $\psi(N)$ β-θ-adheres to some point, say $z_0 \in Z = X$. Since $\psi(S) = S$ has no β-θ-adherent point in X, there is a $R \in \beta R(X,z_0)$ such that $T_{\lambda_0} = \{x_\lambda : \lambda \geq \lambda_0\} \cap R = \emptyset$ for some $\lambda_0 \in I$. Since N converge to x_1 and $\psi(N) = N$, then $\{z_\mu : \mu \geq \mu_1\} \subset T_{\lambda_0} \cup \{x_1\}$ for some $\mu_1 \in J$. But as z_β cannot be x_1 for any $\mu \geq \mu_1$, so $\{\psi(z_\mu) : \mu \geq \mu_1\} = \{z_\mu : \mu \geq \mu_1\} \subset T_{\lambda_0} \subset X - R$. So for this $R \in \beta R(Z,z_0)$ and for $\mu_1 \in J$, there does not exist any $\mu \in J$ such that $\mu > \mu_1$ and $z_\mu \in R$ — a contradiction. So $\psi(N)$ cannot β-θ-adhere to any point in X. Hence ψ has a β-θ-subclosed graph. Similarly, ϕ has also a β-θ-subclosed graph. Clearly, $\Delta = X - \{x_1\}$ as $\phi(x_1) = x_1 \neq x_0 = \psi(x_1)$. But x_1 is a limit point of Δ in Z. So, Δ is not closed in Z — a contradiction. Hence X is β-closed.

Conversely, let X be β-closed. Then by the theorem 4.13, for any space Z and any functions $\phi, \psi : Z \to X$ with β-θ-subclosed graphs, the set $D(\phi, \psi) = \{(z_1, z_2) \in Z \times Z : \phi(z_1) = \phi(z_2)\}$ is closed in $Z \times Z$. Let $\pi_1 : Z \times Z \to Z$ be the first projection and Δ_Z be the diagonal in $Z \times Z$. Since π_1/Δ_Z is a homeomorphism, then $\Delta = \pi_1[D(\phi, \psi) \cap \Delta_Z]$ is closed in Z.

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