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Analytical Approximate Solution of Fractional Wave Equation by the Optimal Homotopy Analysis Method

A. Elsaid[1,*], S. Shamseldeen[1], S. Madkour[1]

[1] *Mathematics and Engineering Physics Department, Faculty of Engineering Mansoura University, Egypt*

Abstract. In this article, we study the space-fractional wave equation with Riesz fractional derivative. The continuation of the solution of this space-fractional equation to the solution of the corresponding integer order equation is proved. The series solution is obtained based on properties of Riesz fractional derivative operator and utilizing the optimal homotopy analysis method (OHAM). Numerical simulations are presented to validate the method and to show the effect of changing the fractional derivative parameter on the solution behavior.

2010 Mathematics Subject Classifications: 35L05, 26A33, 35C10

Key Words and Phrases: Space-fractional wave equation, Riesz, Optimal homotopy analysis method

1. Introduction

Fractional derivatives, as generalizations of classical integer order derivatives, are increasingly used to model numerous problems in different fields of applied science. In recent years, the fractional derivative models are developed to describe the dissipative attenuation in complex materials, such as anomalous diffusion [12] and [15], viscoelastic damping [1] and [11], and wave propagation [4] and [5]. The operators of fractional differentiation and integration are also used for extensions of the diffusion and wave equations [13] and [14]. Studies have been devoted for a type of anomalous diffusion modeled by the fractional diffusion equation with spatial Riesz and Riesz-Feller fractional derivatives [6] and [8].

Yet, few articles dealt with applying iterative techniques to Riesz fractional partial differential equations (FPDEs). This is due to the difficulty in repeated application of Riesz fractional derivative to solution components. This work is based on properties that show repetitive behavior for complex exponential function, hence sine and cosine functions, when subjected to the application of Riesz fractional derivative [6] and [7].

In this work, the motivation is to establish the continuation of the solution of the space-fractional wave equation with spatial derivative in Riesz sense to the exact solution of the

*Corresponding author.

corresponding integer-order equation as the order of the fractional derivative approaches its integer limit. This objective is carried out theoretically then via approximate series solution obtained iteratively by applying the optimal homotopy analysis method (OHAM). We consider the space-fractional wave equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) = R_x^\alpha u(x,t) + P(u), \quad -\infty < x < \infty \ , \ t > 0, \tag{1}$$

subject to the initial conditions

$$\begin{cases} u(x,0) = f_1(x), \\ \frac{\partial}{\partial t}u(x,0) = f_2(x). \end{cases} \tag{2}$$

where R_x^α denotes the Riesz fractional derivative (in space) of order α. The parameter α is restricted to the conditions $0 < \alpha < 2$ and $\alpha \neq 1$. The function P is a continuous function in u, and the two functions f_1 and f_2 are functions in the space of integrable functions $L^1(-\infty, \infty)$.

This paper is organized as follows. In Section two, basic definitions of fractional derivative operators involved are presented. Proof of continuation of solution is presented in Section three. The OHAM is illustrated in Section four. In Section five, the results of numerical experiments are presented, considering the space fractional sine-Gordan equation. Section six contains the conclusion of this work.

2. Fractional derivatives and integrals

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined as*

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, \ x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{3}$$

Definition 3. *The fractional derivative in Riemann-Liouville sense of $f(x), m \in \mathbb{N}$, $x > 0$ is defined as*

$$\mathbf{D}_x^\beta f(t) = \frac{d^m}{dx^m} J^{m-\beta} f(x), \ m-1 < \beta < m. \tag{4}$$

Definition 4. *The fractional derivative in Caputo sense of $f(x) \in C_{-1}^m$, $m \in \mathbb{N}$, $x > 0$ is defined as*

$$^C D_x^\beta f(x) = \begin{cases} J^{m-\beta} \frac{d^m}{dx^m} f(x), & m-1 < \beta < m, \\ \frac{d^m}{dx^m} f(x), & \beta = m. \end{cases} \tag{5}$$

Definition 5. *The Riesz partial fractional derivative* R_x^α *is defined as [8]*

$$R_x^\alpha u(x) = -\frac{1}{2\cos(\alpha\pi/2)}[D_+^\alpha u(x) + D_-^\alpha u(x)], \qquad 0<\alpha<2,\ \alpha\neq 1 \tag{6}$$

where $D_\pm^\alpha u(x)$ *are the Weyl fractional derivatives*

$$D_\pm^\alpha u(x) = \begin{cases} \pm\frac{d}{dx}W_\pm^{1-\alpha}u(x), 0<\alpha<1 \\ \frac{d^2}{dx^2}W_\pm^{2-\alpha}u(x), 1<\alpha<2 \end{cases}, \tag{7}$$

and $W_\pm^\beta$ *denote the Weyl fractional integrals of order* $\beta>0$*, given by*

$$\begin{aligned} W_+^\beta u(x) &= \tfrac{1}{\Gamma(\beta)}\int\limits_{-\infty}^{x}(x-z)^{\beta-1}u(z)dz, \\ W_-^\beta u(x) &= \tfrac{1}{\Gamma(\beta)}\int\limits_{x}^{\infty}(z-x)^{\beta-1}u(z)dz. \end{aligned} \tag{8}$$

When $\alpha=0$ the Weyl fractional derivative degenerates into the identity operator

$$D_\pm^0 u(x) = u(x). \tag{9}$$

For continuity we have

$$D_\pm^1 u(x) = \pm\frac{d}{dx}u(x),\ D_\pm^2 u(x) = \frac{d^2}{dx^2}u(x). \tag{10}$$

Evidently, in case $\alpha=2$, we define

$$R_x^\alpha u(x) = \frac{d^2}{dx^2}u(x). \tag{11}$$

For the case $\alpha=1$ we have

$$R_x^1 u(x) = \frac{d}{dx}Hu(x) \tag{12}$$

$$= \frac{d}{dx}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{u(z)}{z-x}dz, \tag{13}$$

where H is the Hilbert transform and the integral is understood in the Cauchy principal value sense.

3. Continuation of the solution

In this section, we prove the continuation of the solution to fractional-order wave equation with Riesz spatial derivative to the solution of the corresponding integer-order equation.

Theorem 1. *If $f_1(x)$ and $f_2(x)$ are functions in the space of integrable functions $L^1(-\infty,\infty)$, then the exact solution $u_\alpha(x,t)$ of the space fractional wave equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) = R_x^\alpha u(x,t), \quad -\infty < x < \infty \ t > 0, \tag{14}$$

with the initial conditions

$$\begin{cases} u(x,0) = f_1(x), \\ \frac{\partial}{\partial t}u(x,0) = f_2(x). \end{cases} \tag{15}$$

is given by

$$u_\alpha(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\big(E_{2,1}(-\omega^\alpha\ t^2)f_1(v) + t\ E_{2,2}(-\omega^\alpha\ t^2)f_2(v)\big)\cos(\omega(x-v))d\omega dv \tag{16}$$

where $E_{\eta,\gamma}(z)$ is the Mittage Leffler function defined by [16]

$$E_{\eta,\gamma}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\eta n+\gamma)}, \tag{17}$$

where

$$E_{2,1}(-\omega^\alpha\ t^2) = \cos(\omega^{\alpha/2}\ t), \tag{18}$$

$$E_{2,2}(-\omega^\alpha\ t^2) = \frac{\sin(\omega^{\alpha/2}\ t)}{\omega^{\alpha/2}t}. \tag{19}$$

Theorem 2. *Let $\alpha \in (1,2)$, $f_1(x)$ and $f_2(x)$ are functions in the space of integrable functions $L^1(-\infty,\infty)$, and u_α displayed in (16) be the solution of the space-fractional problem (14-15) , then*

$$\lim_{\alpha\to 2} u_\alpha(x,t) = u(x,t),$$

where $u(x,t)$ is the exact solution of the integer-order wave equation

$$\begin{cases} u_{tt}(x,t) = \ u_{xx}(x,t), \ -\infty < x < \infty, \ t > 0, \\ u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x). \end{cases} \tag{20}$$

Proof. Consider the set of functions $\varphi_n(\omega)$ and $\psi_n(\omega)$ for $\omega \in (0,\infty)$, $n \in \mathbb{N}^+$ by

$$\varphi_n(\omega) = \frac{1}{\pi}E_{2,1}(-\omega^{2-\frac{1}{n+1}}t^2)\int_{-\infty}^{\infty} f_1(v)\cos(\omega(x-v))dv, \tag{21}$$

$$\psi_n(\omega) = \frac{1}{\pi} t E_{2,2}(-\omega^{2-\frac{1}{n+1}} t^2) \int\limits_{-\infty}^{\infty} f_2(v) \cos(\omega(x-v)) dv. \tag{22}$$

These two set of functions satisfy Lebesgue dominated convergence theorem as

$$\begin{aligned} |\varphi_n(\omega)| &\leq \frac{1}{\pi} \left| E_{2,1}(-\omega^{2-\frac{1}{n+1}} t^2) \right| \int\limits_{-\infty}^{\infty} |f_1(v)| \, |\cos(\omega(x-v))| \, dv, \\ &\leq \frac{1}{\pi} \left| E_{2,1}(-\omega^{2-\frac{1}{n+1}} t^2) \right| \int\limits_{-\infty}^{\infty} |f_1(v)| \, dv, \end{aligned}$$

and since $f_1 \in L^1(-\infty, \infty)$, there exists $M > 0$ such that $\int\limits_{-\infty}^{\infty} |f_1(v)| \, dv < M$. Hence

$$|\varphi_n(\omega)| \leq \frac{M}{\pi} \left| E_{2,1}(-\omega^{2-\frac{1}{n+1}} t^2) \right|. \tag{23}$$

From [16] Theorem (1.6), there exits $K_1 > 0$ such that

$$|E_{\eta,\gamma}(-z)| \leq \frac{K_1}{1+|z|}, \tag{24}$$

then

$$|\varphi_n(\omega)| \leq \frac{MK_1}{\pi} \frac{1}{1 + \left| \omega^{2-\frac{1}{n+1}} t^2 \right|}, \quad \omega \in (0, \infty), \quad n = 1, 2, ... \tag{25}$$

For bounded time interval $0 < t < T < \infty$, there exists $K_2(\rho) > 0$ such that

$$|\varphi_n(\omega)| \leq g_1(\omega) = \frac{K_2(\rho)}{1 + \omega^{1+\rho}}, \qquad \rho \in (0, 0.5),$$

and $g_1(\omega) \in L^1(0, \infty)$ since

$$\int\limits_{0}^{\infty} |g_1(\omega)| \, d\omega = K_2(\rho) \Gamma(\frac{\rho}{1+\rho}) \Gamma(1 + \frac{1}{1+\rho}). \tag{26}$$

Thus the set of functions $\varphi_n(\omega)$ satisfy Lebesgue dominated convergence theorem. Following the same steps, one can prove that the set of functions $\psi_n(\omega)$ satisfy Lebesgue dominated convergence theorem as well. Now, as

$$\lim_{n \to \infty} \varphi_n(\omega) = \frac{1}{\pi} E_{2,1}(-\omega^2 t^2) \int\limits_{-\infty}^{\infty} f_1(v) \cos(\omega(x-v)) dv, \tag{27}$$

$$\lim_{n\to\infty} \psi_n(\omega) = \frac{1}{\pi} t E_{2,2}(-\omega^2 t^2) \int_{-\infty}^{\infty} f_2(v)\cos(\omega(x-v))dv. \tag{28}$$

then setting $\alpha = 2 - \frac{1}{n+1}$

$$\begin{aligned} u_2(x,t) &= \lim_{\alpha\to 2} u_\alpha(x,t) \\ &= \lim_{n\to\infty} \int_0^{\infty} [\varphi_n(\omega) + \psi_n(\omega)]d\omega \quad (29) \\ &= \int_0^{\infty} \lim_{n\to\infty} [\varphi_n(\omega) + \psi_n(\omega)]d\omega, \quad (30) \end{aligned}$$

which yields

$$u_2(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\cos(\omega\ t) q_1(v) + \frac{\sin(\omega\ t)}{\omega} q_2(v) \right) \cos(\omega(x-v)) d\omega dv,$$

which is the exact solution of the integer-order wave equation (20).

4. Optimal homotopy analysis method (OHAM)

We begin by illustrating the classical homotopy analysis method (HAM). Consider the following nonlinear equation

$$N[u(x,t)] = 0, \tag{31}$$

where N is a nonlinear operator, $u(x,t)$ is the unknown function and x and t denote spatial and temporal independent variables, respectively. By generalizing the traditional homotopy method, Liao [9] constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar H(x,t) N[\phi(x,t;p)], \tag{32}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $H(x,t)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, we have $\phi(x,t;0) = u_0(x,t)$, $\phi(x,t;1) = u(x,t)$, respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. By expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) p^m, \tag{33}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\,|_{p=0}\,. \tag{34}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved by Liao [9], series (33) converges at $p = 1$ and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{35}$$

which must be one of solutions of the original nonlinear equation, as proved by Liao [9]. Using definition (34), the governing equation of the HAM can be deduced from the zero-order deformation equation (32) as follows. Define the vector

$$\overrightarrow{u}_n = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\} \tag{36}$$

From equation (32), the so-called m th-order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{37}$$

where

$$\Re_m[\overrightarrow{u}_{m-1}] = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}\,|_{p=0}, \tag{38}$$

and

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases} \tag{39}$$

Applying the inverse operator L^{-1} to both sides of (37), $u_m(x,t)$ can be easily solved for by symbolic computations software. The HAM has been successfully applied to solve various classes of equations and applied problems [3]-[2].

In the classical HAM, choosing the value of parameter $\hbar$ depends on inspecting the graph of the quantity of interest; the solution or one of its derivatives. Yet, when $H(x,t)$ is fixed, it is obvious that $u_m(x,t)$ contains only one control parameter $\hbar$. Thus, by constructing a formula for the residual error, the OHAM solution is obtained by choosing the value for parameter $\hbar$ that minimizes the error. Here, the averaged residual error defined for ordinary differential equations in [10] is generalized to the case of two variable partial differential equations in the following form

$$E_m(\hbar) = \frac{1}{MK}\sum_{i=0}^{M}\sum_{j=0}^{K}\left[N\sum_{n=0}^{m} u_n\left(\frac{i}{M},\frac{j}{K}\right)\right]^2, \tag{40}$$

which is a nonlinear algebraic equation of one unknown; the convergence-control parameter $\hbar$. Thus the optimal value of $\hbar$ is determined by the minimum of the averaged residual error E_m to ensure the fast convergence of the homotopy series.

To apply the OHAM recursive technique to the problem, a repeated evaluation of Riesz fractional derivative to solution components is needed. This obstacle is overcome by using property of Riesz fractional derivative in the following lemma.

Lemma 3. *Let $\alpha \in (0,2)$, $\alpha \neq 1$. Then*

$$R_x^\alpha(e^{i\omega x}) = -\omega^\alpha e^{i(\omega x)}, \tag{41}$$

or in a trigonometric form

$$R_x^\alpha \sin(\omega x) = -\omega^\alpha \sin(\omega x), \tag{42}$$

$$R_x^\alpha \cos(\omega x) = -\omega^\alpha \cos(\omega x). \tag{43}$$

Proof. See [6] and [7].

5. Numerical simulation

In this section, we consider linear and nonlinear problems to illustrate the efficiency of the method of solution to this type of problems and to illustrate the continuation of the solution we proved in Section 3. In each problem, a table is presented to show the estimated values the optimal convergence control parameter $\hbar$ and the corresponding residual error E_m at different values of the fractional derivative α. These estimated values are calculated via minimizing of the averaged residual error E_m displayed in (40) in the space domain $0 \leq x \leq 2.0$ and the time interval $0 \leq t \leq 2.0$.

Example 1. *Consider problem (1-2) with $p(u) = u$, $f_1(x) = \sin(\pi x/a)$ and $f_2(x) = -\sin(\pi x/a)$*

$$\begin{cases} u_{tt}(x,t) = R_x^\alpha u(x,t) + u, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = \sin(\pi x/a), \quad u_t(x,0) = \sin(\pi x/a), & \end{cases} \tag{44}$$

where a is a real constant.

The auxiliary linear operator is chosen as

$$L[\phi] = \frac{\partial^2}{\partial t^2}(\phi), \tag{45}$$

and the nonlinear operator N is chosen as

$$N[\phi] = \phi_{tt} - R_x^\alpha(\phi) - \phi. \tag{46}$$

The m th-order deformation equation, with $H(x,t) = 1$, for this linear problem is given by

$$\frac{\partial^2}{\partial t^2}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\left(\frac{\partial^2}{\partial t^2}(u_{m-1}) - R_x^\alpha(u_{m-1}) - u_{m-1}\right), \tag{47}$$

with

$$u_0(x,t) = f_1(x) + t\ f_2(x) = (1-t)\sin(\pi x/a) \tag{48}$$

The inverse integral operator is applied to both sides of equation (47) to obtain the series solution terms. The first three terms are given by

$$\begin{aligned}
u_0 &= (1-t)\sin(\frac{\pi x}{a}),\\
u_1 &= -\frac{ht^2}{6}\left(-1+\left(\frac{\pi}{a}\right)^{\alpha}\right)(-3+t)\sin\left(\frac{\pi x}{a}\right),\\
u_2 &= -\frac{ht^2}{120}\left(-1+\left(\frac{\pi}{a}\right)^{\alpha}\right)\\
&\quad \left(20(-3+t)+h\left(-60+20t+\left(5-5\left(\frac{\pi}{a}\right)^{\alpha}\right)t^2+\left(-1+\left(\frac{\pi}{a}\right)^{\alpha}\right)t^3\right)\right)\sin\left(\frac{\pi x}{a}\right).
\end{aligned}$$

Table 1 shows the estimated values of the optimal convergence control parameter $\hbar$ and the corresponding residual error E_m for the linear problem displayed in (44) at different values of the fractional derivative α in the space domain $0 \le x \le 2.0$ and the time interval $0 \le t \le 2.0$.

Table 1: The estimated optimal convergence parameter $\hbar$ and the corresponding residual error E_m for $0 \le x \le 2.0$ and $0 \le t \le 2.0$ at different fractional derivative α for Example (1).

α	$\hbar$ Optimal parameter	E_m Residual Error
1.7	-0.940496	$1.12317E-5$
1.8	-0.938046	$8.19812E-5$
1.9	-0.933025	$1.75198E-4$
2.0	-0.928713	$3.51187E-4$

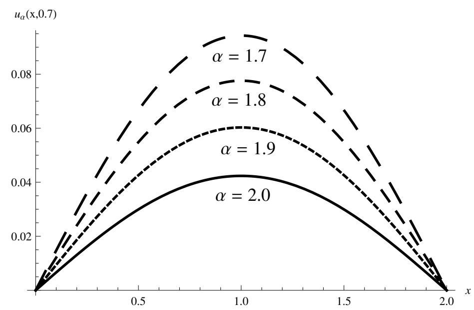

Figure 1: The solution of (44) at $t = 0.5$, $0 \le x \le 2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

The series solution is obtained by $u = u_0 + u_1 + u_2 + u_3 + ...$ Figures (1) and (2) show the effect of the fractional order derivative α on the behavior of the solution at fixed time

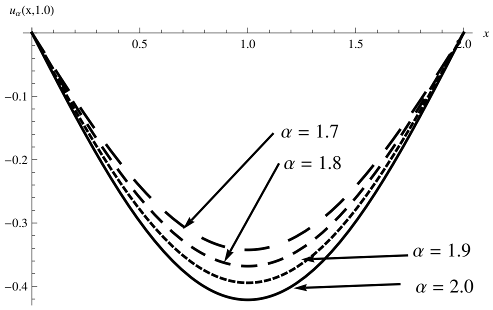

Figure 2: The solution of (44) at $t = 1.0,\ 0 \leq x \leq 2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

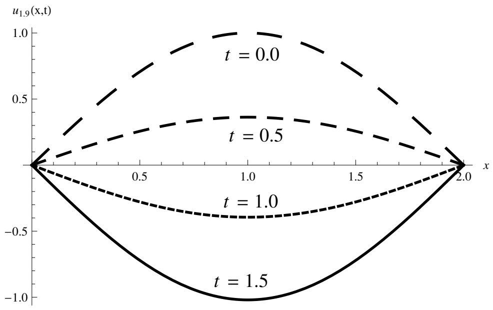

Figure 3: The solution of (44) at different times $t = 0.0, 0.5, 1.0$, and 1.5, $0 \leq x \leq 2$ and the fractional order $\alpha = 1.9$.

$t = 0.5$ and $t = 1.0$, respectively, while Figure (3) illustrates the temporal behavior of the solution at a fixed fractional order, $\alpha = 1.9$. The plots represent the sum of the first four terms (u_0 to u_3) in the OHAM series when $a = 2.0$

Example 2. *Consider problem (1-2) with* $P(u) = u + c\ u^3$, $f_1(x) = \sin(\pi x/a)$ *and* $f_2(x) = -\sin(\pi x/a)$

$$\begin{cases} u_{tt}(x,t) = R_x^\alpha u(x,t) + u + c\, u^3, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = \sin(\pi x/a), \quad u_t(x,0) = \sin(\pi x/a), & \end{cases} \tag{49}$$

where a is a constant.

The auxiliary linear operator is chosen as

$$L[\phi] = \frac{\partial^2}{\partial t^2}(\phi), \tag{50}$$

and the nonlinear operator N is chosen as

$$N[\phi] = \phi_{tt} - R_x^\alpha(\phi) - \phi - c\ \phi^3. \tag{51}$$

Then, m th-order deformation equation for this problem is given by

$$\frac{\partial^2}{\partial t^2}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{52}$$

where $\Re_m[\overrightarrow{u}_{m-1}(x,t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(x,t)] = \frac{\partial^2}{\partial t^2}(u_{m-1}) - R_x^{\alpha}(u_{m-1}) - u_{m-1} - c\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_{m-1-i}u_j u_{i-j}. \tag{53}$$

We choose $H(x,t) = 1$ and

$$u_0(x,t) = f_1(x) + t\ f_2(x) = (1-t)\sin(\pi x/a). \tag{54}$$

By applying the inverse integral operator to both sides of equation (52), we obtain

$$\begin{aligned} u_0 &= (1-t)\sin(\pi x/a) \\ u_1 &= -\frac{ht^2}{120}\left(20\left(-1+\left(\frac{\pi}{a}\right)^{\alpha}\right)(-3+t)\right)\sin\left(\frac{\pi x}{a}\right) \\ &\quad -\frac{ht^2}{120}\left(-3c^2\left(-10+10t-5t^2+t^3\right)\left[1-\cos\left(\frac{2\pi x}{a}\right)\right]\right)\sin\left(\frac{\pi x}{a}\right) \\ &\quad \vdots \end{aligned}$$

Table 2 shows the estimated values of the optimal convergence control parameter $\hbar$ and the corresponding residual error E_m for problem (49) at different values of the fractional derivative α in the space domain $0 \le x \le 2.0$ and the time interval $0 \le t \le 2.0$.

Table 2: The estimated optimal convergence parameter $\hbar$ and the corresponding residual error E_m for $0 \le x \le 2.0$ and $0 \le t \le 2.0$ at different fractional derivative α for Example (2).

α	$\hbar$ Optimal parameter	E_m Residual Error
1.7	-0.620896	$1.66374E-3$
1.8	-0.687193	$3.59934E-3$
1.9	-0.740338	$5.55095E-3$
2.0	-0.736543	$7.51805E-3$

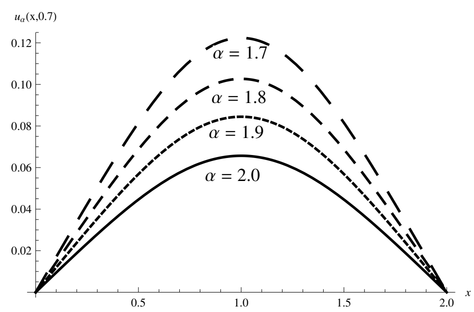

Figure 4: The solution of (49) at $t = 0.5$, $0 \le x \le 2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

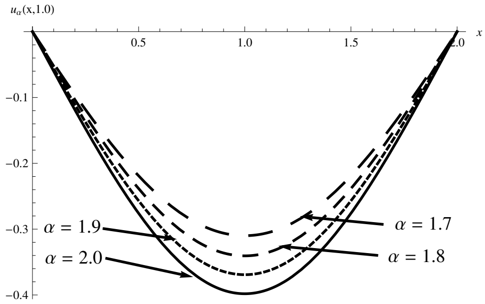

Figure 5: The solution of (49) at $t = 1.0,\ 0 \leq x \leq 2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

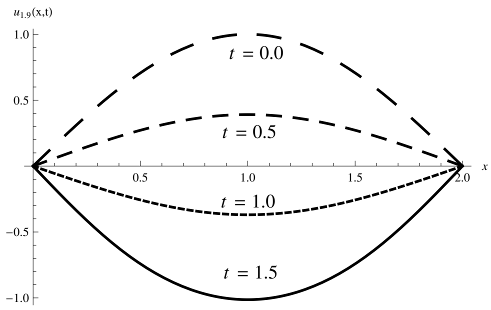

Figure 6: The solution of (49) at different times $t = 0.0,\ 0.5,\ 1.0,$ and $1.5,\ 0 \leq x \leq 2$ and the fractional order $\alpha =\ 1.9$.

and the solution is thus obtained as

$$u = u_0 + u_1 + u_2 + u_3 + ...$$

The solution behavior as the Riesz parameter α changes is shown in Figures (4) and (5) at a fixed time $t = 0.5$ and $t = 1.0$, respectively. As α increases, the amplitude of the sinusoidal behavior in solution decreases. The series displayed in plots is the partial sum of the first four terms; $n = 3$ (summing u_0 to $u3$). Figure (6) shows the evolution with time of the solution at a fixed fractional order $\alpha = 1.9$ in the interval $0 \leq x \leq 2$.

Example 3. *Consider the problem (1-2) with* $p(u) = -\sin(u), f_1(x) = \ \pi + \varepsilon\cos(\mu x)$ *and* $f_2(x) = \ 0$ *(the space-fractional sine-Gordan equation), i.e.,*

$$\begin{cases} u_{tt}(x,t) = \ R_x^\alpha u(x,t) - \ \sin(u), & -\infty < x < \infty,\ t > 0, \\ u(x,0) = \pi + \varepsilon\cos(\mu x), \quad u_t(x,0) = 0, & \end{cases} \tag{55}$$

where ε and μ are real constants.

Here the auxiliary linear operator is

$$L[\phi] = \frac{\partial^2}{\partial t^2}(\phi), \tag{56}$$

and the nonlinear operator N is chosen as

$$N[\phi] = \phi_{tt} - R_x^\alpha(\phi) + \sin(\phi). \tag{57}$$

Then, m th-order deformation equation for this problem is given by

$$\frac{\partial^2}{\partial t^2}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{58}$$

where $\Re_m[\overrightarrow{u}_{m-1}(x,t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(x,t)] = \frac{\partial^2}{\partial t^2}(u_{m-1}) - R_x^\alpha(u_{m-1}) + \sum_{k=0}^{m-1} A_k, \tag{59}$$

where A_k is the Adomian polynomials for $\sin(u)$ [**?**]: $A_0 = \sin(u_0)$, $A_1 = u_1\cos(u_0)$, $A_2 = 1/2(-u_1^2\sin(u_0) + 2u_2\cos(u_0)), \ldots$. We choose $H(x,t) = 1$, and by applying the inverse integral operator to both sides of (58), one can obtain the first four terms as

$$\begin{aligned} u_0 &= \pi, \\ u_1 &= \varepsilon\cos(\mu x), \\ u_2 &= \frac{\epsilon}{2}\left(2 + h\left(2 + t^2\left(-1 + \mu^\alpha\right)\right)\right) cos(\mu x), \\ u_3 &= \frac{\epsilon}{24}\left(24 + 24h(2 + t^2(-1 + \mu^\alpha)) + h^2(24 + 24t^2(-1 + \mu^\alpha) + t^4(-1 + \mu^\alpha)^2)\right) cos(\mu x) \end{aligned}$$

and the solution is $u = u_0 + u_1 + u_2 + u_3 + \ldots..$

Table 3 shows the estimated values of the optimal convergence control parameter $\hbar$ and the corresponding residual error E_m for the problem displayed in (55) at different values of the fractional derivative α in the space domain $0 \le x \le 2.0$ and the time interval $0 \le t \le 2.0$.

Table 3: The estimated optimal convergence parameter $\hbar$ and the corresponding residual error E_m for $0 \le x \le 2.0$ and $0 \le t \le 2.0$ at different fractional derivative α for Example (3).

α	$\hbar$ Optimal parameter	E_m Residual Error
1.7	-0.928016	$1.42983E-4$
1.8	-0.926483	$3.41879E-4$
1.9	-0.925981	$6.10878E-4$
2.0	-0.923954	$9.76395E-4$

The behavior of the solution of the sine-Gordan equation (55) as the Riesz parameter α changes is shown in Figures (7) and (8) at a fixed time $t = 1.0$ and $t = 1.5$, respectively, while the temporal evolution of the solution is depicted in Figure (9) at a fixed fractional order $\alpha = 1.9$. As α increases, the amplitude of the sinusoidal behavior in solution decreases. The series displayed in the figures is the partial sum of the first four terms; $n = 3$ (summing u_0 to $u3$).

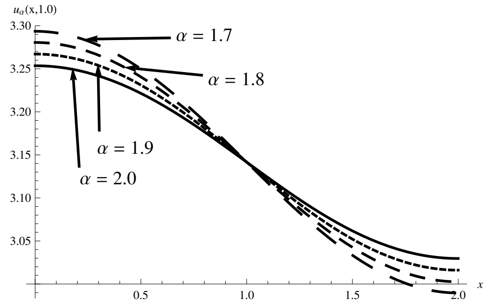

Figure 7: The solution of (55) at $\varepsilon = 0.3, \mu = \pi/2,\ t = 1.0,\ 0 \leq x \leq 0.2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

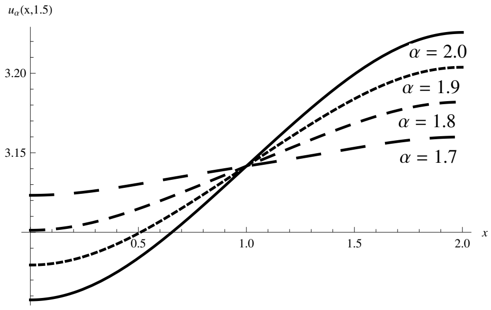

Figure 8: The solution of (55) at $\varepsilon = 0.3, \mu = \pi/2,\ t = 1.5,\ 0 \leq x \leq 0.2$ and different values of the fractional order $\alpha = 1.7,\ 1.8,\ 1.9$ and 2.0.

6. Conclusion

We present a study to the behavior of the solution to the space-fractional wave equation where the spatial derivative is given in Riesz sense. We proved the continuation of the solution of the considered fractional-order wave equation to the solution of the corresponding integer order problem. The iterative series solution for the fractional equation is obtained using the OHAM. The advantage of using this technique is the ability to estimate an approximation to the residual error. The results obtained illustrate graphically the continuation of the solution we proved theoretically.

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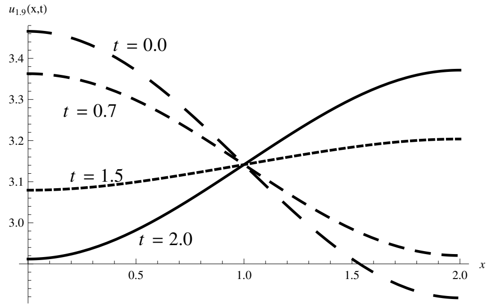

Figure 9: The solution of (55) when $\varepsilon = 0.3, \mu = \pi/2$,the fractional order $\alpha = \ 1.9$ and at different times $t = 0.0,\ 0.7,\ 1.5,$ and 2.0, $0 \le x \le 2$.

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