



On an inverse problem for Sturm-Liouville Equation

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Abstract. In this study, the theorem on necessary and sufficient conditions for the solvability of inverse problem for Sturm-Liouville operator with discontinuous coefficient is proved and the algorithm of reconstruction of potential from spectral data (eigenvalues and normalizing numbers) is given.

2010 Mathematics Subject Classifications: 34A55, 34B24

Key Words and Phrases: Sturm-Liouville operator, inverse problem, necessary and sufficient conditions

1. Introduction

We consider the boundary value problem

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \leq x \leq \pi, \quad (1)$$

$$y'(0) = 0, \quad y(\pi) = 0, \quad (2)$$

where $q(x) \in L_2(0, \pi)$ is a real-valued function, $\rho(x)$ is a piecewise continuous function, λ is a complex parameter. This spectral problem appears while solving wave or heat equations for nonhomogeneous density of the material [1], [2]. Physical applications of discontinuous Sturm-Liouville problem are given in [3]-[8].

For simplicity, we will assume that the density function has only one discontinuity point such that

$$\rho(x) = \begin{cases} 1, & 0 \leq x \leq a, \\ \alpha^2, & a < x \leq \pi, \end{cases} \quad (3)$$

where $0 < \alpha \neq 1$.

Direct problem of spectral analysis for Sturm-Liouville problem is investigated properties of eigenvalues and eigenfunctions, finding normalizing numbers, spectrum set of the boundary value problem, scattering data and some other values. It is important to investigate these properties. Inverse problem of spectral analysis is to find the coefficient of

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the equation for given spectral data. This has to be done uniquely, so that it gives the uniqueness of the inverse problem. In the process of the solution of the inverse problem giving an algorithm for constructing the potential is important. For $\rho(x) \equiv 1$, solutions of inverse problem for equation (1) is given by [9]-[16]. For $\rho(x) \neq 1$, under different boundary conditions similar problem is solved in [17]-[21]. When boundary conditions contain spectral parameter, it is solved by [22], [23].

The inverse problem for this equation is to find necessary and sufficient conditions for any data set to be spectral data. The main of this work is to find these conditions for (1), (2) boundary value problem. Firstly spectral data is defined. Characteristic properties of these values are investigated in [20] and also uniqueness of the solution of the inverse problem is proved.

Consequently, in this work for (1), (2) spectral problem, solution of the inverse problem is given with respect to the spectral data.

For (1), (2) boundary value problem in [20], it is shown that the real numbers $\{\lambda_n^2, \alpha_n\}_{n \geq 1}$ satisfy the following

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad \alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad \{k_n\}, \{t_n\} \in l_2, \tag{4}$$

where λ_n^0 are zeros of the function

$$\begin{aligned} \Delta_0(\lambda) &= \frac{1}{2} \left(1 + \frac{1}{\alpha}\right) \cos \lambda \mu^+(\pi) + \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \cos \lambda \mu^-(\pi), \\ d_n &= \frac{h^+ \sin \lambda_n^0 \mu^+(\pi) + h^- \sin \lambda_n^0 \mu^-(\pi)}{\frac{1}{2} \left(1 + \frac{1}{\alpha}\right) \mu^+(\pi) \sin \lambda_n^0 \mu^+(\pi) + \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \mu^-(\pi) \sin \lambda_n^0 \mu^-(\pi)} \end{aligned}$$

is a bounded sequence.

In [18] it is proved, that the solution $\varphi(x, \lambda)$ of the equation (1) with initial date $\varphi(0, \lambda) = 1, \varphi'(0, \lambda) = 0$ can be represented as

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^{\mu^+(x)} A(x, t) \cos \lambda t dt, \tag{5}$$

where $A(x, t)$ belongs to the space $L_2(0, \pi)$ for each fixed $x \in [0, \pi]$ and is related to the coefficient $q(x)$ of the equation (1) by the formula:

$$\frac{d}{dx} A(x, \mu^+(x)) = \frac{1}{4\sqrt{\rho(x)}} \left(1 + \frac{1}{\sqrt{\rho(x)}}\right) q(x), \tag{6}$$

$$\varphi_0(x, \lambda) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}}\right) \cos \lambda \mu^+(x) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}}\right) \cos \lambda \mu^-(x) \tag{7}$$

is the solution of (1) when $q(x) \equiv 0$,

$$\mu^\pm(x) = \pm x \sqrt{\rho(x)} + a \left(1 \mp \sqrt{\rho(x)}\right). \tag{8}$$

The characteristic function $\Delta(\lambda)$ of the problem (1), (2) is

$$\Delta(\lambda) := \langle \varphi(x, \lambda), \psi(x, \lambda) \rangle = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

where $\Delta(\lambda)$ is independent from $x \in [0, \pi]$. Substituting $x = 0$ and $x = \pi$ into above the equation, we get

$$\Delta(\lambda) = \varphi(\pi, \lambda) = \psi'(0, \lambda).$$

Theorem 1. For each fixed $x \in [0, \pi]$ the kernel $A(x, t)$ from the representation (5) satisfies the following linear functional integral equation

$$\begin{aligned} & \frac{2}{1 + \sqrt{\rho(t)}} A(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a - t)}}{1 + \sqrt{\rho(2a - t)}} A(x, 2a - t) + \\ & + F(x, t) + \int_0^{\mu^+(x)} A(x, \xi) F_0(\xi, t) d\xi = 0, \quad 0 < t < x \end{aligned} \tag{9}$$

where

$$F_0(x, t) = \sum_{n=1}^{\infty} \left(\frac{\varphi_0(t, \lambda_n) \cos \lambda_n x}{\alpha_n} - \frac{\varphi_0(t, \lambda_n^0) \cos \lambda_n^0 x}{\alpha_n^0} \right) \tag{10}$$

$$F(x, t) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) F_0(\mu^+(x), t) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) F_0(\mu^-(x), t) \tag{11}$$

$\{\lambda_n^0\}^2$ are eigenvalues and α_n^0 are norming constants of the boundary value problem (1), (2) when $q(x) \equiv 0$.

Theorem 2. For each fixed $x \in [0, \pi]$ main equation (9) has a unique solution $A(x, \cdot) \in L_{2,\rho}(0, \mu^+(x))$.

The proof of Theorem 1 and Theorem 2 is given in [21].

2. Sufficient conditions for solvability of the inverse problem

Assume that the real numbers $\{\lambda_n^2, \alpha_n\}_{n \geq 1}$ is given by the formula (4). Now, let's construct $F_0(x, t)$ and $F(x, t)$ functions by using the formulas (10), (11) and write the integral equation (9).

We determine $A(x, t)$ from the main equation (9). We shall construct the function $\varphi(x, \lambda)$ with the formula (5) i.e.

$$\varphi(x, \lambda) := \varphi_0(x, \lambda) + \int_0^{\mu^+(x)} A(x, t) \cos \lambda t dt,$$

and the function $q(x)$ with formula

$$q(x) := \frac{4\rho(x)}{\sqrt{\rho(x)} + 1} \frac{d}{dx} A(x, \mu^+(x)). \tag{12}$$

Denote

$$b(x) := \sum_{n=1}^{\infty} \left(\frac{\cos \lambda_n x}{\alpha_n \lambda_n^2} - \frac{\cos \lambda_n^0 x}{\alpha_n^0 \lambda_n^{0^2}} \right).$$

Similar to Lemma 1.3.4 in [15], it is shown that $b(x) \in W_2^1(0, \pi)$. According to (4) and (5) we have

$$F_{0tt}(x, t) = \rho(t)F_{0xx}(x, t), \quad \rho(t)F_{xx}(x, t) = \rho(x)F_{tt}(x, t), \tag{13}$$

$$F_0(x, t)|_{x=0} = 0, \quad F_0(x, t)|_{t=0} = 0, \tag{14}$$

$$\frac{\partial}{\partial x} F_0(\mu^\pm(x), t) = \pm \sqrt{\rho(x)} \frac{\partial}{\partial \xi} F_0(\xi, t)|_{\xi=\mu^\pm(x)}. \tag{15}$$

Using the main equation (9) it can be proved that

$$A(x, 0) = 0, \tag{16}$$

$$\frac{\sqrt{\rho(x)} - 1}{\sqrt{\rho(x)} + 1} \frac{d}{dx} A(x, \mu^+(x)) = \frac{d}{dx} \{A(x, \mu^-(x) + 0) - A(x, \mu^-(x) - 0)\}. \tag{17}$$

2.1. Derivation of the Differential Equation

Lemma 1. *The following relations hold*

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda^2 \rho(x)\varphi(x, \lambda), \tag{18}$$

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = 0. \tag{19}$$

Proof. Assume that $b(x) \in W_2^2(0, \pi)$ and

$$\begin{aligned} J(x, \lambda) := & \frac{2}{1 + \sqrt{\rho(t)}} A(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} A(x, 2a-t) + \\ & + F(x, t) + \int_0^{\mu^+(x)} A(x, \xi) F_0(\xi, t) d\xi = 0, \end{aligned} \tag{20}$$

Differentiating (20) twice with respect to x and t we get

$$J''_{xx}(x, t) - \rho(x)J''_{tt}(x, t) - q(x)J(x, \lambda) \equiv 0.$$

Using the formulas (9), (12)-(15) and (17), we obtain the following homogeneous equation

$$\begin{aligned} & \frac{2}{1 + \sqrt{\rho(t)}} [A_{xx}(x, \mu^+(t)) - \rho(x)A_{tt}(x, \mu^+(t)) - q(x)A(x, \mu^+(t))] + \\ & + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} [A_{xx}(x, 2a-t) - \rho(x)A_{tt}(x, 2a-t) - q(x)A(x, 2a-t)] + \end{aligned}$$

$$+ \int_0^{\mu^+(x)} [A_{xx}(x, \xi) - \rho(x)A_{\xi\xi}(x, \xi) - q(x)A(x, \xi)] F_0(\xi, t)d\xi = 0.$$

We know that from [21] this equation has only trivial solution:

$$A_{xx}(x, t) - \rho(x)A_{tt}(x, t) - q(x)A(x, t) = 0, \quad 0 < t < x. \tag{21}$$

Differentiating (5) twice, integrating by parts twice and using (16) we obtain

$$\begin{aligned} \varphi''(x, \lambda) + \lambda^2\rho(x)\varphi(x, \lambda) - q(x)\varphi(x, \lambda) = & \varphi_0''(x, \lambda) + \int_0^{\mu^+(x)} A_{xx}(x, t) \cos \lambda t dt + \\ & -\lambda\rho(x)A(x, \mu^+(x)) \sin \lambda\mu^+(x) + \sqrt{\rho(x)}A_x(x, \mu^+(x)) \cos \lambda\mu^+(x) + \\ & +\lambda\rho(x) \sin \lambda\mu^-(x) (A(x, \mu^-(x) + 0) - A(x, \mu^-(x) - 0)) + \\ & +\sqrt{\rho(x)} \cos \lambda\mu^-(x) \frac{d}{dx} (A(x, \mu^-(x) + 0) - A(x, \mu^-(x) - 0)) + \\ & +\sqrt{\rho(x)} \cos \lambda\mu^+(x) \left. \frac{\partial A(x, t)}{\partial x} \right|_{t=\mu^+(x)} + \\ & +\sqrt{\rho(x)} \cos \lambda\mu^-(x) \left(\left. \frac{\partial A(x, t)}{\partial x} \right|_{t=\mu^-(x)+0} - \left. \frac{\partial A(x, t)}{\partial x} \right|_{t=\mu^-(x)-0} \right) - \\ & -\varphi_0''(x, \lambda) + \lambda\rho(x) \sin \lambda\mu^+(x)A(x, \mu^+(x)) + \rho(x) \cos \lambda\mu^+(x) \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=\mu^+(x)} - \\ & -\lambda\rho(x) \sin \lambda\mu^-(x) \{A(x, \mu^-(x) + 0) - A(x, \mu^-(x) - 0)\} + \\ & +\rho(x) \cos \lambda\mu^-(x) \left[\left. \frac{\partial A(x, t)}{\partial t} \right|_{t=\mu^-(x)-0} - \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=\mu^-(x)+0} \right] - \\ & -\rho(x) \int_0^{\mu^+(x)} A_{tt}''(x, t) \cos \lambda t dt - \\ & -q(x) \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) \cos \lambda\mu^+(x) + \right. \\ & \left. + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) \cos \lambda\mu^-(x) + \int_0^{\mu^+(x)} A(x, t) \cos \lambda t dt \right]. \end{aligned}$$

Hence using (12), (17) and (21) we arrive at (18). The relations (19) follow from (5) for $x = 0$. Lemma 1 is proved in the case $b(x) \in W_2^2(0, \pi)$.

The proof of Lemma 1 in the case $b(x) \in W_2^1(0, \pi)$ is carried out by a standard method (see e.g. [8] p. 40).

As in the theory of Sturm-Liouville problems (see [15], Lemma 1.5.8 and Corollary 1.5.1) the following lemmas can be proved.

Lemma 2. For each function $g(x) \in L_{2,\rho}(0, \pi)$,

$$\int_0^\pi \rho(x)g^2(x)dx = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi \rho(t)g(t)\varphi(t, \lambda_n)dt \right)^2. \quad (22)$$

Corollary 1. For arbitrary functions $f(x), g(x) \in L_{2,\rho}(0, \pi)$,

$$\int_0^\pi \rho(x)f(x)g(x)dx = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \int_0^\pi \rho(t)f(t)\varphi(t, \lambda_n)dt \int_0^\pi \rho(t)g(t)\varphi(t, \lambda_n)dt. \quad (23)$$

Using the below lemmas the following lemma is proved with standard method.

Lemma 3. The following relation holds

$$\int_0^\pi \rho(x)\varphi(t, \lambda_n)\varphi(t, \lambda_k)dt = \begin{cases} 0, & n \neq k \\ \alpha_n, & n = k. \end{cases} \quad (24)$$

2.2. Derivation of Boundary Condition

Lemma 4. For all $n \geq 1$ the equality

$$\varphi(\pi, \lambda_n) = 0$$

holds.

Proof. Since

$$\begin{aligned} -\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) &= \lambda_n^2 \rho(x)\varphi(x, \lambda_n), \\ -\varphi''(x, \lambda_m) + q(x)\varphi(x, \lambda_m) &= \lambda_m^2 \rho(x)\varphi(x, \lambda_m), \end{aligned}$$

we get

$$\begin{aligned} \frac{d}{dx} (\varphi(x, \lambda_n)\varphi'(x, \lambda_m) - \varphi'(x, \lambda_n)\varphi(x, \lambda_m)) &= \\ = (\lambda_n^2 - \lambda_m^2) \rho(x)\varphi(x, \lambda_n)\varphi(x, \lambda_m) \end{aligned} \quad (25)$$

From (25) we have

$$\begin{aligned} (\lambda_n^2 - \lambda_m^2) \int_0^\pi \rho(x)\varphi(x, \lambda_n)\varphi(x, \lambda_m)dx &= \\ = \varphi(\pi, \lambda_n)\varphi'(\pi, \lambda_m) - \varphi'(\pi, \lambda_n)\varphi(\pi, \lambda_m). \end{aligned}$$

By (24) we get

$$\varphi(\pi, \lambda_n)\varphi'(\pi, \lambda_m) - \varphi'(\pi, \lambda_n)\varphi(\pi, \lambda_m) = 0. \quad (26)$$

Clearly, $\varphi'(\pi, \lambda_n) \neq 0$, for all $n \geq 1$. Indeed, if we suppose that $\varphi'(\pi, \lambda_m) = 0$ for a certain m , then $\varphi(\pi, \lambda_m) \neq 0$, and in view of (26) $\varphi'(\pi, \lambda_n) = 0$ for all n .

On the other hand,

$$\varphi'(\pi, \lambda_n) = \varphi'_0(\pi, \lambda_n) + O(e^{Im\lambda|\mu^+(x)}), \quad |\lambda| \rightarrow \infty$$

i.e. for any n , $\varphi'(\pi, \lambda_n) \approx \varphi'_0(\pi, \lambda_n^0) \neq 0$ as $n \rightarrow \infty$, that contradicts the condition $\varphi'(\pi, \lambda_n) = 0$, $n \neq m$. Thus, $\varphi'(\pi, \lambda_n) \neq 0$, for all $n \geq 1$ and from (26) we have

$$\frac{\varphi(\pi, \lambda_n)}{\varphi'(\pi, \lambda_n)} = \frac{\varphi(\pi, \lambda_m)}{\varphi'(\pi, \lambda_m)} = H,$$

i.e. for any n , $\varphi(\pi, \lambda_n) = H\varphi'(\pi, \lambda_n)$. Since $\varphi(\pi, \lambda_n) = o(1)$ as $n \rightarrow \infty$, we have $H = 0$ i.e. $\varphi(\pi, \lambda_n) = 0$.

Thus, we prove that the numbers $\{\lambda_n^2, \alpha_n\}_{n \geq 1}$ are spectral data of the constructed boundary value problem (1), (2). Then, the following theorem is proved.

Theorem 3. For the sequences $\{\lambda_n^2, \alpha_n\}_{n \geq 1}$, where $\lambda_n \neq \lambda_m$ for $n \neq m$, $\alpha_n > 0$ for all n to be spectral data of a problem $L(q(x))$ of the form (1)-(3) with $q(x) \in L_2(0, \pi)$, it is necessary and sufficient to satisfy conditions

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad \alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad \{k_n\}, \{t_n\} \in l_2$$

Here λ_n^0 are the zeros of the function

$$\Delta_0(\lambda) = \frac{1}{2} \left(1 + \frac{1}{\alpha}\right) \cos \lambda \mu^+(\pi) + \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \cos \lambda \mu^-(\pi),$$

$$\alpha_n^0 = \int_0^\pi \varphi_0^2(x, \lambda_n) \rho(x) dx,$$

$$\varphi_0(x, \lambda) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}}\right) \cos \lambda \mu^+(x) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}}\right) \cos \lambda \mu^-(x),$$

$$\mu^\pm(x) = \pm x \sqrt{\rho(x)} + a \left(1 \mp \sqrt{\rho(x)}\right),$$

d_n is a bounded sequence; $\{k_n\}, \{t_n\} \in l_2$.

Algorithm of the construction of the function $q(x)$ by spectral data $\{\lambda_n^2, \alpha_n\}$ follows from the proof of the Theorem 3:

- 1) By the given numbers $\{\lambda_n^2, \alpha_n\}_{n \geq 1}$ the functions $F_0(x, t)$ and $F(x, t)$ are constructed by the formulas (10) and (11), respectively;
- 2) The function $A(x, t)$ is found from equation (9);
- 3) $q(x)$ is calculated by the formula (12).

Acknowledgements

This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK).

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