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Linear essential spectrum compressors

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Abstract. Let A be a unital C^*-algebra of real rank zero and B be a unital semisimple complex Banach algebra. We characterize linear maps from A onto B that compress different essential spectral sets such as the (left, right) essential spectrum, the semi-Fredholm spectrum, and the Weyl spectrum. Essentially spectrally bounded linear mappings from A onto B are also characterized.

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1. Introduction

Linear preserver problems is an active research area in matrix and operator theory. These problems involve linear or additive maps that leave invariant certain relations, or subsets, or functions. Over the past decades much work has been done on linear preserver problems on matrix or operator spaces. Often, the characterization of such linear preservers reveal the algebraic structures, in many cases, they are in fact Jordan homomorphisms; see surveys papers [3, 12, 14, 16, 19] and the references therein.

Throughout, A and B will denote infinite dimensional unital semisimple Banach algebras over the field $\mathbb{C}$ of complex number, unless specified otherwise. The unit is denoted by $\mathbf{1}$. A linear mapping $\varphi : A \to B$ is said to be Jordan homomorphism if $\varphi(a^2) = \varphi(a)^2$ for all $a \in A$, or equivalently

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$$

for all $a, b \in A$. Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. For further properties of Jordan homomorphisms, we refer the reader to [11, 13]. The map φ is said to be essentially spectrally bounded if there exists a positive constant M such that

$$r_e(\varphi(a)) \leq M r_e(a)$$

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for all $a \in A$, where $r_e(.)$ stands for the essential spectral radius. In [6], the authors characterized linear maps on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ that are essentially spectrally bounded, extending some recent results obtained in [5] concerning linear essential spectral radius (essential spectrum) preservers. They proved that a linear surjective up to compact operators map from $\mathcal{L}(\mathcal{H})$ into itself is essentially spectrally bounded if and only if it preserves the ideal of compact operators and the induced linear map on the Calkin algebra is either a continuous epimorphism or a continuous anti-epimorphism multiplied by a nonzero scalar. Recently, in [7], as a local version, the authors studied linear maps on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ that compress the local spectrum and the ones that are locally spectrally bounded.

The object of this note is to study essential spectrum compressors and essentially spectrally bounded linear maps between Banach algebras.

The paper is organized as follows. In the next section, we characterize essentially spectrally bounded linear maps from a unital purely infinite C^*-algebra of real rank zero onto a unital semisimple Banach algebra. In section 3 we describe linear maps φ from a C^*-algebra of real rank zero onto a semisimple Banach algebra that preserve Atkinson elements and the ones that compress different essential spectral sets such as the (left, right) essential spectrum, the semi-Fredholm spectrum. While the last section is devoted to the characterization of such maps φ that compress the Weyl spectrum.

2. Essentially spectrally bounded linear maps

First, let us recall the following useful facts about Fredholm theory in semisimple Banach algebras that will be often used in the sequel.

Let A be a semisimple Banach algebra. The socle of A, $\mathrm{Soc}\,(A)$, is defined to be the sum of all minimal left (or right) ideals of A. The ideal of inessential elements of A is given by

$$I(A) := \bigcap\{P : P \in \Pi_A : \mathrm{Soc}\,(A) \subseteq P\},$$

where Π_A denotes the set of all primitive ideals of A. It is a closed ideal of A. We call $\mathcal{C}(A) := A/I(A)$ the generalized Calkin algebra of A. It should be noted that a semisimple Banach algebra is finite dimensional if and only if it coincides with its socle; see for instance [2, Theorem 5.4.2]. Since our algebras are always supposed to be infinite dimensional, the generalized Calkin algebra introduced above is not trivial.

An element $a \in A$ is called left semi-Fredholm (resp. right semi-Fredholm) if it is left (resp. right) invertible modulo $\mathrm{Soc}\,(A)$, and is called Fredholm if it is invertible modulo $\mathrm{Soc}\,(A)$. The element a is said to be Atkinson if it is left or right semi-Fredholm. It is known that left (resp. right) invertible modulo $\mathrm{Soc}\,(A)$ is equivalent to left (resp. right) invertible modulo $I(A)$.

For every $a \in A$ we set

$$\sigma_e(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Fredholm}\},$$

$$\sigma_{le}(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not left semi-Fredholm}\},$$
$$\sigma_{re}(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not right semi-Fredholm}\},$$

and

$$\sigma_{SF}(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Atkinson}\}.$$

These are called respectively the essential spectrum, the left essential spectrum, the right essential spectrum, and the semi-Fredholm spectrum of a.

For an element $a \in A$, the essential spectral radius is defined by

$$r_e(a) := \max\{|\lambda| : \lambda \in \sigma_e(a)\}.$$

It coincides with the limit of the convergent sequence $(\|a^n\|_e^{\frac{1}{n}})_{n\geq 1}$, where $\|a\|_e := \|\pi(a)\|$ is the essential norm of a and π is the canonical projection from A onto $\mathcal{C}(A)$. We refer the reader to [17, 18] and the monographs [1, 4] for basic facts concerning Atkinson and Fredholm theory in Banach algebras.

A linear map $\varphi : A \to B$ is said to be surjective up to inessential elements if $B = \varphi(A) + I(B)$. It is called spectrally bounded if there exists a positive constant M such that $r(\varphi(a)) \leq Mr(a))$ for all $a \in A$, where $r(.)$ denotes the classical spectral radius.

The following, quoted in [6], is needed in what follows.

Lemma 1. *Let A be a unital purely infinite C^*-algebra with real rank zero and let B be a unital semi-prime Banach algebra. If $\varphi : A \to B$ be a surjective spectrally bounded linear map, then there exist a central invertible element c, viz., $\varphi(\mathbf{1})$, and a Jordan epimorphism $J : A \to B$ such that $\varphi(x) = cJ(x)$ for all $x \in A$.*

Proof. See [6, Lemma 2.1]

The problem of characterizing essentially spectrally bounded it seems to be difficult even when A and B are supposed to be C^*-algebras of real rank zero. Recall that a C^*-algebra A is of real rank zero if the set of all finite real linear combinations of orthogonal projections is dense in the set of all self adjoint elements of A; see [8]. However, We give a positive answer to this problem when A is a purely infinite C^*-algebra of real rank zero. A C^*-algebras A is purely infinite if it has no characters and if, for every pair of positive elements a, b in $\mathcal{A}$ with $a \in \overline{\mathcal{A}b\mathcal{A}}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $a = \lim_n x_n^* b x_n$; see [15].

The main result of this section is the following. It characterizes essentially spectrally bounded linear maps.

Theorem 1. *Let φ be a linear surjective up to inessential elements map from a purely infinite C^*-algebras with real rank zero A into semisimple a Banach algebra B. If φ is essentially spectrally bounded, then*

$$\varphi(I(A)) \subseteq I(B)$$

and the induced mapping $\widehat{\varphi}: \mathcal{C}(A) \to \mathcal{C}(B)$ *defined by*

$$\widehat{\varphi}(\pi(a)) := \pi(\varphi(a)),\ (a \in A),$$

is a continuous Jordan epimorphism multiplied by an invertible central element of $\mathcal{C}(B)$.

Proof. Assume that there is a positive constant M such that $r_e(\varphi(x)) \leq M r_e(x)$ for all $x \in A$. We first show that φ maps $I(A)$ into $I(B)$. So pick an inessential element $a \in I(A)$, and let us prove that $\varphi(a)$ is inessential as well. Let y be an arbitrary element in B and note that, since φ is surjective up to inessential elements, there exist $x \in A$ and $y_0 \in I(B)$ such that $y = \varphi(x) + y_0$. For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} r(\lambda\pi(\varphi(a)) + \pi(y)) &= r(\pi(\lambda\varphi(a) + y)) = r_e(\lambda\varphi(a) + y) \\ &= r_e(\varphi(\lambda a + x) + y_0) \\ &= r_e(\varphi(\lambda a + x)) \\ &\leq M r_e(\lambda a + x) = M r_e(x) = M r(\pi(x)). \end{aligned}$$

Since $\lambda \mapsto r(\lambda\pi(\varphi(a)) + \pi(y))$ is a subharmonic function on $\mathbb{C}$, Liouville's Theorem implies that

$$r(\pi(\varphi(a)) + \pi(y)) = r(\pi(y)).$$

As y is arbitrary in B, it follows from semi-simplicity of $\mathcal{C}(B)$ and the Zemánek's characterization of the radical [2, Theorem 5.3.1] that $\pi(\varphi(a)) = 0$ and $\varphi(a) \in I(B)$.

Therefore $\varphi(I(A)) \subseteq I(B)$, and φ induces a linear map $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ defined by $\widehat{\varphi}(\pi(x)) := \pi(\varphi(x))$ for all $x \in A$. The map $\widehat{\varphi}$ is obviously surjective and spectrally bounded. Note that, by [15, Proposition 4.3], the quotient of a C^*-algebra of real rank zero by a closed ideal is a C^*-algebra of real rank zero and the quotient of a purely infinite C^*-algebra by a closed ideal is a purely infinite C^*-algebra. Thus, the desired conclusion follows by applying Lemma 1.

For an infinite-dimensional complex Hilbert space $\mathcal{H}$, $\mathrm{Soc}\,(\mathcal{L}(\mathcal{H})) = \mathcal{F}(\mathcal{H})$ is the ideal of all finite rank operators on $\mathcal{H}$, $I(\mathcal{L}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$ is the closed ideal of all compact operators on $\mathcal{H}$, and the generalized Calkin algebra $\mathcal{C}(\mathcal{L}(\mathcal{H}))$ coincides with the usual Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and it is prime. Thus, a Jordan homomorphism $\widehat{\varphi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is either a homomorphism or an anti-homomorphism by [11].

More generally, if A is a C^*-algebra, then $\mathrm{Soc}\,(A)$ is the set of all finite rank element in A, and $I(A) = \overline{\mathrm{Soc}\,(A)} = \mathcal{K}(A)$, the set of all compact elements in A; see [4]. Recall that an element x of A is said to be finite (resp. compact) in A if the wedge operator $x \wedge x : A \to A$, given by $x \wedge x(a) = xax$, is a finite rank (resp. compact) operator on A. Note that even if the C^*-algebra A is prime, the generalized Calkin algebra $\mathcal{C}(A) = A/\mathcal{K}(A)$ is not necessary prime. For example, consider A the C^*-algebra generated by $\mathcal{K}(\mathcal{H})$ and two orthogonal infinite dimensional projections on a Hilbert space $\mathcal{H}$, and note that $\mathcal{C}(A) \cong \mathbb{C}^2$ is not prime. However, when A is factor, the ideal $\mathcal{K}(A)$ is the largest ideal of type I, and $\mathcal{C}(A)$ is a prime C^*-algebra.

Corollary 1. *Let φ be a surjective up to inessential elements linear map from a purely infinite C^*-algebra A with rank real zero into a factor B. Then the following assertions are equivalent.*

(i) φ is essentially spectrally bounded.

(ii) $\varphi(I(A)) \subseteq I(B)$ and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is either a continuous epimorphism or a continuous anti-epimorphism up to a nonzero complex scalar.

Proof. We only need to proof that (i)⇒(ii) as (ii)⇒(i) follows easily. If φ is essentially spectrally bounded then, by Theorem 1 and the fact that the center of $\mathcal{C}(B)$ is trivial, we infer that $\widehat{\varphi}$ is a continuous Jordan epimorphism multiplied by a nonzero complex number c. As the algebra $\mathcal{C}(B)$ is prime, then by [11] the map $\widehat{\varphi}$ is, in fact, either an epimorphism or an anti-epimorphism multiplied by c.

3. Atkinson elements preserving linear maps

Recall that a linear map φ from A into B is said to preserve Fredholm elements (resp. preserve Fredholm elements in both directions) if $\varphi(a)$ is a fredholm element whenever (resp. if and only if) a is. The map φ is said to compress the essential spectrum if $\sigma_e(\varphi(a)) \subseteq \sigma_e(a)$ for all $a \in A$. In a similar way, we define linear maps compressing different essential spectra or preserving left semi-Fredholm elements, right semi-Fredholm elements, and Atkinson elements.

The following describes linear mapping preserving Atkinson elements.

Theorem 2. *Let A be a C^*-algebra with real rank zero and B be a semisimple Banach algebra. Let $\varphi : A \to B$ be a surjective up to inessential elements linear map, and consider the following statements.*

(i) φ preserves Fredholm elements.

(ii) $\varphi(\mathbf{1})$ is a Fredholm element and φ preserves left Fredholm elements.

(iii) $\varphi(\mathbf{1})$ is a Fredholm element and φ preserves right Fredholm elements.

(iv) $\varphi(\mathbf{1})$ is a Fredholm element and φ preserves Atkinson elements.

If any of these statements holds, then $\varphi(I(A)) \subseteq I(B)$ and the induced mapping $\widehat{\varphi}$: $\mathcal{C}(A) \to \mathcal{C}(B)$ is a continuous Jordan epimorphism multiplied by an invertible element in $\mathcal{C}(B)$.

Proof. Note that, if any one of these statements holds then $\varphi(\mathbf{1})$ is a Fredholm element and so there exist $b \in B$ and $j, j' \in I(B)$ such that

$$b\varphi(\mathbf{1}) = \mathbf{1} - j \;\text{ and }\; \varphi(\mathbf{1})b = \mathbf{1} - j'.$$

For every $a \in A$, set $\psi(a) = b\varphi(a)$. The equality $b\varphi(\mathbf{1}) = \mathbf{1} - j$ implies that ψ compresses one of the essential spectral sets $\sigma_e(.)$, $\sigma_{le}(.)$, $\sigma_{re}(.)$, or $\sigma_{SF}(.)$; and so it is essentially spectrally bounded linear map since each of them contains the boundary of the essential spectrum. The same argument as in the proof of Theorem 1 entails that $\psi(I(A)) \subseteq I(B)$; which implies that $\varphi(I(A)) \subseteq I(B)$ since $b\varphi(i) = \psi(i) \in I(B)$ and

$$\varphi(i) = (\varphi(\mathbf{1})b + j')\varphi(i) = \varphi(\mathbf{1})b\varphi(i) + j'\varphi(i) \in J$$

for all $i \in I(A)$. As, by [9, Corollary 3.2], the induced map $\widehat{\psi}$ is a Jordan epimorphism; the proof is therefore complete.

As a consequences, we describe linear mapping compressing certain essential spectral sets.

Theorem 3. *Let A be a C^*-algebra with real rank zero and B be a semisimple Banach algebra. Let $\varphi : A \to B$ be a surjective up to inessential elements linear map, and consider the following statements.*

(i) φ compresses the essential spectrum.

(ii) φ compresses the left essential spectrum.

(iii) φ compresses the right essential spectrum.

(iv) φ compresses the semi-Fredholm spectrum.

If any of these statements holds, then $\varphi(I(A)) \subseteq I(B)$ and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is a continuous Jordan epimorphism multiplied by a central invertible element in $\mathcal{C}(B)$.

Proof. It is a direct consequence of the above Theorem.

One get the followings two corollaries.

Corollary 2. *Let A be a C^*-algebra with real rank zero and let B be a factor. A surjective up to inessential elements linear map $\varphi : A \to B$ compresses the essential spectrum (the semi-Fredholm spectrum) if and only if $\varphi(I(A)) \subseteq I(B)$, and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is either a continuous epimorphism or a continuous anti-epimorphism.*

Proof. Checking the "if" part is straightforward. So, assume that ϕ compresses the essential spectrum, and let us establish the "only if part". Note that, by the above theorem together with the same argument as in the proof of Corollary 1, $\varphi(I(A)) \subseteq I(B)$, and the induced map $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is either a continuous epimorphism or a continuous anti-epimorphism multiplied by a nonzero scalar c. Obviously $c = 1$ because

$$\{c\} = \sigma_e(\varphi(\mathbf{1})) \subset \sigma_e(\mathbf{1}) = \{1\}.$$

The case when φ compresses the semi-Fredholm spectrum is dealt with similarly.

Corollary 3. *Let φ be a surjective up to inessential elements linear map from a C^*-algebra A with real rank zero into a factor B. If A contains a semi-Fredholm element a such that $\varphi(a)$ is not Fredholm, then φ compresses the left essential spectrum (the right essential spectrum) if and only if $\varphi(I(A)) \subseteq I(B)$, and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is an continuous epimorphism.*

Proof. By similar arguments in the proof of the above Corollary, we only have to show that $\widehat{\varphi}$ cannot be an anti-epimorphism. So, assume that ϕ compresses the left essential spectrum, and suppose to the contrary that $\widehat{\varphi}$ is an anti-epimorphism. If the element a is left semi-Fredholm, then $\varphi(a)$ is also left semi-Fredholm, and there exists $b \in A$ such that $\pi(b)\pi(a) = \pi(\mathbf{1})$. Thus,

$$\pi(\mathbf{1}) = \widehat{\varphi}(\pi(b)\pi(a)) = \widehat{\varphi}(\pi(a))\widehat{\varphi}(\pi(b))$$

which show that $\varphi(a)$ is right semi-Fredholm. This implies that $\varphi(a)$ is Fredholm, which is a contradiction. By similarity, in the case when a is right semi-Fredholm we get a contradiction too.

Similar arguments yield that $\widehat{\varphi}$ cannot be an anti-epimorphism in the case when φ compresses the right semi-Fredholm spectrum; and the proof is complete.

4. Linear Weyl spectrum compressors

Now, let us recall the following useful facts about index theory in primitive Banach algebra that we will need in the sequel.

In what follows and unless otherwise specified, we assume that A is a primitive Banach algebra with nontrivial socle. Fix $e \in A$ a minimal idempotent (such an element exists if and only if $\mathrm{Soc}\,(A) \neq 0$), and let $\rho : A \to \mathcal{L}(Ae)$, defined as $\rho(a)(x) := ax\,(x \in Ae)$, denote the left regular representation of A on the Banach space Ae. For an element $a \in A$, we denote as usual the index of a by

$$\mathrm{ind}\,(a) := \dim(\ker(\rho(a)) - \mathrm{codim}\,(\rho(a)(Ae)).$$

It is well known that $\mathrm{ind}\,(a)$ is independent of the choice of e; see for instance [4]. The element a is said to be Weyl element if it is Fredholm with index zero. The Weyl spectrum of a is given by

$$W(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Weyl}\},$$

and it coincides with

$$\bigcap \{\sigma(a + b) : b \in I(A)\};$$

see [4].

Note that, for the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$, the set of all Fredholm operators strictly contains the set of Fredholm operators of zero index, but there exist infinite dimensional Banach spaces for which every Fredholm operator has index zero; see [10].

In the remainder of this paper we will assume that our primitive Banach algebras have non-trivial index function.

The following result describe linear maps φ from a purely infinite C^*-algebra with real rank zero onto a semisimple Banach algebras that compress the Weyl spectrum (i.e., $W(\varphi(a)) \subseteq W(a)$ for all $a \in A$).

Theorem 4. *Let φ be a linear surjective up to inessential elements map from a purely infinite C^*-algebras A with real rank zero into a primitive Banach algebra B. If φ compresses the Weyl spectrum, then*

$$\varphi(I(A)) \subseteq I(B)$$

and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is a Jordan epimorphism multiplied by an invertible central element of $\mathcal{C}(B)$.

Proof. Denote by ηK the polynomial convex hull of a compact subset K of $\mathbb{C}$. Since $\eta W(x) = \eta\sigma_e(x)$ for all $x \in A$ (see [17, Corollary 6.21]), it follows that the map φ is essentially spectrally bounded; and the desired conclusion follows from Theorem 1.

Recall that a Weyl operator on a complex Hilbert space is a Fredholm operator with zero index. As consequence of the above theorem, one gets the following result.

Corollary 4. *Let $\mathcal{H}$ and $\mathcal{H}'$ be infinite dimensional complex Hilbert spaces and let $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ be a surjective up to compact operators linear map. If φ compresses the Weyl spectrum, then*

$$\varphi(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H}')$$

and the induced mapping $\widehat{\varphi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}')$ is either a continuous epimorphism or a continuous anti-epimorphism.

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References

[1] P. Aiena. *Fredholm and local spectral theory, with applications to Multipliers.* Kluwer Academic Publishers, 2004.

[2] B. Aupetit. *A primer on spectral theory.* Springer-verlag, New York, 1991.

[3] B. Aupetit. *Spectrum-preserving linear map between Banach algebra or Jordan-Banach algebra.* J. London Math. Soc., 62(3):917-924, 2000 .

[4] B. A. Barnes, G. J. Murphy, M. R. F. Smyth, and T. T. West. *Riesz and Fredholm theory in Banach algebra.* Pitman, London, 1982.

[5] M. Bendaoud, A. Bourhim, and M. Sarih. *Linear maps preserving the essential spectral radius.* Linear algebra Appl., 428:1041-1045, 2008 .

[6] M. Bendaoud and A. Bourhim. *Essentially spectrally bounded linear maps.* Proc. Amer. Math. soc., 137(10):3329–3334, 2009.

[7] M. Bendaoud and M. Sarih . *Locally spectrally bounded linear maps.* Math. Bohem, 136(1):81–89, 2011.

[8] L. G. Brown and G. K. Pedersen. *C^*-algebras of rank real zero.* J. Funct. Anal., 99:131-149, 1991.

[9] J. Cui and J. Hou. *Linear maps between Banach algebras compressing certains spectral functions.* Rocky Mountain J. Math. Soc., 34(2):565-584, 2004.

[10] W. T. Gowers and B. Maurey. *The unconditional basic sequence problem.* J. Amer. Math. soc., 6:851-874, 1993.

[11] I. N. Herstein. *Jordan homomorphisms.* Trans. Amer. Math. Soc., 81(2):331-341, 1956.

[12] J .C. Hou. *Rank preserving linear maps on $B(X)$.* Sci. China Ser. 32(8):929-940, 1989.

[13] N. Jacobson and C. E. Rickart. *Jordan homomorphism of rings.* Trans. Amer. Math. Soc., 69:749-502, 1950.

[14] A. Jafarian and A. R. Sourour. *Spectrum preserving linear maps.* J. Funct. Anal., 66(2):255-261, 1986.

[15] E. Kirchberg and M. Rørdam. *Non-simple purely infinite C^*-algebras.* Amer. J. Math., 122(3): 637-666, 2000.

[16] C. K. Li and N. K. Tsing. *Linear preserver problems: a bref introduction and some special thechniques.* Linear Algebra Appl., 162/164:217-235, 1992.

[17] J. W. Rowell. *Unilateral Fredholm theory and unilateral spectra.* Proc. Roy. Irish. Acad., 84(1)69-85, 1984.

[18] C. Schmoger. *Atkinson theory and holomorphic functions in Banach algebras.* Proc. Roy. Irish. Acad., 91(1):113-127, 1991.

[19] A. R. Sourour. *Inversibility preserving linear maps on $\mathcal{L}(X)$.* Trans. Amer. Math. Soc., 348(1):13-30, 1996.