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On the inverse problem of the Scattering Theory for a class of systems of dirac equations with discontiunous coefficient

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Abstract. In this paper it is devoted to study the inverse scattering problem for a singular boundary value problem of generalized form of system Dirac type. The new representation for the solutions of the differential equations system is considered, the scattering function is defined and its properties are given. The main equation is obtained for the solution of the inverse problem and it is shown the uniqueness of the solution of the inverse problem of scattering theory on the half line $[0,\infty)$.

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Key words: Dirac operator on the half line, scattering function, inverse problem of scattering theory, uniqueness of the solution to inverse problem.

1. Introduction

We consider on the half line $(0,\infty)$ the system of Dirac equations

$$BY' + \Omega(x)Y = \lambda\rho(x)Y \tag{1.1}$$

and the boundary condition

$$Y_1(0) - hY_2(0) = 0, \tag{1.2}$$

where h is an arbitrary real number, λ is spectral parameter, $p(x)$ and $q(x)$ are real-valued measurable functions,

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}.$$

Also, the coefficient $\rho(x)$ is a piecewise constant function takes the form

$$\rho(x) = \begin{cases} \alpha, & 0 \le x < a, \\ 1, & x \ge a, \end{cases} \tag{1.3}$$

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and $1 \neq \alpha > 0$. Assume that the condition

$$\int_0^\infty \|\Omega(x)\| \, dx < \infty \tag{1.4}$$

is satisfied for Euclidean norm.

The aim of this paper is to show the uniqueness of solution of the inverse problem for the boundary value problem (1.1), (1.2) with discontinuous coefficients on the half line $(0,\infty)$.

The inverse scattering problem for classical Sturm-Liouville and Dirac operators on the half line was solved completely in [1]- [7]. In the case that coefficients have discontinuous points, it is come up new changes in the solution of problem. For example, when the potential has a discontinuous point at $x = a$, the solution of inverse problem on the half line $(0,\infty)$ is turned to the solutions of two inverse problems in the intervals $[0,a]$ and $[a,\infty)$ (see [8]). In this case it is used the new integral representation for the solution (see [9], [10]), not operator transformation. We showed the literature about the inverse scattering problem on the half line. The inverse problem of scattering theory for Sturm-Liovuille problem with discontinuous coefficients on the half line was investigated in [11], [12]. The references about the inverse problems for Dirac operators on the finite and the half-infinite intervals were given in [13]. The results obtained in this work were presented in the conference [14].

Let suppose that

$$\mu(x) = \begin{cases} a + \alpha(x-a), & 0 \le x \le a, \\ x, & x > a. \end{cases}$$

We denote the solution of the equation (1.1) satisfying the condition

$$\lim_{x\to\infty} f(x,\lambda) e^{-i\lambda x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

by $f(x,\lambda)$. When $\Omega(x) \equiv 0$, it is easily obtained the solution of the equation (1.1) having this property in this form

$$f^0(x,\lambda) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda\mu(x)}.$$

As in [9] and [10], let $F(x,\lambda)$ be a solution of the equation (1.1) satisfying the condition

$$\lim_{x\to\infty} F(x,\lambda) e^{-\lambda B x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is obvious that

$$f(x,\lambda) = F(x,\lambda) \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Therefore, it suffices to check that $F(x,\lambda)$ has the form

$$F(x,\lambda) = e^{-\lambda B\mu(x)} + \int_{\mu(x)}^\infty K(x,t) e^{-\lambda B t} dt. \tag{1.5}$$

By the method of variation of parameters, it is obtained the integral equation for $F(x,\lambda)$:

$$F(x,\lambda)=e^{-\lambda B\mu(x)}-\int_x^{\infty}B\Omega(t)e^{\lambda B\mu(x)-\lambda B\mu(t)}F(t,\lambda)\,dt. \tag{1.6}$$

In order for the function $F(x,\lambda)$ to satisfy this integral equation, it is necessary that the equality

$$\begin{aligned}\int_{\mu(x)}^{\infty}K(x,t)e^{-\lambda Bt}dt &= -\int_x^{\infty}B\Omega(t)\exp\{\lambda B\mu(x)-\lambda B\mu(t)\}\times\\ &\left[e^{-\lambda B\mu(t)}+\int_{\mu(t)}^{\infty}K(t,s)e^{-\lambda Bs}ds\right]dt\end{aligned} \tag{1.7}$$

holds. Conversely, if the matrix function $K(x,t)$ satisfies this equality, then the matrix function $F(x,\lambda)$ satisfies the integral equation (1.6).

We transform the right hand side of the equality (1.7) such that it is similar to the left hand side of this equality. Let's assume the following expressions:

$$K_{\pm}(x,t)=\frac{1}{2}\left[K(x,t)\pm BK(x,t)B\right].$$

It is clearly from the expressions of the matrix functions $K_{\pm}(x,t)$ that

$$\begin{aligned}K(x,t) &= K_{+}(x,t)+K_{-}(x,t),\\ BK_{+}(x,t) &= \frac{1}{2}\left[BK(x,t)-K(x,t)B\right]=-K_{+}(x,t)B,\\ BK_{-}(x,t) &= \frac{1}{2}\left[BK(x,t)+K(x,t)B\right]=-K_{-}(x,t)B.\end{aligned}$$

By transforming the right hand of (1.6), it is obtained for the matrix functions $K_{\pm}(x,t)$ the following integral equations:

$$K_{+}(x,t)=-\frac{1}{2\alpha}B\Omega\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)-\int_x^{\frac{t+\alpha x+\alpha a-a}{2\alpha}}B\Omega(\zeta)K_{-}(\zeta,t-\alpha\zeta+\alpha x)\,d\zeta,$$

if $0<x<a,\ \ \alpha x-\alpha a+a<t<-\alpha x+\alpha a+a$;

$$\begin{aligned}K_{+}(x,t) &= -\frac{1}{2}B\Omega\left(\frac{t+\alpha x-\alpha a+a}{2}\right)-\int_x^{a}B\Omega(\zeta)K_{-}(\zeta,t-\alpha\zeta+\alpha x)\,d\zeta\\ &\quad-\int_a^{\frac{t+\alpha x-\alpha a+a}{2}}B\Omega(\zeta)K_{-}(\zeta,t-\zeta+\alpha x-\alpha a+a)\,d\zeta,\end{aligned}$$

if $0 < x < a, \quad t > -\alpha x + \alpha a + a$;

$$\begin{aligned} K_-(x,t) &= -\int_x^a B\Omega(\zeta)K_+(\zeta, t+\alpha\zeta-\alpha x)\,d\zeta \\ &\quad -\int_a^\infty B\Omega(\zeta)K_+(\zeta, t+\zeta-\alpha x+\alpha a-a)\,d\zeta, \end{aligned}$$

if $0 < x < a, \quad t > \alpha x - \alpha a + a$;

$$\begin{aligned} K_+(x,t) &= -\frac{1}{2}B\Omega\left(\frac{x+t}{2}\right) - \int_x^{\frac{x+t}{2}} B\Omega(\zeta)K_-(\zeta, t+x-\zeta)\,d\zeta, \\ K_-(x,t) &= -\int_x^\infty B\Omega(\zeta)K_+(\zeta, t-x+\zeta)\,d\zeta, \end{aligned}$$

if $t > x > a$.

The solvability of these equations system can be established by the method of successive approximations. It is obtained the following theorem.

Theorem 1.1. *[9] Assume that the condition (1.4) is satisfied. Then for $Im\lambda \geq 0$ the equation (1.1) has an unique solution in the form*

$$f(x,\lambda) = f^0(x,\lambda) + \int_{\mu(x)}^\infty K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt, \tag{1.8}$$

where the elements of the matrix function $K(x,t)$ are summable on the positive half line and $K(x,t)$ satisfies the following property

$$\int_{\mu(x)}^\infty \|K(x,t)\|\,dt \leq e^{\sigma(x)} - 1,$$

here $\sigma(x) = \int_x^\infty \|\Omega((t))\|\,dt$.

Also, if $\Omega(x)$ is absolute continuous, then it is obtained from the equations system above the relations

$$\begin{aligned} &BK_x(x,t) + \Omega(x)K(x,t) = -\rho(x)K_t(x,t)B, \\ &\rho(x)\{BK(x,\mu(x)) - K(x,\mu(x))B\} = \Omega(x). \end{aligned} \tag{1.9}$$

Let $y(x,\lambda)$ and $z(x,\lambda)$ be vector functions. The expression

$$W\left[y(x,\lambda),z(x,\lambda)\right]=y^T(x,\lambda)Bz(x,\lambda)=(y_1,y_2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=y_1z_2-y_2z_1$$

is called Wronskian of the vector functions $y(x,\lambda)$ and $z(x,\lambda)$.

Since $p(x)$ and $q(x)$ are real valued functions, the vector functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ constitute fundamental system of solutions of the equation (1.1) for real λ. Wronskian of this functions doesn't depend on x and is equal to $2i$

$$W\left[f(x,\lambda),\overline{f(x,\lambda)}\right]=2i.$$

Denote by $\varphi(x,\lambda)$ the solution of the equation (1.1) satisfying the initial conditions

$$\varphi_1(0,\lambda)=h,\qquad \varphi_2(0,\lambda)=1. \tag{1.10}$$

Let us define the function

$$\Delta(\lambda)=f_1(0,\lambda)-hf_2(0,\lambda). \tag{1.11}$$

2. The Scattering Function

It is proved the following lemma.

Lemma 2.1. *The identity*

$$\frac{2i\varphi(x,\lambda)}{\Delta(\lambda)}=\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda) \tag{2.1}$$

holds for all real λ, *where*

$$S(\lambda)=\frac{\overline{f_1(0,\lambda)}-h\overline{f_2(0,\lambda)}}{f_1(0,\lambda)-hf_2(0,\lambda)} \tag{2.2}$$

and

$$|S(\lambda)|=1$$

Proof. Since $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ constitute the fundamental system of solutions of equation (1.1) on the half line $(0,\infty)$ for real λ, it is written

$$\varphi(x,\lambda)=c_1(\lambda)f(x,\lambda)+c_2(\lambda)\overline{f(x,\lambda)}, \tag{2.3}$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are functions, which we have to find. Substituting $x=0$ and taking into account the initial conditions (1.10), it is obtained

$$\begin{aligned}c_1(\lambda)f_1(0,\lambda)+c_2(\lambda)\overline{f_1(0,\lambda)}&=h,\\ c_1(\lambda)f_2(0,\lambda)+c_2(\lambda)\overline{f_2(0,\lambda)}&=1.\end{aligned}$$

From here, it is found

$$c_1(\lambda) = -\frac{\overline{f_1(0,\lambda)} - h\overline{f_2(0,\lambda)}}{2i}, \qquad c_2(\lambda) = \frac{f_1(0,\lambda) - hf_2(0,\lambda)}{2i}.$$

For all real λ, $\Delta(\lambda) \neq 0$. In fact, assume the contrary that $f_1(0,\lambda) = hf_2(0,\lambda)$ for $\lambda_0 \in (-\infty,\infty)$. It is clearly that $\overline{f_1(0,\lambda_0)} = h\overline{f_2(0,\lambda_0)}$. Then it is found

$$W\left[f(0,\lambda_0), \overline{f(0,\lambda_0)}\right] = 2i$$

or

$$f_1(0,\lambda_0)\overline{f_2(0,\lambda_0)} - f_2(0,\lambda_0)\overline{f_1(0,\lambda_0)} = 2i.$$

If we substitute the expression of $f_1(0,\lambda_0)$ and $\overline{f_1(0,\lambda_0)}$ above, it is found a contradiction.

Substituting the constants $c_1(\lambda)$, $c_2(\lambda)$ in (2.3) and dividing the equality by $\Delta(\lambda)$, the identity (2.1) is obtained. From (2.2)

$$S(\lambda) = \frac{\overline{f_1(0,\lambda)} - h\overline{f_2(0,\lambda)}}{f_1(0,\lambda) - hf_2(0,\lambda)} = \overline{\left\{\frac{f_1(0,\lambda) - hf_2(0,\lambda)}{\overline{f_1(0,\lambda)} - h\overline{f_2(0,\lambda)}}\right\}} = \overline{[S(\lambda)]} = [S(\lambda)]^{-1}.$$

The lemma is proved.

The function $S(\lambda)$ is called *the scattering function* of the boundary value problem (1.1)-(1.2).

In particular if $\Omega(x) \equiv 0$, the equality (2.1) has the form

$$\frac{2i\varphi^0(x,\lambda)}{\Delta(\lambda)} = \overline{f^0(x,\lambda)} - S_0(\lambda)f^0(x,\lambda) \tag{2.4}$$

where the vector function $\varphi^0(x,\lambda)$ is a solution of the equation (1.1) satisfying the initial conditions

$$\varphi_1^0(0,\lambda) = h, \qquad \varphi_2^0(0,\lambda) = 1$$

and

$$S_0(\lambda) = \frac{\overline{f_1^0(0,\lambda)} - h\overline{f_2^0(0,\lambda)}}{f_1^0(0,\lambda) - hf_2^0(0,\lambda)} = e^{-2i\lambda a(1-\alpha)}\frac{1+ih}{1-ih}.$$

We saw in the proof of Lemma 2.1 that the function $\Delta(\lambda)$ had no real zeros. From the expression (1.8) of the solution, it is clear that $f_1(0,\lambda)$ and $\overline{f_1(0,\lambda)}$ can be continued as analytical and are continuous on the whole line. This properties holds for $\Delta(\lambda)$. As $|\lambda| \to \infty$

$$f(0,\lambda) \to \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and thus the zeros of $\Delta(\lambda)$ in the upper plane are not more than countable and constitute a bounded set.

Let us show that $\Delta(\lambda)$ has no zeros on the upper half plane. Assume the contrary. Let μ $(Im\mu > 0)$ be one of the zeros of the function $\Delta(\lambda)$.The function $f^*(x,\mu)$ denotes the transposed matrix function of $\overline{f(x,\mu)}$.

Now differentiating the equation

$$Bf'(x,\mu) + \Omega(x) f(x,\mu) = \rho(x)\mu f(x,\mu)$$

with respect to μ, one obtains the following equation

$$-\dot{f}^*(x,\mu)B + f^*(x,\mu)\Omega(x) = \rho(x)\overline{\mu} f^*(x,\mu).$$

Taking this into account, multiplying the first equation by $f^*(x,\mu)$ and the second equation by $f(x,\mu)$, and subtracting the first equality from the second one, and finally integrating this relation according to x from 0 to ∞, we get

$$W\left\{\overline{f(x,\mu)}, f(x,\mu)\right\}|_{x=0} + (\overline{\mu}-\mu)\int_0^\infty f^*(x,\mu) f(x,\mu)\rho(x)dx = 0.$$

On the other hand we have

$$\Delta(\mu) \equiv f_1(0,\mu) - hf_2(0,\mu) = 0$$

or

$$f_1(0,\mu) = hf_2(0,\mu).$$

Hence, we get

$$W\left\{\overline{f(x,\mu)}, f(x,\mu)\right\}|_{x=0} = \overline{f_1(0,\mu)} f_2(0,\mu) - \overline{f_2(0,\mu)} f_1(0,\mu) = 0$$

and then

$$(\overline{\mu}-\mu)\int_0^\infty f^*(x,\mu) f(x,\mu)\rho(x)dx = 0.$$

It is found $\mu = \overline{\mu}$ from here. It is contrary to assumption.

Thus, we arrived the following result.

Lemma 2.2. $\Delta(\lambda)$ *is analytic in the upper half plane* $(Im\lambda > 0)$, *is continuous function on the whole line and has no zeros on the upper half plane.*

From the results in Lemma 2.1 and Lemma 2.2, we obtain that the function $S(\lambda)$ is continuous and for $|\lambda| \to \infty$ the following asymptotic form holds

$$S(\lambda) = S_0(\lambda) + O\left(\frac{1}{\lambda}\right)$$

and accordingly $S(\lambda) - S_0(\lambda) \in L_2(-\infty,\infty)$.

3. Derivation of The Main Equation

In this chapter, we show that if the scattering function of the boundary value problem (1.1), (1.2) are known, then we can construct an integral equation for the unknown function $K(x,t)$. We obtain the integral equation which has an important role in the solution of the inverse boundary value problem (1.1)-(1.2).

To show it, the identity (2.1) in Lemma 2.1 is used. Let's substitute the expression (1.8) of the function $f(x,\lambda)$

$$\begin{aligned}
&\frac{2i\varphi(x,\lambda)}{\Delta(\lambda)}+S_0(\lambda)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)}-\begin{pmatrix}1\\-i\end{pmatrix}e^{-i\lambda\mu(x)}\\
=\ &\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\i\end{pmatrix}e^{-i\lambda t}dt-S_0(\lambda)\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt\\
&+\left[S_0(\lambda)-S(\lambda)\right]\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)}+\left[S_0(\lambda)-S(\lambda)\right]\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt.
\end{aligned}$$

Multiplying this equality by $\frac{1}{2\pi}(1,-i)e^{i\lambda y}$ and integrating it to λ, from $-\infty$ to ∞ we get

$$\begin{aligned}
&Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{2i\varphi(x,\lambda)}{\Delta(\lambda)}+S_0(\lambda)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)}-\begin{pmatrix}1\\-i\end{pmatrix}e^{-i\lambda\mu(x)}\right](1,-i)e^{i\lambda y}d\lambda\\
=\ &Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\i\end{pmatrix}(1,-i)e^{-i\lambda(t-y)}dtd\lambda\\
&-Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_0(\lambda)\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)e^{i\lambda(t+y)}dtd\lambda+\\
&+Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0(\lambda)-S(\lambda)\right]\int\limits_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)e^{i\lambda(t+y)}dtd\lambda\\
&+Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0(\lambda)-S(\lambda)\right]\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)e^{i\lambda(\mu(x)+y)}d\lambda.
\end{aligned}\tag{3.1}$$

It is easily shown that

$$\begin{aligned} Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)\,e^{-i\lambda(t-y)}d\lambda &= Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\lambda(t-y)}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}d\lambda \\ &= \delta\left(t-y\right)I_2 \equiv \delta_2\left(t-y\right), \quad I_2 = Re\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}, \end{aligned}$$

where $\delta\left(x\right)$ is the Dirac delta function.

Thus

$$\begin{aligned} \int\limits_{\mu(x)}^{\infty}K\left(x,t\right)Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)\,e^{-i\lambda(t-y)}d\lambda dt &= \int\limits_{\mu(x)}^{\infty}K\left(x,t\right)\delta_2\left(t-y\right)dt \\ &= K\left(x,y\right) \end{aligned}$$

and

$$Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_0\left(\lambda\right)\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)\begin{pmatrix}1\\ -i\end{pmatrix}(1,-i)\,e^{i\lambda(t+y)}dt d\lambda =$$

$$\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_0\left(\lambda\right)\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}e^{i\lambda(t+y)}d\lambda dt.$$

Now, we calculate the integral

$$\int\limits_{-\infty}^{\infty}S_0\left(\lambda\right)e^{i\lambda(t+y)}d\lambda.$$

Substituting $S_0\left(\lambda\right)$ here, we find

$$\int\limits_{-\infty}^{\infty}S_0\left(\lambda\right)e^{i\lambda(t+y)}d\lambda = \frac{1+ih}{1-ih}\delta\left(t+y-2a\left(1-\alpha\right)\right).$$

Taking this values into account on the right hand of (3.1), we get

$$K\left(x,y\right)+Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)\begin{pmatrix}1\\-i\end{pmatrix}\left(1,-i\right)e^{i\lambda\left(t+y\right)}dt\,d\lambda$$

$$+Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\begin{pmatrix}1 & -i\\-i & -1\end{pmatrix}e^{i\lambda\left(\mu(x)+y\right)}d\lambda$$

$$-\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)Re\frac{1}{2\pi}\begin{pmatrix}1 & -i\\-i & -1\end{pmatrix}\frac{1+ih}{1-ih}\delta\left(t+y-2a\left(1-\alpha\right)\right)dt=K\left(x,y\right)$$

$$+\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)F_0\left(t+y\right)dt+F_0\left(\mu\left(x\right)+y\right)-Re\frac{1+ih}{1-ih}K\left(x,2a\left(1-\alpha\right)-y\right),$$

where

$$F_0\left(x\right)=Re\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\begin{pmatrix}1 & -i\\-i & -1\end{pmatrix}e^{i\lambda x}d\lambda \tag{3.2}$$

and $K\left(x,2a\left(1-\alpha\right)-y\right)=0$ for $y>\mu\left(x\right)$.

Hence, the right hand of (3.1) has the form

$$K\left(x,y\right)+F_0\left(\mu\left(x\right)+y\right)+\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)F_0\left(t+y\right)dt$$

for $y>\mu\left(x\right)$.

Since integrand on the left hand of (3.1) is analytic, it is obtained that the left hand is equal to zero. Hence for $y>\mu\left(x\right)$ we get

$$K\left(x,y\right)+F_0\left(\mu\left(x\right)+y\right)+\int\limits_{\mu(x)}^{\infty}K\left(x,t\right)F_0\left(t+y\right)dt=0 \tag{3.3}$$

from (3.1), where $F_0\left(x\right)$ is defined by (3.2).

The integral equation (3.3) is called the main equation of the boundary value problem (1),(2).

Eventually we proved the following theorem.

Theorem 3.1. *For each $x\geq 0$, the kernel $K\left(x,y\right)$ of special solution of (1.8) satisfies the main equation.*

4. Solvability of The Main Equation

Theorem 4.1. *For each fixed $x \geq 0$, the main equation has an unique vector solution with elements in $L_2\left(\mu(x), \infty\right)$.*

Proof. Suppose that the scattering function $S(\lambda)$ is given. It is found the function $F_0(x)$ by the formula (3.2) and the main equation is constructed by aid of this. Let us rewrite it in the more convenient form

$$K\left(x, t+\mu(x)\right)+F_0\left(t+2\mu(x)\right)+\int_0^{\infty} K\left(x, \zeta+\mu(x)\right) F_0\left(\zeta+t+2\mu(x)\right) d\zeta=0 \tag{4.1}$$

and seek its solution $K\left(x, y+\mu(x)\right)$ for every $x \geq 0$ in the same space $L_2\left(\mu(x), \infty\right)$.

We consider the operator $\mathbf{F}_{0x}$

$$\mathbf{F}_{0x} f=\int_0^{\infty} f(\zeta) F_0\left(\zeta+t+2\mu(x)\right) d\zeta$$

acting in the space $L_2(0, \infty)$, which appears in the main equation. It is showed that the operator $\mathbf{F}_{0x} f$ is compact in each space $L_2(0, \infty)$ for every choice of $\mu(x) \geq 0$. Taking $f(t)= K\left(x, t+\mu(x)\right)$ the integral equation (4.1) can be written as

$$f(t)+\mathbf{F}_{0x} f(t)=-F_0\left(t+2\mu(x)\right).$$

For solvability of this equation, it is necessary that the homogeneous equation

$$f(t)+\mathbf{F}_{0x} f(t)=0$$

has no nonzero solutions in the corresponding space. The operator $\mathbf{F}_{0x}$ has the same properties of $F_{s,a}^{+}$ defined in ([4] s.202). The kernels of both of two operators are defined the functions $S(\lambda)$ having same properties. Hence the proof of lemma is obtained as result of Lemma 3.3.3 in ([4]).

For every $x \geq 0$ the main equation (3.3) hasn't any solution except for $K(x, t)$ satisfying the relation (1.9) according to Theorem 4.1. It is arrived the following result from here.

Theorem 4.2. *The scattering function determines the boundary value problem (1.1),(1.2) uniquely.*

Proof. Clearly, when it is given the scattering function $S(\lambda)$, the function $F_0(x)$ is found by the formula (3.2). By aid of this function, it is constructed the main equation (3.3) according to unknown $K\left(x, y\right)$. It is seen from (4.1) that the main equation has an unique solution. The potential $\Omega(x)$ which has the form (1.9) is established uniquely by $K\left(x, y\right)$. It is constructed the equation (1.1) by given algorithm. The theorem is proved.

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