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Convergence of Singular Integral Operators in Weighted Lebesgue Spaces

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Abstract. In this paper, the pointwise approximation to functions $f \in L_{1,w}\langle a,b\rangle$ by the convolution type singular integral operators given in the following form:

$$L_\lambda(f;x) = \int_a^b f(t) K_\lambda(t-x)\,dt, \quad x \in \langle a,b\rangle, \ \lambda \in \Lambda \subset \mathbb{R}_0^+$$

where $\langle a,b\rangle$ stands for arbitrary closed, semi closed or open bounded interval in $\mathbb{R}$ or $\mathbb{R}$ itself, $L_{1,w}\langle a,b\rangle$ denotes the space of all measurable but non-integrable functions f for which $\left|\frac{f}{w}\right|$ is integrable on $\langle a,b\rangle$ and $w : \mathbb{R} \to \mathbb{R}^+$ is a corresponding weight function, at a μ-generalized Lebesgue point and the rate of convergence at this point are studied.

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1. Introduction

In paper [16], Taberski analyzed the pointwise convergence of integrable functions and the approximation properties of derivatives of integrable functions in $L_1\langle -\pi,\pi\rangle$ by a two parameter family of convolution type singular integral operators of the form:

$$T_\lambda(f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda(t-x)\,dt, \ x \in \langle -\pi,\pi\rangle, \ \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{1}$$

where $K_\lambda(t)$ is the kernel fulfilling appropriate assumptions and $\lambda \in \Lambda$ and Λ is a given set of non-negative numbers with accumulation point λ_0. Later on, the weighted

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pointwise convergence and rate of convergence of a family of m-singular integral operators in $f \in L_p(\mathbb{R})$ were investigated by Mamedov [14].

Then, Gadjiev [9] and Rydzewska [15] studied the pointwise convergence theorems and the order of pointwise convergence theorems for operators type (1) at a generalized Lebesgue point and $\mu-$generalized Lebesgue point of $f \in L_1(-\pi,\pi)$ based on Taberski's analysis, respectively.

Further, in [10] and [12] Karsli extended the results of Taberski [16], Gadjiev [9] and Rydzewska's [15] studies by considering the more general integral operators of the form:

$$T_\lambda(f;x) = \int_a^b f(t)\, K_\lambda(t-x)\, dt, \; x \in \langle a,b\rangle\,, \; \lambda \in \Lambda \subset \mathbb{R}_0^+ \tag{2}$$

for functions in $L_1\langle a,b\rangle$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$ such as $[a,b]$, (a,b), $[a,b)$ or $(a,b]$.

In [11, 13], Karsli analyzed the pointwise convergence theorems and the rate of pointwise convergence theorems for a family of nonlinear singular integral operators at a $\mu-$generalized Lebesgue point and at a generalized Lebesgue point of $f \in L_1\langle a,b\rangle$, respectively.

Bardaro and Cocchieri [2] evaluated the degree of pointwise convergence of Fejer-Type singular integrals at the generalized Lebesgue points of the functions $f \in L_1(\mathbb{R})$. In an another study, Bardaro [3] studied similar convergence results about moment type operators. Besides, the pointwise convergence of family of nonlinear Mellin type convolution operators at Lebesgue points was also analyzed by Bardaro and Mantellini [4].

Bardaro, Karsli and Vinti [5] obtained some approximation results related to the pointwise convergence and the rate of pointwise convergence for non-convolution type linear operators at a Lebesgue point based on Bardaro and Mantellini's study [4]. After that, they got similar results for its nonlinear counterpart in [6] while in an another study [7], the pointwise convergence and the rate of pointwise convergence results for a family of Mellin type nonlinear m-singular integral operators at m-Lebesgue points of f were investigated.

Almali [1] studied the problem of pointwise convergence of non-convolution type integral operators at Lebesgue points of some classes of measurable functions.

In this paper, we also investigated the pointwise convergence and the rate of convergence of the operators similar to the study of Karsli [12]. However, the difference between this study and Karsli [12] is that while Karsli [12] considers the pointwise convergence of the integral operators to the functions $f \in L_1\langle a,b\rangle$, our study covers the case $f \notin L_1\langle a,b\rangle$.

The main contribution of this paper is obtaining the pointwise convergence of the convolution type singular integral operators of the form:

$$L_\lambda(f;x) = \int_a^b f(t)\, K_\lambda(t-x)\, dt, \; x \in \langle a,b\rangle\,, \; \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{3}$$

where the symbol $\langle a,b\rangle$ stands for an arbitrary closed, semi closed or open bounded interval in $\mathbb{R}$ or $\mathbb{R}$ itself, to the function $f \in L_{1,w}\langle a,b\rangle$ where $L_{1,w}\langle a,b\rangle$ is the space of all measurable and non-integrable functions f for which $\left|\frac{f}{w}\right|$ is integrable on $\langle a,b\rangle$ and $w:\mathbb{R}\to\mathbb{R}^{+}$ is a corresponding weight function, at a common μ-generalized Lebesgue point of $\frac{f}{w}$ and w.

The paper is organized as follows: In Section 2, we introduce the fundamental definitions. In Section 3, we prove the existence of the operators type (3). In Section 4, we present two theorems concerning the pointwise convergence of $L_{\lambda}(f;x,y)$ to $f(x_0,y_0)$ whenever (x_0,y_0) is a common $\mu-$generalized Lebesgue point of $\frac{f}{w}$ and w. In section 5, we give two theorems concerning the rate of pointwise convergence.

2. Preliminaries

Definition 1. *A point $x_0\in\langle a,b\rangle$ is called $\mu-$generalized Lebesgue point of the function $f\in L_1\langle a,b\rangle$, if*

$$\lim_{h\to 0}\frac{1}{\mu(h)}\int_0^h |f(x_0+t)-f(x_0)|\,dt=0,$$

where the function $\mu:\mathbb{R}\to\mathbb{R}$ is increasing and absolutely continuous on $[0,b-a]$ and $\mu(0)=0$ [15, 12].

Example 1. *Consider the function $f\in L_1(\mathbb{R})$ defined by*

$$f(t)=\begin{cases} e^{-t}, & \text{if } t\in(0,1]\\ 0, & \text{if } t\in\mathbb{R}\backslash(0,1].\end{cases}$$

One can compute that $x_0=0$ is not a Lebesgue point of f. On the other hand, by taking $\mu(t)=\sqrt{t}$ we see that $x_0=0$ is a $\mu-$generalized Lebesgue point of f.

Now, we define a new class for the weighted approximation.

Definition 2. ($Class\ A_w$) *Let $\Lambda\subset\mathbb{R}_0^{+}$ be an index set and λ_0 is an accumulation point of it. Further, let $K_\lambda:\mathbb{R}\to\mathbb{R}$ be an integrable function for each $\lambda\in\Lambda$ and* $\varphi(t)=\sup\limits_{x\in\langle a,b\rangle}\left[\frac{w(t+x)}{w(x)}\right]$, $\forall t\in\langle a,b\rangle$.

It is said that $K_\lambda(t)$ belongs to class A_w, if it satisfies the following conditions:

a. $\|K_\lambda\varphi\|_{L_1(\mathbb{R})}\le M<\infty$, $\forall\lambda\in\Lambda$.

b. $\lim\limits_{\lambda\to\lambda_0}\left[\sup\limits_{|t|>\xi}|K_\lambda(t)|\right]=0,\ \forall\xi>0.$

c. $\lim\limits_{\lambda\to\lambda_0}\left[\int\limits_{|t|>\xi}|K_\lambda(t)|\,dt\right]=0,\ \forall\xi>0.$

d. For a given $\delta_0>0$, $|K_\lambda(t)|$ is non-decreasing function with respect to t on $\langle-\delta_0,0]$ and non-increasing function with respect to t on $[0,\delta_0\rangle$.

e. At some $x \in \mathbb{R}$, $K_\lambda(x)$ *tends to infinity as* λ *tends to* λ_0.

f. $\lim\limits_{\lambda\to\lambda_0}\left|\int\limits_{\mathbb{R}} K_\lambda(t)\,dt - 1\right| = 0.$

Throughout this paper K_λ belongs to class A_w.

3. Existence of the Operators

Main results in this work are based on the following theorem.

Theorem 1. *Suppose that* $f \in L_{1,w}\langle a,b\rangle$. *Then the operator* $L_\lambda(f;x)$ *defines a continuous transformation acting on* $L_{1,w}\langle a,b\rangle$.

Proof. Let $\langle a,b\rangle$ be an arbitrary closed, semi closed or open bounded interval in $\mathbb{R}$. By the linearity of the operator $L_\lambda(f;x)$, it is sufficient to show that the expression:

$$\|L_\lambda\|_{1,w} = \sup_{f\neq 0}\frac{\|L_\lambda(f;x)\|_{L_{1,w}\langle a,b\rangle}}{\|f\|_{L_{1,w}\langle a,b\rangle}}$$

remains finite. Here, the norm of $f \in L_{1,w}\langle a,b\rangle$ is given by the following equality (see, for example [14]):

$$\|f\|_{L_{1,w}\langle a,b\rangle} = \int_a^b \left|\frac{f(x)}{w(x)}\right| dx.$$

Let us define a new function g by

$$g(t) := \begin{cases} f(t), & if \quad t \in \langle a,b\rangle \\ 0, & if \quad t \in \mathbb{R}\setminus\langle a,b\rangle. \end{cases}$$

Now, using Fubini's Theorem [8] we can write

$$\begin{aligned} \|L_\lambda(f;x)\|_{L_{1,w}\langle a,b\rangle} &= \int_a^b \frac{1}{w(x)}\left|\int_a^b f(t)K_\lambda(t-x)dt\right| dx \\ &= \int_a^b \frac{1}{w(x)}\left|\int_{-\infty}^{\infty} g(t+x)\frac{w(t+x)}{w(t+x)}K_\lambda(t)dt\right| dx \\ &= \int_a^b \frac{1}{w(x)}\left|\int_{-\infty}^{\infty} w(t+x)\frac{g(t+x)}{w(t+x)}K_\lambda(t)dt\right| dx \end{aligned}$$

$$\leq \int_a^b \frac{1}{w(x)}\left[\int_{-\infty}^{\infty} w(t+x)\left|\frac{g(t+x)}{w(t+x)}\right| |K_\lambda(t)|\,dt\right] dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} |K_\lambda(t)| \left[\int_a^b \frac{w(t+x)}{w(x)} \left| \frac{g(t+x)}{w(t+x)} \right| dx \right] dt \\
&\leq M \|f\|_{L_{1,w}\langle a,b\rangle} .
\end{aligned}
$$

Now, the proof of the indicated case is completed. The assertion can be proved with the above method for the case $\langle a, b\rangle = \mathbb{R}$. Thus the proof is completed.

4. Convergence at Characteristic Points

In this section, two theorems concerning pointwise convergence of the operators of type (3) will be presented. In the following theorem we suppose that $\langle a, b\rangle$ is an arbitrary closed, semi closed or open bounded interval in $\mathbb{R}$.

Theorem 2. *Suppose that $w(t)$ and $|K_\lambda(t-x)|$ are almost everywhere differentiable functions on $\mathbb{R}$ with respect to variable t such that the following inequality:*

$$
\frac{d}{dt} w(t) \frac{d}{dt} |K_\lambda(t-x)| > 0, \text{ for any fixed } x \in \langle a, b\rangle \tag{4}
$$

holds. If $x_0 \in \langle a, b\rangle$ is a common $\mu-$generalized Lebesgue point of functions $f \in L_{1,w}\langle a, b\rangle$ and $w \in L_1\langle a, b\rangle$, then

$$
\lim_{(x,\lambda)\to(x_0,\lambda_0)} L_\lambda(f; x) = f(x_0)
$$

on any planar set Z on which the function

$$
\int_{x_0-\delta}^{x_0+\delta} |K_\lambda(t-x)|\, w(t) \left| \{\mu(|x_0-t|)\}'_t \right| dt + 2\, |K_\lambda(0)|\, w(x)\, \mu(|x_0-x|), \ 0 < \delta < \delta_0,
$$

where δ_0 is a fixed positive real number, is bounded as (x, λ) tends to (x_0, λ_0).

Proof. Let $x_0 + \delta < b$, $x_0 - \delta > a$ and $|x_0 - x| < \frac{\delta}{2}$ for a given $\delta > 0$. The proof will be stated for the case $0 < x_0 - x < \frac{\delta}{2}$. The proof of the reverse case is similar.

Set $I = |L_\lambda(f; x) - f(x)|$. Using Theorem 2 in [12] we may write the integral I as follows:

$$
\begin{aligned}
I &= \left| \int_a^b f(t)\, K_\lambda(t-x)\, dt - f(x_0) \right| \\
&= \left| \int_a^b \left[\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)} \right] w(t)\, K_\lambda(t-x)\, dt + \frac{f(x_0)}{w(x_0)} \left[\int_a^b w(t)\, K_\lambda(t-x)\, dt - w(x_0) \right] \right|
\end{aligned}
$$

$$\leq \int_a^b \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right| w(t) |K_\lambda(t-x)| \, dt + \left|\frac{f(x_0)}{w(x_0)}\right| \left|\int_a^b w(t) K_\lambda(t-x) \, dt - w(x_0)\right|$$

$$= \left\{ \int_a^{x_0-\delta} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} + \int_{x_0+\delta}^{b} \right\} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right| w(t) |K_\lambda(t-x)| \, dt$$

$$+ \left|\frac{f(x_0)}{w(x_0)}\right| \left|\int_a^b w(t) K_\lambda(t-x) \, dt - w(x_0)\right|$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

It is sufficient to show that $I_i \to 0$ $(i = \overline{1,5})$ as $(x, \lambda) \to (x_0, \lambda_0)$ provided $(x, \lambda) \in Z$.

Let us consider the integral I_5. By Theorem 2 in [12], we get

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \int_a^b w(t) K_\lambda(t-x) \, dt = w(x_0).$$

Therefore, we have

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} I_5 = 0.$$

Now, we consider the integrals I_1 and I_4. From hypothesis (4) and condition (d) of class A_w, we have

$$\begin{aligned} I_1 &= \int_a^{x_0-\delta} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right| w(t) |K_\lambda(t-x)| \, dt \\ &\leq \sup_{t\in\langle a,x_0-\delta\rangle} w(t) \sup_{t\in\langle a,x_0-\delta\rangle} |K_\lambda(t-x)| \int_a^{x_0-\delta} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right| dt \end{aligned}$$

Hence we obtain

$$\begin{aligned} I_1 &\leq w(x_0-\delta) \sup_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)| \int_a^b \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right| dt \\ &\leq w(x_0-\delta) \sup_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)| \left\{ \|f\|_{L_{1,w}\langle a,b\rangle} + \left|\frac{f(x_0)}{w(x_0)}\right| (b-a) \right\}. \end{aligned}$$

Using similar strategy, we have the following inequality for the integral I_4

$$I_4 \leq w(x_0+\delta) \sup_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)| \left\{ \|f\|_{L_{1,w}\langle a,b\rangle} + \left|\frac{f(x_0)}{w(x_0)}\right| (b-a) \right\}.$$

Combining these integrals, we get the following inequality:

$$I_1 + I_4 \leq \{w(x_0 - \delta) + w(x_0 + \delta)\} \sup_{|\xi| > \frac{\delta}{2}} |K_\lambda(\xi)| \left\{ \|f\|_{L_{1,w}\langle a,b\rangle} + \left| \frac{f(x_0)}{w(x_0)} \right| (b-a) \right\}.$$

In view of condition (b) of class A_w, $I_1 + I_4 \to 0$ as $\lambda \to \lambda_0$.

Now, we consider the integral I_2. By the definition of $\mu-$generalized Lebesgue point, for every $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that the expression:

$$\int_{x_0-h}^{x_0} \left| \frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)} \right| dt < \varepsilon\mu(h) \tag{5}$$

holds for all $0 < h \leq \delta < \delta_0$.

Define the function $F(t)$ by

$$F(t) = \int_{t}^{x_0} \left| \frac{f(y)}{w(y)} - \frac{f(x_0)}{w(x_0)} \right| dy. \tag{6}$$

From (6), we have

$$dF(t) = -\left| \frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)} \right| dt. \tag{7}$$

From (5) and (6), for all t satisfying $0 < x_0 - t \leq \delta < \delta_0$ we have

$$F(t) \leq \varepsilon\mu(x_0 - t). \tag{8}$$

By virtue of (6) and (7) we have

$$\begin{aligned} I_2 &= \int_{x_0-\delta}^{x_0} \left| \frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)} \right| |K_\lambda(t-x)| w(t) dt \\ &= \int_{x_0-\delta}^{x_0} |K_\lambda(t-x)| w(t) d[-F(t)]. \end{aligned}$$

Using integration by parts and applying (8), we have the following inequality:

$$|I_2| \leq \varepsilon\mu(\delta) |K_\lambda(x_0 - \delta - x)| w(x_0 - \delta) + \varepsilon \int_{x_0-\delta}^{x_0} \mu(x_0 - t) |d|K_\lambda(t-x)| w(t)|.$$

It is easy to see that

$$|I_2| \leq \varepsilon\mu(\delta) |K_\lambda(x_0 - \delta - x)| w(x_0 - \delta)$$

$$+\varepsilon \int\limits_{x_0-x-\delta}^{x_0-x} \mu\left(x_0-x-t\right) \left| d\left[\bigvee_{x_0-x-\delta}^{t} \left|K_\lambda\left(u\right)\right| w\left(u+x\right) \right] \right|.$$

If we use hypothesis (4) and integration by parts, then we have the following expression:

$$\left|I_2\right| \leq -\varepsilon \int\limits_{x_0-x-\delta}^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{t} \left|K_\lambda\left(u\right)\right| w\left(u+x\right) \right] \left\{\mu\left(x_0-x-t\right)\right\}_t' dt.$$

Using condition (d) of class A_w, we obtain

$$\left|I_2\right| \leq \varepsilon \int\limits_{x_0-\delta}^{x_0} \left|K_\lambda\left(t-x\right)\right| w\left(t\right) \left|\left\{\mu\left(x_0-t\right)\right\}_t'\right| dt + 2\left|K_\lambda\left(0\right)\right| w\left(x\right) \mu\left(x_0-x\right).$$

Using preceding method, we can estimate the integral I_3 as

$$\left|I_3\right| \leq \varepsilon \int\limits_{x_0}^{x_0+\delta} \left|K_\lambda\left(t-x\right)\right| w\left(t\right) \left|\left\{\mu\left(t-x_0\right)\right\}_t'\right| dt.$$

Combining $\left|I_2\right|$ and $\left|I_3\right|$, we obtain

$$\left|I_2\right| + \left|I_3\right| \leq \varepsilon \int\limits_{x_0-\delta}^{x_0+\delta} \left|K_\lambda\left(t-x\right)\right| w\left(t\right) \left|\left\{\mu\left(\left|t-x_0\right|\right)\right\}_t'\right| dt + 2\varepsilon\left|K_\lambda\left(0\right)\right| w\left(x\right) \mu\left(\left|x_0-x\right|\right).$$

Note that the above inequality is obtained for the case $0 < x - x_0 < \frac{\delta}{2}$. Therefore, if the points $(x,\lambda) \in Z$ are sufficiently near to (x_0,λ_0), we have $\left|I_2\right| + \left|I_3\right|$ tends to zero. Thus, the proof is completed.

In the next theorem we suppose that $\langle a,b\rangle = \mathbb{R}$.

Theorem 3. *Suppose that $w(t)$ and $|K_\lambda(t-x)|$ are almost everywhere differentiable functions on $\mathbb{R}$ with respect to variable t such that the following inequality:*

$$\frac{d}{dt}w(t)\frac{d}{dt}\left|K_\lambda(t-x)\right| > 0, \text{ for any fixed } x \in \mathbb{R}. \tag{4}$$

holds. If $x_0 \in \mathbb{R}$ is a common $\mu-$generalized Lebesgue point of functions $f \in L_{1,w}(\mathbb{R})$ and $w \in L_1(\mathbb{R})$, then

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} L_\lambda(f;x) = f(x_0)$$

on any planar set Z on which the function

$$\int\limits_{x_0-\delta}^{x_0+\delta} \left|K_\lambda(t-x)\right| w(t) \left|\left\{\mu(\left|x_0-t\right|)\right\}_t'\right| dt + 2\left|K_\lambda\left(0\right)\right| w\left(x\right) \mu\left(\left|x_0-x\right|\right), \ 0<\delta<\delta_0,$$

where δ_0 is a fixed positive real number, is bounded as (x,λ) tends to (x_0,λ_0).

Proof. Making similar calculations as in Theorem 2 we have

$$\begin{aligned} |L_\lambda(f;x) - f(x_0)| \quad \leq \quad & \sup_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)| \{w(x_0-\delta) + w(x_0+\delta)\} \|f\|_{L_{1,w}(\mathbb{R})} \\ & + \{w(x_0-\delta) + w(x_0+\delta)\} \left|\frac{f(x_0)}{w(x_0)}\right| \int\limits_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)| \, d\xi \end{aligned}$$

$$+\varepsilon \int\limits_{x_0-\delta}^{x_0+\delta} |K_\lambda(t-x)| \, w(t) \left|\{\mu(|x_0-t|)\}_t'\right| dt + 2\varepsilon \, |K_\lambda(0)| \, w(x) \, \mu(|x_0-x|)$$

$$+\left|\frac{f(x_0)}{w(x_0)}\right| \left| \int\limits_{-\infty}^{\infty} w(t) \, K_\lambda(t-x) \, dt - w(x_0) \right|.$$

Using conditions (b) and (c) of class A_w and Theorem 3 in [12] we obtain $|L_\lambda(f;x) - f(x_0)| \to 0$ as $(x,\lambda) \to (x_0,\lambda_0)$. Hence the proof is completed.

Example 2. *The following Gauss-Weierstrass type kernel and weight functions satisfy the hypothesis of Theorem 3, respectively. Let* $\Lambda = (0,\infty)$, $\lambda_0 = 0$ *and* $K_\lambda : \mathbb{R} \to \mathbb{R}^+$ *for each* $\lambda \in \Lambda$ *is given by*

$$K_\lambda(t) = \frac{1}{\lambda\sqrt{\pi}} e^{-\frac{t^2}{\lambda^2}}$$

and $w : \mathbb{R} \to \mathbb{R}^+$ *is given by*

$$w(t) = \begin{cases} 1, & \text{if } t = 0 \\ \frac{1}{\sqrt{|t|(1+|t|)}}, & \text{otherwise.} \end{cases}$$

For detailed analysis of the above functions, authors refer to [1].

5. Rate of Convergence

In this section, two theorems concerning rate of pointwise convergence will be given.

Theorem 4. *Suppose that the hypothesis of Theorem 2 is satisfied.*
Let

$$\Delta(x,\lambda,\delta) = \int\limits_{x_0-\delta}^{x_0+\delta} |K_\lambda(t-x)| \, w(t) \left|\{\mu(|x_0-t|)\}_t'\right| dt + 2\,|K_\lambda(0)| \, w(x) \, \mu(|x_0-x|),$$

where $0 < \delta \leq \delta_0$, *and the following conditions are satisfied:*

i. $\Delta(x,\lambda,\delta) \to 0$ *as* $(x,\lambda) \to (x_0,\lambda_0)$ *for some* $\delta > 0$.

ii. *For every* $\xi > 0$

$$|K_\lambda(\xi)| = o(\Delta(x, \lambda, \delta))$$

as $(x, \lambda) \to (x_0, \lambda_0)$.

iii.

$$\left| \int_a^b w(t) K_\lambda(t - x)\, dt - w(x_0) \right| = o(\Delta(x, \lambda, \delta))$$

as $(x, \lambda) \to (x_0, \lambda_0)$.

Then at each common $\mu-$*generalized Lebesgue point of functions* $f \in L_{1,w}\langle a, b\rangle$ *and* $w \in L_1\langle a, b\rangle$ *we have as* $(x, \lambda) \to (x_0, \lambda_0)$

$$|L_\lambda(f; x) - f(x_0)| = o(\Delta(x, \lambda, \delta)).$$

Proof. Under the hypothesis of Theorem 2, we may write

$$\begin{aligned} |L_\lambda(f; x) - f(x)| \;&\le\; \{w(x_0 - \delta) + w(x_0 + \delta)\} \\ &\quad\times \sup_{|\xi| > \frac{\delta}{2}} |K_\lambda(\xi)| \left\{ \|f\|_{L_{1,w}\langle a,b\rangle} + \left| \frac{f(x_0)}{w(x_0)} \right| (b - a) \right\} \end{aligned}$$

$$+\varepsilon \int_{x_0-\delta}^{x_0+\delta} |K_\lambda(t - x)|\, w(t) \left| \{\mu(|x_0 - t|)\}_t' \right| dt + 2\varepsilon\, |K_\lambda(0)|\, w(x)\, \mu(|x_0 - x|)$$
$$+ \left| \frac{f(x_0)}{w(x_0)} \right| \left| \int_a^b w(t) K_\lambda(t - x)\, dt - w(x_0) \right|.$$

From (i)-(iii) we have the desired result i.e.:

$$|L_\lambda(f; x) - f(x_0)| = o(\Delta(x, \lambda, \delta)).$$

Theorem 5. *Suppose that the hypothesis of Theorem 3 is satisfied.*

Let

$$\Delta(x, \lambda, \delta) = \int_{x_0-\delta}^{x_0+\delta} |K_\lambda(t - x)|\, w(t) \left| \{\mu(|x_0 - t|)\}_t' \right| dt + 2\, |K_\lambda(0)|\, w(x)\, \mu(|x_0 - x|)$$

where $0 < \delta \le \delta_0$, *and the following conditions are satisfied:*

i. $\Delta(x, y, \lambda, \delta) \to 0$ *as* $(x, \lambda) \to (x_0, \lambda_0)$ *for some* $\delta > 0$.

ii. *For every* $\xi > 0$

$$|K_\lambda(\xi)| = o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

iii. *For every* $\xi > 0$

$$\lim_{\lambda\to\lambda_0}\left[\int_{|t|>\xi} |K_\lambda(t)|\, dt\right] = o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

iv.

$$\left|\int_{-\infty}^{\infty} w(t)\, K_\lambda(t-x)\, dt - w(x_0)\right| = o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

Then at each common $\mu-$*generalized Lebesgue point of functions* $f \in L_{1,w}(\mathbb{R})$ *and* $w \in L_1(\mathbb{R})$ *we have as* $(x,\lambda) \to (x_0,\lambda_0)$

$$|L_\lambda(f;x) - f(x_0)| = o(\Delta(x,\lambda,\delta)).$$

Proof. Under the hypothesis of Theorem 3, we write

$$\begin{aligned}|L_\lambda(f;x) - f(x_0)| \quad &\le \quad \sup_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)|\{w(x_0-\delta)+w(x_0+\delta)\}\,\|f\|_{L_{1,w}(\mathbb{R})}\\ &\quad + \{w(x_0-\delta)+w(x_0+\delta)\}\left|\frac{f(x_0)}{w(x_0)}\right| \int_{|\xi|>\frac{\delta}{2}} |K_\lambda(\xi)|\, d\xi\end{aligned}$$

$$\begin{aligned}&+ \quad \varepsilon \int_{x_0-\delta}^{x_0+\delta} |K_\lambda(t-x)|\, w(t) \left|\{\mu(|x_0-t|)\}_t'\right| dt + 2\varepsilon\, |K_\lambda(0)|\, w(x)\, \mu(|x_0-x|)\\ &+ \left|\frac{f(x_0)}{w(x_0)}\right| \left|\int_{-\infty}^{\infty} w(t)\, K_\lambda(t-x)\, dt - w(x_0)\right|\end{aligned}$$

and from (i)-(iv) we have the desired result i.e.:

$$|L_\lambda(f;x) - f(x_0)| = o(\Delta(x,\lambda,\delta)).$$

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