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Stability Results in Terms of Two Measures for Set Differential Equations involving Causal Operators

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Abstract. Differential equations involving causal operators is an area of research that unifies many types of mathematical models such as ordinary differential equations, integro differential equations, delay differential equations and so on. Also, Stability in terms of two measures is another concept that unifies various types of stability. It has been observed that set differential equations generalizes ordinary differential equations (ODEs) and the study of ODEs can be done in semilinear metric space. In this paper, combining all the fore mentioned notions an attempt is made to obtain stability results in terms of two measures for set differential equations involving causal operators.

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1. Introduction

During the past couple of decades the theory of set differential equations attracted the attention of many researchers and much of the basic theory is given in [9]. Some of the papers dealing with stability of set differential equations are [7, 5]. The reason for the continued interest in this area of research is that studying differential equations in metric space is gaining attention and also for the following reasons: the base space $K_c(\mathbb{R}^n)$, consisting of all compact convex subsets of $\mathbb{R}^n$ endowed with Hausdorff metric, is a semi linear metric space. If the Hukuhara derivative and the Hukuhara integral are restricted to $\mathbb{R}$ they become the conventional derivative and integral and one can observe that the theory of ordinary differential equations (ODE's) can be developed in a semi linear metric space. Similarly when we restrict to $\mathbb{R}^n$, then the theory of vector differential equations can

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be studied in a semi linear space. Further, it is useful in studying multi valued inclusions and also is related to fuzzy differential equations. Also, one can observe that the solution U(t) of the set differential equation has the interesting nature that as time increases the diameter of U(t) is non decreasing. It has been observed that this is due to the fact that, in the generation of the set differential equation (SDE) from an ODE, certain undesirable elements may enter the solution U(t) and hence the norm used may not be suitable to develop stability without some adjustment. Therefore while studying the stability theory for SDE's the concept of Hukuhara difference in the initial values was introduced in [11] so as to see that the properties of the solutions of the ODE's are preserved to a certain extent in the SDE. For a detailed description through an example see [11].

The stability theory via the Lyapunov function has been extensively studied and is applied in various models due to the fact that the Lyapunov function helps to study the qualitative behavior of the solution without knowing the solution.

Of late it has been observed that the Lyapunov function can be utilized to construct simpler scalar differential equations to study complex systems. Also, Lyapunov function can be used as a generalized distance and can be utilized to study the qualitative and quantitative behavior of solutions of differential equations.

Owing to the developments in the study of many physical phenomena, new concepts of stability such as partial stability, practical stability, eventual stability have been introduced. All these developments posed the question of unifying all the definitions under one set up. This lead to the introduction of stability theory in terms of two measures [10].

The notion of causal operators has been introduced by Tonneli [1]. A causal operator or a non anticipative operator is an operator whose prior information or memory is known till the present time 't'. Ordinary differential equations, delay differential equations, integro differential equations, impulsive differential equations etc. are some of the differential equations involving causal operators. The unifying character of the causal operator makes it an interesting topic of study. In [2, 3, 4] the mathematical model of SDE involving causal operator with memory was introduced. The existence and uniqueness results, comparison theorems and stability results has been studied. Much of the basic theory has been established in [8].

In this paper an attempt has been made to combine all the above set ups and develop some stability results in two measures for set differential equations involving causal operators.

2. Preliminaries

In this section, we begin with the definition of Hausdorff metric, Hukuhara difference and proceed to define the semi metric space $K_c(\mathbb{R}^n)$. Next we proceed to define the Hukuhara derivative, Hukuhara integral and a partial order in $K_c(\mathbb{R}^n)$ [9]. Further, we state some important properties that are useful tools in establishing our main results.

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$.

We define the Hausdorff metric by

$$D[A,B] = \max[\ \sup_{x\in B} d(x,A),\ \ \sup_{y\in A} d(y,B)\ ], \tag{1}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A and B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space. It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semi linear metric space which can be embedded as a complete cone into a corresponding Banach space. The Hausdorff metric (1) satisfies the following properties:

$$D[A+C, B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{4}$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

Let $A, B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0+} \frac{F(t_0+h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0+} \frac{F(t_0) - F(t_0-h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the IVP for set differential equation

$$D_H U = F(t,U),\ \ U(t_0) = U_0 \in K_c(\mathbb{R}^n),\ \ t_0 \ge 0, \tag{5}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

The mapping $U \in C^1[J,\ K_c(\mathbb{R}^n)], J = [t_0, t_0 + a]$ is said to be a solution of IVP (5) on J if it satisfies (5) on J.
Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s)ds,\ \ t \in J. \tag{6}$$

Hence, we can associate with the IVP (5) the Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s))ds,\ \ t \in J. \tag{7}$$

where the integral is the Hukuhara integral which is defined as,

$$\int F(s)ds = \{\int f(s)ds : f \textit{ is any continuous selector of } F\}$$

Observe also that U(t) is a solution of IVP (5) on J iff it satisfies (7) on J.

We now proceed to define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. We begin with the definition of a cone in this set up.

Let $K(K^o)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of set $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative (positive) vector of n components satisfying $u_i \geq 0$ $(u_i > 0)$ for i=1...n. Then K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

Definition 1. *For any U and $V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V\,(U > V)$. Similarly we can define $U \leq V\,(U < V)$.*

To introduce the causal operator [9, 10, 8] which is also known as a non anticipative operator we first introduce the following notation and notion.
Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$, We define a norm $D_0 : E \times E \to R_+$ as follows: for $U, V \in E$,

$$D_0[U, V] = \sup_{t_0 \leq t \leq T} D[U(t), V(t)] \tag{8}$$

where D is the Hausdorff metric.

Definition 2. *By a causal operator or a Volterra operator or a nonanticipative operator we mean a mapping Q: $E \to E$ satisfying the property that*

$$if\ U(s) = V(s),\ t_0 \leq s \leq t < T\ then\ (QU)(s) = (QV)(s),\ t_0 \leq s \leq t < T.$$

To develop the stability results in terms of two measures we need new concepts that are introduced in [5], which we present below.

$$\mathbb{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(u)\ \textit{is strictly increasing in } u \textit{ and } a(0) = 0\}$$

$$\mathbb{L} = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+], \sigma(u)\ \textit{is strictly decresing in } u \textit{ and } \lim_{u\to\infty}\ \sigma(u) = 0\}$$

$$\mathbb{C}K = \{a \in C[\mathbb{R}^2_+, \mathbb{R}_+], a(t,s) \in \mathbb{K}\ for\ each\ t\ and\ a(t,s)\ is\ continious\ for\ each\ s\}$$

$$\Gamma = \{h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] : \inf_{(t,U)}\ h(t,U) = 0\}$$

$$\Gamma_0 = \{h \in \Gamma : \inf_{U}\ h(t,U) = 0,\ for\ each\ t \in \mathbb{R}_+\}$$

We next give the definitions that must be satisfied by a function V when the notion of two measures is involved, thus introducing a Lyapunov like function. Let $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+]$, then V is said to be

Definition 3. *h-Positive definite if there exists a $\rho > 0$ and a function $b \in \mathbb{K}$ such that $b(h(t,U)) \leq V(t,U)$, whenever $h(t,U) < \rho$.*

Definition 4. *h-decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{K}$ such that $V(t,U) \leq a(h(t,U))$, whenever $h(t,U) < \rho$.*

Definition 5. *h- weakly decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{CK}$ such that $V(t,U) \leq a(t,h(t,U))$, whenever $h(t,U) < \rho$.*

Let $h_0, h \in \Gamma$, Then we say that

Definition 6. *h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathbb{CK}$ such that $h_0(t,U) \leq \rho$ implies $h(t,U) \leq \phi(t,h_0(t,U))$.*

Definition 7. *h_0 is uniformly finer than h if ϕ is independent of t in the above definition.*

3. Stability Results

In this section, we develop the stability results for set differential equations involving causal operators given by

$$D_H U = (QU)(t) \qquad U(t_0) = U_0, \tag{9}$$

where $Q \in C[E,E]$ is a causal operator $E = C[J, K_c(\mathbb{R}^n)]$, and J=[0,T].
We assume that the operator Q is smooth enough to gaurantee existence, uniqueness of solutions and continuous dependence of solutions $U(t) = U(t,t_0,U_0)$ of (9) with respect to the initial values.

Now we state from [10] the various stability concepts for the system (9) in terms of two measures $h_0, h \in \Gamma$.

Assume that the system (9) admits the trivial solution $U(t) = \theta$ through (t_0, θ), then the differential system (9) is

Definition 8. *$(h_0,h)-$equi stable if for each $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta = \delta(t_0,\epsilon)$ that is continuous in t_0 for each ϵ such that*

$$h_0(t_0,U_0) < \delta \Rightarrow h(t,U(t)) \leq \epsilon, \qquad t \geq t_0,$$

where $U(t,t_0,U_0)$ is any solution of (9).

Definition 9. *$(h_0,h)-$ uniformly stable if the δ in Definition 8 is independent of t_0.*

Definition 10. *$(h_0,h)-$ equi attractive, if for each $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a positive constant $\delta_0 = \delta(t_0)$ and $T = T(t_0,\epsilon)$ such that*

$$h_0(t_0,U_0) < \delta_0 \Rightarrow h(t,U(t)) < \epsilon, \quad t \geq t_0 + T.$$

.

Definition 11. *$(h_0,h)-$ uniformly attractive, if Definition 10 holds with δ_0 and T being independent of t_0.*

Definition 12. $(h_0, h)-$ *equi asymptotically stable if the Definition* 8 *and Definition* 10 *hold simultaneously.*

Definition 13. $(h_0, h)-$ *uniformly asymptotically stable if the Definition* 9 *and Definition* 11 *hold simultaneously.*

In order to use the method of Lyapunov function, it is necessary to select minimal subset of E over which the derivative of the Lyapunov function can be conveniently estimated. To define this set, we consider
$V \in C[\mathbb{R}_+ \times B_b, \mathbb{R}_+]$, where $B_b = B(\theta, b) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq b\}$

$$E_1 = \{U \in E; V(s, U(s)) \leq V(t, U(t)),\ t_0 \leq s \leq t\},$$

where $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+]$ is a Lyapunov function.

In order to discuss the stability properties of (9) let us assume that the solutions of (9) exist and are unique for all $t \geq t_0$. In addition, in order to match the behavior of solutions of (9) with those of the corresponding ordinary differential equations with causal map, we assume that $U_0 = V_0 + W_0$ so that Hukuhara difference $U_0 - V_0 = W_0$ exists. Consequently, in what follows, we consider the solutions $U(t) = U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$. Hence we have the initial value problem.

$$D_H U = (QU)(t) \qquad U(t_0) = W_0, \quad \textit{where} \ \ U_0 = V_0 + W_0 \tag{10}$$

This idea is clearly explained with example in the paper [7].
Now we will state the comparison result, which is analogous to the comparison result in [6].

Theorem 1. *Suppose that the following hypotheses hold*
(i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, *V(t,U) is locally Lipschitzian in U,*
(ii) for $t \geq t_0$ *and* $U \in E_1$,

$$D^+V(t, U(t)) \leq g(t, V(t, U(t))$$

Where

$$D^+V(t, U(t)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, U(t) + h(QU)(t)) - V(t, U(t))]$$

(iii) $r(t) = r(t, t_0, u_0)$ *is the maximal solution of the scalar differential equation*

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$ *existing on* $[t_0, \infty)$ *then if* $U(t; t_0, U_0)$ *is any solution of IVP (9) existing on* $[t_0, \infty)$,

$$V(t_0, U_0) \leq u_0 \ \ \textit{implies} \ \ V(t, U(t)) \ \leq \ r(t), \qquad t \geq t_0.$$

Corollary 1. *If, in addition to the assumptions of Theorem* 1 *with g(t,u)=0 and* $U(t) \in E_1$ *then* $V(t, U(t)) \leq V(t_0, U_0)$, $\ t \geq t_0$, *where* $U(t)$ *is any solution of IVP (9).*

Now we prove the main stability results for the system (9)

Theorem 2. *Suppose that the following hypothesis hold*
(i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+],\ h \in \Gamma,$ *V(t,U) is locally Lipschitzian in U and h-positive definite,*
(ii) $D^+V(t,U(t)) \leq 0$ *for* $(t,U) \in S(h,\rho)$ *where*

$$S(h,\rho) = \{(t,U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n),\ h(t,U) < \rho,\ \rho > 0\}$$

and $U \in E_1$ *then*
(A) if, in addition, $h_0 \in \Gamma$, h_0 *is finer than h and V(t,U) is* h_0*-weakly decrescent, then the system (9) is* (h_0,h)*- equistable.*
(B) if, in addition, $h_0 \in \Gamma$, h_0 *is uniformly finer than h and V(t,U) is* h_0*- decrescent, then the system (9) is* (h_0,h)*-uniformly stable.*

Proof. Given that V(t,U) is h_0 -weakly decrescent, then by the definition for $t_0 \in \mathbb{R}_+$, $W_0 \in K_c(\mathbb{R}^n)$, there exist constsnt $\delta_0 = \delta_0(t_0) > 0$ and a function $a \in \mathbb{CK}$ such that

$$V(t_0,W_0) \leq a(t_0,h_0(t_0,W_0)) \tag{11}$$

provided $h_0(t_0,W_0) < \delta_0$ *and* $W_0 = U_0 - V_0$. Also from hypothesis V(t,U) is h-positive definite, implies that there exist constant $\rho_0 \in (0,\rho)$ and a function $b \in \mathbb{K}$ such that

$$b(h(t,U)) \leq V(t,U) \ whenever\ h(t,U) \leq \rho_0 \tag{12}$$

and by the assumption that h_0 is finer than h, there exists a constant $\delta_1 = \delta_1(t_0) > 0$ and function $\phi \in \mathbb{CK}$ such that

$$h(t_0,W_0) \leq \phi(t_0,h_0(t_0,W_0))\ if\ h_0(t_0,W_0) < \delta_1, \tag{13}$$

where δ_1 is chosen so that $\phi(t_0,\delta_1) < \rho_0$.
Let $\epsilon \in (0,\rho_0)$ and $t_0 \in \mathbb{R}_+$ be given, then by the assumption on 'a', there exists a $\delta_2 = \delta_2(t_0,\epsilon) > 0$ that is continuous in t_0 such that

$$a(t_0,\delta_2) < b(\epsilon). \tag{14}$$

Now choose $\delta(t_0) = min\{\delta_0,\delta_1,\delta_2\}$ then $h_0(t_0,W_0) < \delta,\ where\ W_0 = U_0 - V_0$.

$$b(h(t_0,W_0)) \leq V(t_0,W_0) \leq a(t_0,h_0(t_0,W_0)) < b(\epsilon)$$

Hence $h(t_0,W_0) < \epsilon$.
Now we claim that for every solution $U(t) = U(t,t_0,W_0) = U(t,t_0,U_0 - V_0)$ of (10) with $h_0(t_0,W_0) < \delta$ implise

$$h(t,U(t)) < \epsilon,\ t \geq t_0, \tag{15}$$

Suppose (15) is not true, then there would exists a $t_1 > t_0$ such that

$$h(t_1,U(t_1)) = \epsilon, \quad and \quad h(t,U(t)) < \epsilon\ for\ t \in [t_0,t_1)$$

for some $U(t) = U(t, t_0, W_0)$ of (10).
Set $m(t) = V(t, U(t))$ for $t \in [t_0, t_1]$. Since V(t, U) is locally Lipschitzian in U, it follows from Corollory 1 that m(t) is non increasing in $[t_0, t_1]$ and $V(t, U(t)) \leq V(t_0, W_0)$, $\delta \leq t \leq t_1$. Thus it follows that

$$b(\epsilon) = b(h(t_1, U(t_1)) \leq V(t_1, U(t_1)) \leq V(t_0, W_0) < b(\epsilon),$$

which is a contraduction to (15) and hence the system (10) is (h_0, h)-equistable.
Now we prove the second part of the theorem.
Since $V(t, U)$ is h- positive definite, there exists a constant $0 < \rho_0 \leq \rho$, $0 < \delta_0$ and a function $a \in \mathbb{K}$ such that

$$b(h(t, U)) \leq V(t, U), \quad (t, U) \in S(h, \rho_0) \tag{16}$$

and since V(t,U) is h_0-decrescent, there exist constants $0 < \rho_0 \leq \rho$, $\delta_0 > 0$ *and* $b \in \mathbb{K}$, such that

$$V(t, U) \leq a(h_0(t, U)), \ if \ \ h_0(t, U) < \delta_0. \tag{17}$$

Also we have h_0 is uniformly finer than h, there exists a constant $\delta_1 > 0$, independent of 't' and a function $\phi \in \mathbb{CK}$ such that

$$h(t_0, W_0) \leq \phi(t_0, h_0(t_0, W_0)) \ \ if \ \ h_0(t_0, W_0) < \delta_1, \tag{18}$$

where δ_1 chosen so that $\phi(t_0, \delta_1) < \rho_0$.
Let $\epsilon \in (0, \rho_0)$ and $t_0 \in \mathbb{R}_+$ be given. By the assumption on 'a' there exists a $\delta_2 = \delta_2(\epsilon) > 0$ such that $a(\delta_2) < b(\epsilon)$. Choose $\delta = min\{\delta_0, \delta_1, \delta_2\}$ then

$$h_0(t_0, W_0) < \delta, \ where \ W_0 = U_0 - V_0. \tag{19}$$

We claim that for every solution $U(t) = U(t, t_0, W_0) = W(t, t_0, \ U_0 - V_0)$ of (10) with

$$h_0(t_0, W_0) < \delta \implies h(t, U(t)) < \epsilon, \quad t \geq t_0. \tag{20}$$

Suppose it is not true, then there would exists a $t_1 > t_0$ such that

$$h(t_1, U(t_1)) = \epsilon \ \ and \ \ h(t, U(t)) < \epsilon \quad for \ \ t \in [t_0, t_1)$$

for some $U(t) = U(t, t_0, W_0)$ of (10).
Set $m(t) = V(t, U(t))$ for $t \in [t_0, t_1)$. Since V(t, U) is locally Lipschitzian in U, it follows from Corollary 1 that m(t) is non increasing in $[t_0, t_1]$ and it follows that

$$b(\epsilon) = b(h(t_1, U(t_1)) \leq V(t_1, U(t_1)) \leq V(t_0, W_0) < b(\epsilon),$$

which is a contraduction to (15) hence the system (10) is (h_0, h)-uniformly stable.

Now we will prove uniform asymptotic stability result.

Theorem 3. *Suppose that*
(i) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than h; (ii)* $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+],\ h \in \Gamma$, *V(t,U) is locally Lipschitzian in U and h-positive definite,* h_0*-decrescent and*

$$D^+V(t,U(t)) \leq -c(h_0(t,U))\ for\ (t,U) \in S(h,\rho)$$

where $c \in \mathbb{K}$

$$S(h,\rho) = \{(t,U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), h(t,U) < \rho, \rho > 0\}$$

and $U \in E_1$ *then*
the system (10) *is* (h_0, h)*-uniformly asymptotically stable.*

Proof. Since $V(t,U)$ is h-positive definite and h_0 decrescent, there exists a constant $0 < \rho_0 \leq \rho$, and functions $a, b \in \mathbb{K}$ such that

$$b(h(t,U)) \leq V(t,U), \quad (t,U) \in S(h,\rho_0), \tag{21}$$

and

$$V(t,U) \leq a(h_0(t,U)), \quad if\ h_0(t,U) \leq \delta_0, \tag{22}$$

then by the Theorem 2 the system (10) is $(h_0.h)$- uniformly stable.
Now we will prove asymptotic stability.
Let $\epsilon = \rho_0$ then there exists $\delta_1 = \delta_1(\rho) > 0$ such that

$$h_0(t_0, W_0) < \delta_1 \Rightarrow h(t,U(t)) < \rho_0, \quad t \geq t_0, \tag{23}$$

where $U(t) = U(t, t_0, W_0)$ is any solution of (10).
Let $0 < \epsilon < \rho_0$ and $\delta = \delta(\epsilon)$ be the same δ as in the definition of (h_0, h)- uniform stability. Assume that $h_0(t_0, U_0) < \delta^* = min\{\delta_0, \delta_1\}$ and set $T = T(\epsilon) = \frac{a(\delta^*)}{c(\delta)} + 1$
To prove (h_0, h)- uniform asymptotic stability, it is enough to show that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$h_0(t^*, U(t^*)) < \delta, \tag{24}$$

Suppose (24) is not true, then there exists a solution $U(t) = U(t, t_0, W_0)$ of (10) with $h_0(t_0, W_0) < \delta^*$ such that

$$h_0(t, U(t)) \geq \delta, \quad t \in [t_0, t_0 + T]. \tag{25}$$

Let $m(t) = V(t, U(t))$, then it follows from hypothesis (ii) that

$$D^+m(t) \leq -c(h_0(t, U(t)), \qquad t \geq t_0. \tag{26}$$

Then from the equation (22) we get,

$$\int_{t_0}^{t_0+T} C(h_0(s, U(s))ds \leq m(t_0) \leq a(\delta^*),$$

also from (25) we have

$$\int_{t_0}^{t_0+T} C(h_0(s, U(s))ds \geq C(\delta)T > a(\delta^*)$$

which is a contraduction. Hence proof of the theorem is complete.

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