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Origin of Neural Firing and Synthesis in Making Comparisons

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We are all agreed that your theory is crazy. The question that divides us is whether it is crazy enough to have a chance of being correct.
-Niels Bohr

Abstract. The nervous system uses its own kind of mathematical function patterns for both external and internal realities. The conscious part of the nervous system is there to respond to what happens outside by regulating externally received information signals from the senses and the skin and muscles of the body itself. To do that, it needs to communicate with its subconscious using the familiar language of neural firing. In this paper, we show that because reciprocal pairwise comparisons are performed at the neural level, the division algebra of the octonions, in which commutativity and associativity are not satisfied, provides the structure needed to represent mental processes and that these processes could be represented in G_2-manifolds.

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1. Introduction

We use our nervous system to understand the external reality scientifically studied in physics in sorting out natural law. In the past century or so the nervous system has also been used to study internal reality through the workings of the brain itself and how it gives rise to thought, feeling and memory. It has been discovered that animals brought up in a stimulating environment have many more synapses and as a result, a heavier

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brain than their unstimulated counterparts. Thus, the brain itself changes over time that makes it harder to describe it well as a closed system with all its activities. To understand change we need to *compare* different states of the brain that derive from its workings. To appreciate how basic comparisons are, we refer to what philosopher Arthur Schopenhauer wrote,"Every truth is the reference of a judgment to something outside it, and intrinsic truth is a contradiction".

The nervous system is a response mechanism that carries out its work electrochemically at the molecular level. It does that subconsciously through feedback and is also modified by the functions of the body where it resides, sustained through chemicals that give rise to its electrical firing actions and reactions. The nervous system uses its own kind of mathematical function patterns for both external and internal realities. That is the subject of this paper [13, 14].

We cannot think or feel or apprehend something that our brain and nervous system are incapable of grasping. Our neurons fire one at a time and the simultaneous firings of many neurons give rise to all that happens to us and inside us. The limiting factors in understanding is the way our nervous system works. Our collective responses form a box that we cannot go beyond at will by using imagination that is a by-product of that instrument itself.

A well-known example of thinking outside the box, beyond our present state of mind, is a 9-point exercise.

The question is, what is it beyond what we know about the fundamentals of space and time and energy, physics, and particularly an objective and subjective philosophy and psychology lurking behind the scene ready to be discovered as our existing way of knowledge evolved through research and exploration. Yet it is all a consequence of electrical firing and neurons that together capture what is out there. We can only know it if we are able to detect and synthesize it into a whole. Still we cannot know what we are not able to know through the firing of neurons. That is the total sum of the reality that we know. The question is whether there is something out there or inside us that we are ever incapable of knowing. To identify such a thing, we must be aware of it. To be aware of something we must be able to create it as a firing neuron. But neurons fire in special ways that satisfy the requirements of arithmetic and that knowledge is tied to the firings and to combining them arithmetically in particular ways.

The power of our brain depends on the firing and complexity of connecting ten to the eleventh power of the number of neurons which takes more than a lifetime of approximately 3.5 to the ninth seconds to use them. But many are connected and fire at the same instant of time.

The conscious part of the nervous system is there to respond to what happens outside by regulating externally received information signals from the senses and the skin and muscles of the body itself.To do that, it needs to communicate with its subconscious using the familiar language of neural firing.

The subconscious is able to *compare* the states of the body from instant to instant. Comparisons of both variables and of functions are made to determine the magnitude of influence that a dominant element has over a lesser element. If the dominant element is x times the dominated element, then the latter is $1/x$ times the dominant one.This reciprocal value is assigned automatically in a group of many comparisons as we show below. Because of this reciprocal relationship, we need to work with division and more generally with a division algebra.

Since we must work in division algebras we briefly summarize what is known about their origin. It was proved by Adolf Hurwitz [8], that there are exactly four normed division algebras: the real numbers ($\mathbb{R}$), the complex numbers ($\mathbb{C}$), the quaternions ($\mathbb{H}$), and the octonions ($\mathbb{O}$). The Octonions were discovered by John T. Graves in 1844 [7] who called them octaves. Before John Graves had a chance to publish his work, Arthur Cayley published a paper in 1845 [4] in which the octonions were mentioned. They were later called Cayley-numbers by others. An excellent exposition of octonions can be found in [2], and of Quaternions in [11]. The book by Conway and Smith [5] explains in some detail both quaternions and octonions.

Immediately beyond these four division algebras are the16 dimensional sedenions. In fact, one can get an algebra of dimension 2^n for any non-negative $n > 3$, but these algebras are perhaps less interesting because they are no longer division algebras that is $ab = 0$ no longer implies that either $a = 0$ or $b = 0$.

Brain activity creates consciousness. According to Mark Robert Waldman consciousness exists at eight inclusive levels: reality (which Plato called Phenomena) consisting of the three space physical dimensions and time, and seven states of consciousness (called by Plato Nuomena) that cover instinctual awareness (wakefulness), habitual responsiveness, intentional decision-making, free-floating imagination, self-reflective awareness, transformational awareness and enlightenment. Waldmans model of human consciousness consolidates more than 31,000 studies contained in the database of the National Library of Medicine. The first four levels seem to correspond to the perceptual-cognitive-active loop mentioned by Goertzel [6].

" The quaternion structure is interpreted as a "perceptual-cognitive-active loop," representing the basic structure of engagement with the world. This structure is seen to lead naturally to a certain type of adaptive learning, analogous to backtracking in artificial intelligence.

The remaining levels appear to need the octonions as Goertzel [6] writes:

The octonion structure is seen to ensue from adding to the quaternions an extra mental process that observes the others. This extra element is called the "inner eye" and is hypothesized to provide for reflexive consciousness and higher-order thought."

We are left with the need for the eight dimensional octonions that arise from the different physics manifestations of our world, the functions of the nervous system and from the mathematical account of proportionality between perception and physical reality. Physically, as opposed to behaviorally (which is our concern here) the multiplication of octonions is used to describe rotations in 7 dimensions with stretch and contraction as the additional 8th dimension. The role of the octonions is more than just describing geometric properties. The octonions are the foundation for their group of automorphisms known as G_2 on which physics models of the universe, superstring theory, are built. We discuss this later in the paper.

The following question is a bold assertion just short of sounding crazy.

Why are octonions necessary and sufficient to represent all brain activity and the feelings and knowledge we use to explain all we can ever conceptualize?

The necessity follows from the fact that if comparisons are taken as the most fundamental axiom of all knowledge, decision making and action,octonions are the largest division algebra. The other three algebras are special cases of octonions. The proof of sufficiency is vast and empirical as we shall see below. In passing, we note that the Fourier transform is needed to transform electrochemical vibrations to the real world of space and time. Such a transformation is not needed to transform thoughts themselves and we need a different way to write about the synthesis of firings of neurons. We believe that the property that makes the octonions sufficient to represent thoughts is density, i.e., all thoughts are represented by a finite combination of the solutions of the fundamental equation of proportionality given below. The reason why there is no need for a transformation such as the Fourier transform is that in the physical domain the solution of the fundamental equation is not itself dense. Thus, it needs the transformation that leads to Dirac type distributions that are known to be dense and sufficient to represent reality.

2. How Reciprocal Comparisons Work

A basic concept at the core of understanding is the reciprocal property. Essentially this property asserts that, for example, when comparing two stones according to weight with the aid of the hands and one stone is judged to be five times heavier than the other, then the other is automatically one fifth the weight of the first. Both stones participate in the judgment, the smaller one serving as a unit of reference. All our senses can make such comparisons and so does our mind in comparing abstract ideas with respect to common properties. This process must also take place at the very elementary molecular level. From such comparisons among objects in pairs, a relative scale of measurement is derived among the objects.

Reciprocal comparison is an inherent conscious ability of all human beings which enables us to scale things encountered in practice. The logical question then is: Is our judgment sufficiently accurate to ensure that if stone A is, for example, five times heavier than stone B and stone B is three times heavier than stone C that we would judge A to be fifteen times heavier than C? Most likely not. To improve the accuracy of the scale derived from the paired comparisons we should also compare A with C. We would then need to say something about the inconsistency in performing comparisons. In general, if indicates the relative dominance of object i over object j when comparing n objects in pairs, the comparisons are said to be consistent if the relation $a_{ij}\ a_{jk} = a_{ik}$ holds for all $i, j, k = 1, 2, , n$. The reciprocal property mentioned above is given by $a_{ji} = 1/a_{ij}$ and follows from consistency but it does not imply it and is thus a weaker condition that can be safely say predates consistency. The usual criterion of transitivity is implied by consistency but not conversely.

Consider n stocks, $A_1, ..., A_n$, with known worth $w_1, ..., w_n$, respectively, and suppose that pairwise ratios are formed to show the worth of each stock with respect to all others as in Table 1. We can recover the scale w using the equation next to it as in Table 2 which briefly says that $Aw = nw$.

Table 1. Matrix of ratios

$$A = \begin{array}{c} \\ A_1 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & \cdots & A_n \end{array} \\ \begin{bmatrix} \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \end{array}$$

Table 2. Equation to get back the scale

$$\begin{bmatrix} \frac{w_1}{w_1} & \cdots & \frac{w_1}{w_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

Thus, to recover the scale w from the matrix of ratios A, one must solve the eigenvalue problem $Aw = nw$ or $(A - nI)w = 0$. This is a system of homogeneous linear equations. It has a nontrivial solution if and only if the determinant of $A - nI$ vanishes, that is, n is an eigenvalue of A and w is its eigenvector. It turns out that n is the maximum eigenvalue and the entries of w which are the priorities we want are always positive.

In the general case when only judgments but not the numbers themselves are available, the precise value of w_i/w_j which is a dimensionless number and thus belongs to an absolute scale that is invariant under the identity transformation, is not known, but instead only an estimate of it can be given as a numerical judgment from the fundamental scale. The maximum eigenvalue is no longer equal to n but is replaced by the maximum eigenvalue of the matrix of judgments. Tables 1 and 2 become Tables 3 and 4, respectively, and the solution is obtained from the equation $A'w' = \lambda_{max} w'$.

Table 3. Matrix of ratios

$$A' = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix}$$

Table 4. Equation to get back the scale

$$A'w = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \lambda_{max} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

3. Generalizing from Discrete to Continuous Judgments

There is a way to formulate the problem of automatic decisions that relates to the stimulus-response equation $w(as) = bw(s)$ It involves generalizing the discrete, eigenvalue-oriented decision-making of the AHP to the continuous case where Fredholms equation is the continuous version of the discrete eigenvalue formulation. In that generalization, it turns out that the functional equation $w(as) = bw(s)$ is a necessary condition for Fredholms integral equation of the second kind to be solvable. Instead of finding the eigenvector of a pairwise comparison matrix one uses the kernel of an operator. Operations on the matrix translate to operations on the kernel. From the matrix formulation leading to the solution of a principal eigenvalue problem

we have

$$\int_a^b K(s,t)w(t)dt = \lambda_{max}w(s) \text{ or } \lambda \int_a^b K(s,t)w(t)dt = w(s)$$

$$\int_a^b w(s)ds = 1$$ where the positive matrix A is replaced by a positive kernel $K(s,t) > 0$, the continuous version of pairwise comparisons and the eigenvector w by the eigenfunction $w(s)$. Note that the entries in a matrix depend on the two variables i and j that assume discrete values. Thus, the matrix itself depends on these discrete variables and its generalization, the kernel function, depends on two (continuous) variables. The reason for calling it a kernel is the role it plays in the integral, where without knowing it, we cannot determine the exact form of the solution. The standard way in which the first equation is written is to move the eigenvalue to the left-hand side which gives it the reciprocal form. In general, by abuse of notation, one continues to use the symbol λ to represent the reciprocal value and with it one includes the familiar condition of normalization $\int_a^b w(s)ds = 1$. Here also, the kernel K(s,t)is said to be 1) consistent and therefore also reciprocal, if$K(s,t)K(t,u) = K(s,u)$, for all s, t and u, or 2) reciprocal, but perhaps not consistent, if $K(s,t)K(t,s) = 1$ for all s, t.

A value of λ for which Fredholms equation has a nonzero solution $w(t)$ is called a characteristic value (or its reciprocal is called an eigenvalue) and the corresponding solution is called an eigenfunction. An eigenfunction is determined to within a multiplicative constant. If $w(t)$ is an eigenfunction corresponding to the characteristic value λ and if C is an arbitrary constant, we see by substituting in the equation that $Cw(t)$ is also an eigenfunction corresponding to the same λ. The value $\lambda = 0$ is not a characteristic value because we have the corresponding solution $w(t) = 0$ for every value of t , which is the trivial case, excluded in our discussion.

3.1. How Neurons Compare Charges

Turning to neurons, consider a neuron that compares neurotransmitter-generated charges in increments of time. Let $[0,T]$ be a time interval, let $0 = t_0 < t_1 < ...t_{n-1}$ be a partition of the interval $[0,T]$, and $I_k(t_{k-1}, t_k), k = 1, 2, , n$. Let , $w(t), t \in [0,T]$ be a single firing (voltage discharge) of a neuron in spontaneous activity. Let $G(t), t \in [0,T]$ be the cumulative response of the neuron in spontaneous activity over time. Then, we have $dG(t)/dt = w(t)$. Note that G(t) is monotonically increasing and hence, $w(t) > 0$ and $w(0) = 0$. Let

$$K = (I_i, I_j) = \frac{G(t_i) - G(t_{i-1})}{G(t_j) - G(t_{j-1})}$$

be the relative comparison of the response of the neuron during a time interval I_i with another time interval I_j . We have

$$\frac{1}{n}\sum_{j=1}^{n} K(I_i, I_j)[G(t_j) - G(t_{j-1})] = G(t_i) - G(t_{i-1}), i = 1, 2, ..., n. \tag{1}$$

If $G(t)$ is of class $C^1(0,T]$, then as $\Delta t_k \equiv t_k - t_{k-1} \to 0$ for all $k, K(I_i, I_j) \to K(s,t) = \frac{w(s)}{w(t)}, s, t \in (0,T]$. In addition, because the left-hand side of (1) is an average, we obtain as $\Delta t_k \to 0$ for all k and as $n \to \infty$:

$$\frac{1}{T}\int_0^T k(s,t)w(t)dt = w(s)$$

It can be easily shown that T is the principal eigenvalue of the consistent kernel $K(s,t)$. In general, if $K(s,t)$ is reciprocal but not consistent, the homogeneous equation takes the form given by

$$w(s) = \lambda_0 \int_\Omega K(s,t)w(t)dt. \tag{2}$$

Because positive reciprocal kernels are non-factorable (the property that corresponds to irreducibility for non-negative matrices), there exists a unique simple eigenvalue λ_0^{-1} whose modulus dominates the moduli of all other eigenvalues.The corresponding eigenfunction $w(s)$ is the response function of the neuron in spontaneous activity.

If $K(s,t)$ is consistent, it can be written as a ratio $K(s,t) = k(s)/k(t)$ and $w(s) = k(s)/\int_\Omega w(s)ds$.

Saaty ([12] , p.113) has shown that: A necessary and sufficient condition for $w(s)$ to be an eigenfunction solution of Fredholm equation of the second kind (2) with a consistent kernel that is homogeneous of order one is that it satisfies the functional equation

$$w(as) = bw(s) \tag{3}$$

4. The Solution of the Functional Equation $w(as) = bw(s)$

In Saaty ([12], pp. 113-121), the solutions of (3) are given in the real ($\mathbb{R}$), the complex ($\mathbb{C}$), the quaternionsl ($\mathbb{H}$), and the octonions ($\mathbb{O}$) spaces. These are the only normed division (or composition) algebras. A composition algebra A is an algebra whose norm is multiplicative. Let H be a composition subalgebra of A and $x \in A$ with x orthogonal to H. If H is associative then $H \oplus xH$ is a composition algebra.

Hurwitz [8] proved that the only composition algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ .The smallest composition algebra is $\mathbb{R}$. Since $\mathbb{R}$ is associative and i is orthogonal to $\mathbb{R}, \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is a composition algebra. Now as $\mathbb{C}$ is associative and j is orthogonal to $\{1, i\}$, hence to $\mathbb{C}$, thus $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ is a composition algebra. Finally, since $\mathbb{H}$ is associative and l is orthogonal to $\{1, i, j, k\}$, hence to $\mathbb{H}, \mathbb{O} = \mathbb{H} \oplus l\mathbb{H}$ is a composition algebra. The process of doubling the composition algebra to obtain another composition algebra ends with the octonions because $\mathbb{O}$ is not associative. Since the dimension of a composition algebra is a power of two, these are the only composition algebras.

The solution of the equation $w(as) = bw(s)$ in octonions, of which solutions in the other three spaces are special cases, is given by

$$w(r) = r^{\left(\frac{ln\ b}{ln\ a}\right)} P\left(\frac{ln\ r}{ln\ a}\right) \oplus P\left(\frac{ln\ r}{ln\ a}\right) r^{\left(\frac{ln\ b}{ln\ a}\right)} = w_G(r) \oplus w_D(r) \tag{4}$$

where a and b are constants in, $C, r\epsilon\ \mathbb{O}$ and $P\left(\dfrac{ln\ r}{ln\ a}\right)$ is a periodic function of period 1, e.g., $cos(2\pi u)$. A constant in $\mathbb{O}$ is of the form $a = a_0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7, a_0\epsilon\mathbb{R}$. Using the transformation $\frac{ln\ b}{ln\ a} = r$ or $r = a^u$ we have

$$\tilde{w}(u) a^{\left(\frac{ln\ b}{ln\ a}\right)u} P(u) \oplus a^{\left(\frac{ln\ b}{ln\ a}\right)u} = \tilde{w}_G(u) \oplus \tilde{w}_D(u) \tag{5}$$

The direct sum of functions is a formal representation, not exactly a sum of functions. Let $\mathbb{U}$ be the domain of definition of the functions given by (5) and let $\mathbb{U}_{\mathbb{G}}$ and $\mathbb{U}_{\mathbb{D}}$ be the domains of definition of the functions $\hat{w}_G(u)$ and $\tilde{w}_D(u)$, respectively. If $\mathbb{U}_{\mathbb{G}} \cap \mathbb{U}_{\mathbb{D}} \neq \varnothing, \tilde{w}_G(u) = \tilde{w}_D(u) = 0$ for all $u\epsilon\mathbb{U}_{\mathbb{G}} \cap \mathbb{U}_{\mathbb{D}}$.

If the solution to the equation $w(as) = bw(s)$ satisfies $\tilde{w}(uv) = \tilde{w}(u)\tilde{w}(v)$, then it would belong to the group of automorphisms of the octonions G_2 . If these functions were dense in the space of continuous functions defined on the octonions, then all the functions could be expressed as linear combinations of the solution of the equation and they could generate the group of automorphisms. Then, any representation of brain activity with octonions can be expressed with the solution of the equation $w(as) = bw(s)$. Thus, the question is: When does $\tilde{w}(uv) = \tilde{w}(u)\tilde{w}(v)$ hold for the solution of $w(as) = bw(s)$?

Theorem 1. *If the periodic function of period 1, $P(u)$ satisfies the semigroup condition, $P(u + v) = P(u)P(v)$ then $\tilde{w}(uv) = \tilde{w}(u)\tilde{w}(v)$.*

Proof. Let $w(r) = r^{\left(\frac{ln\ b}{ln\ a}\right)} P\left(\dfrac{ln\ r}{ln\ a}\right) r\epsilon\mathbb{O}$. we have

$$w(st) = st^{\left(\frac{ln\ b}{ln\ a}\right)} P\left(\frac{ln\ (st)}{ln\ a}\right) = s^{\left(\frac{ln\ b}{ln\ a}\right)} t^{\left(\frac{ln\ b}{ln\ a}\right)} P\left(\frac{ln\ s}{ln\ a} + \frac{ln\ t}{ln\ a}\right)$$

Substituting $S = a^u$ and $t = a^v$ we have $\tilde{w}(uv) = a^{\left(\frac{ln\ b}{ln\ a}\right)u} a^{\left(\frac{ln\ b}{ln\ a}\right)v} P(u+v)$ and $\tilde{w}(uv) = \tilde{w}(u)\tilde{w}(v)$ obtains if $P(u+v) = P(u)P(v)$. Q.E.D

This theorem implies that the solution of the functional equation (3) belongs to the group of automorphism G_2.

The solution of the equation $w(as) = bw(s)$ in the group of automorphisms of the octonions G_2 is given by the damped exponential function

$$\tilde{w}(u) = b^u e^{2n\pi u} \tag{6}$$

Proof. The solution of the functional equation $P(u+v) = P(u)P(v)$ is given by $P(u) = e^u$. Since $P(.)$ must be a periodic function of period 1, then $P(u) = e^{2n\pi u}$ is a possible solution in the space of the octonions if the period is the unit octonion, i.e., $1_u = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$ and $P(u + 1_u) = e^{2n\pi(u+1_u)} = e^{2n\pi u + 2n\pi 1_u} = e^{2n\pi u}$ Since $e^{2n\pi 1_u} = e^{2n\pi(e_1+e_2+e_3+e_4+e_5+e_6+e_7)} = 1$.

Thus, the solution of the equation $w(as) = bw(s)$ in the group of automorphisms of the octonions G_2 can be written as $\tilde{w}(u) = a^{\left(\frac{ln\ b}{ln\ a}\right)u} e^{2n\pi u} \oplus e^{2n\pi u} a^{\left(\frac{ln\ b}{ln\ a}\right)v}$. However, because $a^{\left(\frac{ln\ b}{ln\ a}\right)u} e^{2n\pi u} = b^u e^{2n\pi u} = e^{(2n\pi + lnb)u}$ there is no need to use the direct sum and we have $\tilde{w}(u) = b^u e^{2n\pi u}$.
Manogue and Schray [9] showed that the automorphisms of $\mathbb{O}$ can be generated by octonions of the form $e^{\hat{u}\theta} + cos(\theta) + \hat{u}sin(\theta)$ with $\hat{u}$ a pure imaginary, unit octonion, but where θ must be restricted to be a multiple of $\pi/3$, corresponding to the sixth roots of unity, i.e., $e^{k\left(\frac{\pi}{3}\right)\hat{u}}, k = 0, ..., 5$. Thus , if $2n\pi + lnb = k\left(\frac{\pi}{3}\right)$ or $b = e^{(k-6n)\frac{\pi}{3}}$ the solution of the functional equation (3) can generate the group of automorphism G_2.

5. Synthesis of Neural Responses

Let us assume that the response of the brain to each distinct type of information obtained from senses or generated form within the brain satisfies the equation $w(as) = bw(s)$, and hence, the response can be modeled by the function $\tilde{w}(u) = b^u e^{2n\pi u}$. Now all the different types of responses need to be synthesized to create a consistent picture of the functioning of the brain at any point in time. The synthesizing principle is a solution of the functional equation $w(as) = bw(s)$ in operator form given by Brillouet-Bellout [3]

$$w(\alpha X) = \beta w(X). \tag{7}$$

The operator W, defined from a normed linear space E to another normed linear space G, $W : E \to G$, is a way of thinking about synthesis of neural responses. For example, if

α is a root of 1 of order n, and $\beta^n = 1$ there exists a unique p in $\{0.1, ..., n-1\}$ such that $\beta = \alpha^p$ and the solution of the functional equation in operator form (7) is given by

$$w(X_1, ..., X_q) = \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} X_1^{n_1} X_2^{n_2} ... X_q^{n_q} \tag{8}$$

where $J = \{(n_1, ..., n_q) \in \mathbb{N}^q | n_1 + ... + n_q = p + jn, for\ some\ \ j \in \mathbb{N}$ and $a_{n_1,...,n_q} \in\ G$

The solution (8) will allow us to synthesize the functions given by (6). Each of the components of (8) could be the functions given by (6). Hence, for the group of automorphisms of the octonions G_2 , synthesis of the firing of neurons under the conditions of Equation (8) is given by

$w(X_1, ..., X_q) = \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} X_1^{n_1} ... X_q^{n_q}$. Thus for $\tilde{w}_i(u)\epsilon X_i, i = 1, 2, ..., q$ we have

$$\begin{aligned} W(\tilde{w}_1, ..., \tilde{w}_q) &= \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} \tilde{w}_1(u_1)^{n_1} \tilde{w}_q(u_q)^{n_q} \\ &= \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} \left[{b_1}^{u_1} e^{2n\pi u_1}\right]^{n_1} ... \left[{b_q}^{u_q} e^{2n\pi u_q}\right]^{n_q} \\ &= \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} \prod_{i=1}^{q} e^{n_i(2n\pi + lnb_i)u_i}, u_i \epsilon \mathbb{O} \end{aligned} \tag{9}$$

Note the similarity of expression (9) with the Fourier series approximation of a real periodic function of period L in $\mathbb{O}$ (Martinez et al. [10]. For $L = 1$ the Fourier series approximation of the function $P(u)$, for $P(u)$, real,is given by $\tilde{P}(u) = c_0 + \frac{1}{7} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{k=1}^{7} c_n^k e^{e_k 2n\pi u}$

where

$c_n^k = \frac{1}{2} \int_{-1/2}^{+1/2} e^{-e_k 2n\pi x} dx$. Our periodic function of period 1 defined in $\mathbb{O}$ already has that form i.e., $P(u) = e^{2n\pi u}, u\epsilon\mathbb{O}$. Thus, the following result is intuitive.

Theorem 2. *The functions given by* $\tilde{w}(u) = b^u e^{2n\pi u}, u\epsilon\mathbb{O}$ *are dense in the space of continuous functions in* $\mathbb{O}$.

The density of $\tilde{w}(u) = b^u e^{2n\pi u}, u \in \mathbb{O}$ allows to say that all brain activity can be represented by these functions and that the density extends beyond the firing of a neuron. That is, one would expect that the synthesis of these functions is also dense in the space of continuous functions defined in $\mathbb{O}$.

Theorem 3. *The synthesis of neural activity given by*

$(\tilde{w}_1, ..., \tilde{w}_q)(u) = \sum_{(n_1,...,n_q)\epsilon J} a_{n_1,...,n_q} \prod_{i=1}^{q} e^{n_i(2n\pi + lnb_i)u_i}, u_i \in \mathbb{O}$ *is dense in the space of continuous functions in* $\mathbb{O}$.

6. The Discrete Nature of the Brain and the Creation of Consciousness: Why do we Need Octonions to Represent Neural Activity?

Physicists have been trying to find the smallest particle in the universe and the forces that keep them together for a long time. Although the theories that try to explain how these particles manage to stay together to create life are mathematically represented under the assumption of continuity, matter is not continuous, but brain activity is continuous, if life, as we understand it, exists in the system studied.

According to B. S. Acharya [1] the known physics in our universe is well modeled by the Standard Model of Particle Physics along with General Relativity. This mathematical model is based on (1) Maxwell equations, (2) Yang-Mills equations, (3) Dirac equation, (4) Higgs equation and (5) Einstein equations. String theory is based on equations that describe 2-dimensional surfaces embedded in space-time. The equations that describe low energy harmonics of such strings include the five set of equations just mentioned. Incorporating the property of supersymmetry creates superstring theory that solves some fundamental problems of string theory and creates a symmetry between fermions and bosons.There are five superstring theories and they are defined in ten dimensions, but only time and three-dimensional space have been observed,and thus, the extra six dimensions are hidden. Since these superstring theories are interrelated, a new theory emerged known as M theory that resides in eleven dimensions. Representations in M theory use G_2 -manifolds. G_2 -manifolds are models of the extra dimensions in the M theory. Smooth G_2 -manifolds are not that relevant in particle physics, but they are important in superstring theory.

In differential geometry, a G_2 -manifold is a seven-dimensional Riemannian manifold (a manifold with an inner product defined on the tangent space at each point) with holonomy group contained in G_2 - one of the five exceptional Lie groups that can be described as the automorphism group of the octonions.In differential geometry, the holonomy of a connection on a smooth manifold relates to the curvature of the connection and measures the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported.

Thus, we think that this is the connection between the solution of the functional equation $w(as) = bw(s)$ and superstring theory. The firing of the neurons through the continuous paired comparison process generate a smooth G_2 -manifold in which cognition must take place, and hence, the representations of our thoughts must take place in smooth G_2 -manifolds.

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