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Topp-Leone Inverse Weibull Distribution: Theory and Application

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Abstract. In this article, the discussion has been carried out through the generalization of Inverse Weibull distribution. We introduce a new three parameter life model called the Topp-Leone Inverse Weibull distribution. We provide comperhensive result of the mathematical characteristic, including moments, quantile function, random number generator, survival function, hazard rate function, and mode. Distributional properties of order statistics are analyzed. The parameters of the proposed model are estimated by the method of maximum likelihood. Simulation study is performed to investigate the performance of the maximum likelihood estimators. To assess the flexibility, empirical results of new model are obtained by modeling two real data sets.

Key Words and Phrases: Reliability Analysis, Order Statistics, Maximum Likelihood, Simulation Study

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1. Introduction

Recently, a considerable number of authors are generalizing classical distributions to extended their form which are more flexible to model real data. Inverse Weibull distribution has wider application in the field of reliability and biological studies due to its failure rate. Keller and Kanath [4] introduced the Inverse Weibull distribution to study the shape of the density and the failure rate function. The Inverse Weibull distribution provides a good fit of several data in terms of times to breakdown of an insulating fluid, the subject

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leaded to the action of constant tension, see Nelson [23]. Calabria and Pulcini [16] discussed the maximum likelihood and least squares estimation of its parameters. Calabria and Pulcini [17] considered Bayes 2-sample prediction of the distribution. Mahmoud et al. [10] discussed moments of order statistics of Inverse Weibull distribution and obtained BLUE (best linear unbiased estimator) for both location and scale parameters. Aleem and Pasha [11] derived single, product and ratio moment of Inverse Weibull Distribution. Aleem [12] worked on the product, ratio, and single moments of lower record values of Inverse Weibull distribution. Hanook *et al*. [21] derived Beta Inverse Weibull distribution. Shahbaz *et al*. [13] proposed the Kumaraswamy Inverse Weibull distribution using distribution function of kumaraswamy family of distributions.
Ali *et al*. [1] proposed Topp-Leone family of distribution. The distribution and density function of proposed family is given by

$$F_{TL-G}(y) = [G(y)]^{\alpha}[2 - G(y)]^{\alpha} = [1 - (\bar{G}(y))^2]^{\alpha} :, \quad x \in \Re, \ \ \alpha > 0, \tag{1}$$

the density of Toop-Leone family is

$$f_{TL-G}(y) = 2\alpha g(y)\bar{G}(y)[G(y)]^{\alpha-1}[2 - G(y)]^{\alpha-1}, \quad \alpha > 0,$$

or

$$f_{TL-G}(y) = 2\alpha g(y)\bar{G}(y)[1 - (\bar{G}(y))^2]^{\alpha-1}, \quad \alpha > 0, \tag{2}$$

where $g(y) = G'(y)$ and $\bar{G}(y) = 1 - G(y)$.
This present article is designed as follows; Section 2, we derive three parameter life model called Topp-Leone Inverse Weibull distribution, the pdf and cdf expansion. The main mathematical properties of the proposed model including, moments, survival function, hazard rate function, quantile function, and mode are discussed in Section 3. Section 4 is based on the distributional properties of order statistics. Estimation of parameters is determined in Section 5. To analyse the flexibility of maximum likelihood estimators, simulation study is provided in Section 6. In Section 7, we prove empirically that the proposed distribution is a very competitive model to other classical models by means of two real data sets. Finally, extensive concluding remarks are offered in Section 8.

2. The Topp-Leone Inverse Weibull Distribution

In this section, we derive three parameter Topp-Leone Inverse Weibull distribution. To construct the density and distribution function, consider pdf and cdf of Inverse Weibull distribution is given by

$$F(y) = e^{-\dfrac{\beta}{y^{\gamma}}}. \tag{3}$$

From above equation the density of Inverse Weibull distribution is given by

$$f(y) = \frac{\beta\gamma}{y^{\gamma+1}} e^{-\dfrac{\beta}{y^{\gamma}}}, \quad \beta, \gamma > 0, \ \ y \in \Re^{+}. \tag{4}$$

By inserting (3) and (4) into (1) and (2), we have cdf and pdf of the proposed model given by

$$F(y) = [1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^\alpha, \tag{5}$$

and

$$f(y) = \frac{2\alpha\beta\gamma}{y^{\gamma+1}} e^{-\frac{\beta}{y^\gamma}} (1 - e^{-\frac{\beta}{y^\gamma}})[1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^{\alpha-1}, \quad \beta, \gamma > 0 \;\; y \in \Re^+. \tag{6}$$

The proposed model have two shape and one scale parameter. We will use the notation $TLIW(\alpha, \beta, \gamma)$ to denote the density (6).

It is observed from (5) that the proposed model is a special case of exponentiated generalized class of distribution derived by Cordeiro *et al.* [7]. The proposed model of exponentiated generalized class of distribution is given by

$$F(x, \alpha, \beta) = [1 - (1 - G(x))^\alpha]^\beta, \qquad x \in \Re^+ \tag{7}$$

If we replace $\alpha = 2$, $\beta = \alpha$ and $e^{-\frac{\beta}{y^\gamma}}$ in (5), the above distribution converts to Topp-Leone Inverse Weibull distribution. Corderio *et al.* [7] used the method of adding parameter leads to the exponentiated type of distribution which was introduced by Lehmann [6] and studied by Nadarajah and Kotz [22]. Where Ali *et al.* [1] proposed Topp-Leone family of distribution by using survival function instead of distribution function. The proposed model provides some ideal sub models. For $\gamma = 1$ the proposed distribution in (5) converts to Topp-Leone Inverted Exponential distribution. For $\beta = 1$ and $\gamma = 1$ the distribution (5) reduces to Topp-Leone Standard Inverted Exponential distribution.

2.1. Shape

For real value of α, using following series representation of Prudnikov *et al.* [3]

$$(1 + x)^\alpha = \sum_{j=0}^{\infty} \frac{(1)^j \Gamma(\alpha + 1)}{j! \Gamma(\alpha + 1 - j)} x^j.$$

The cdf of $TLIW$ distribution given in (5) is expressed as infinite sum given as follows

$$F(y) = [1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^\alpha = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\alpha + 1)}{j! \Gamma(\alpha + 1 - j)} (1 - e^{-\frac{\beta}{y^\gamma}})^{2j},$$

or

$$F(y) = \sum_{j=0}^{\infty} \sum_{m=0}^{2j} (-1)^{j+m} \frac{\Gamma(\alpha + 1)}{j! \Gamma(\alpha + 1 - j)} \binom{2j}{m} [e^{-\frac{\beta}{y^\gamma}}]^m = \sum_{j=0}^{\infty} \sum_{m=0}^{2j} a(j, m) [e^{-\frac{\beta}{y^\gamma}}]^m, \tag{8}$$

where $a(j,m) = (-1)^{j+m} \dfrac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)} \binom{2j}{m}$.

Again for (6), follows series representation the density function of $TLIW$ distribution is written as follows

$$f(y) = \sum_{j=0}^{\infty} (-1)^j \frac{2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)} \frac{\beta\gamma}{y^{\gamma+1}} \{e^{-\frac{\beta}{y^\gamma}}\}\{1-e^{-\frac{\beta}{y^\gamma}}\}^{2j+1},$$

or

$$f(y) = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} (-1)^{j+m} \frac{2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)} \binom{2j+1}{m} \frac{\beta\gamma}{y^{\gamma+1}} \{e^{-\frac{\beta}{y^\gamma}}\}^{m+1} = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} b(j,m) h_{m+1}(y), \quad (9)$$

where $b(j,m) = (-1)^{j+m} \dfrac{\Gamma(\alpha)}{j!\Gamma(\alpha-j)(m+1)} \binom{2j+1}{m}$ and $h_{m+1}(y) = (m+1)\dfrac{\beta\gamma}{y^{\gamma+1}}\{e^{-\frac{\beta}{y^\gamma}}\}^{m+1}$ is exponentiated-G distribution with power function m.

The density and distribution function of $TLIW$ distribution are given (8) and (9) shows that the $TLIW$ distribution is expressed as weighted sum of exponentiated family of distribution.

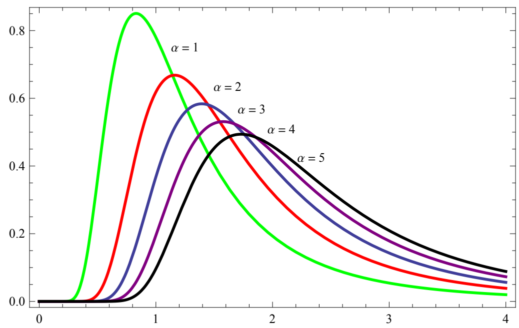

Figure 1: Graph for pdf of $TLIW$ for $\beta = 1.5$, $\gamma = 1.5$ and for different values of α

In Figure 1, we can see that for the lower values of α, peak are increased. For $\alpha > 1$ a slow decrease are observed.
In Figure 2, we can clearly see that at $\gamma = 3.5$ function shows high peak, but as the value of γ are decreasing a rapid change appears it starts decreasing but no change appears in the location of the curve.
Figure 3 shows the graphs for the different values of β and fixed values of α and γ.
Figure 4 shows the plot of probability density function of Topp-Leone Inverse Weibull distribution for the different values of α and β when γ is fixed. This plot shows the different shapes for different values of parameters. For this plot it is surely clear that Topp-Leone Inverse Weibull distribution is unimodel distribution.

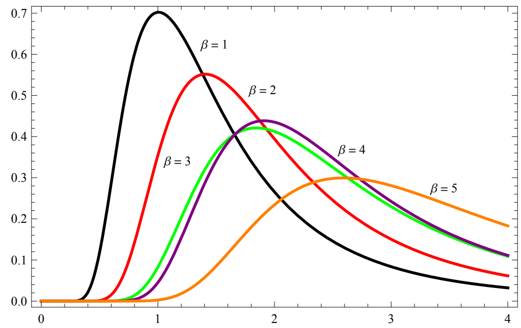

Figure 2: Graph for pdf of $TLIW$ for $\alpha = 2$, $\gamma = 1.5$ and for different values of β

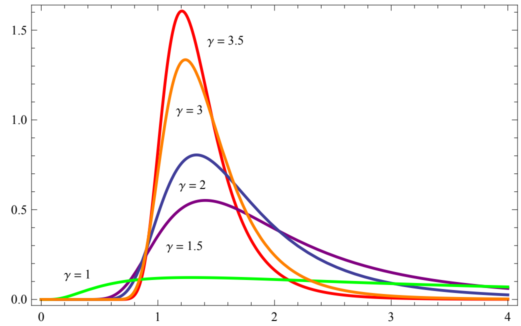

Figure 3: Graph for pdf of $TLIW$ for $\alpha = 2$, $\beta = 2$ and for different values of γ

3. Properties of TLIW Distribution

In this section, we discuss important and useful statistical characteristics of the proposed distribution.

3.1. Quantile and Median

The qth percentile of the distribution can be obtained by solving y_q for variable Y. The qth percentile is obtained by solving $Q_{(y)} = F(y)^{-1}$ as:

$$y_q = -\left(\frac{\beta}{\ln[1 - \sqrt{1 - q^{\frac{1}{\alpha}}}]}\right)^{\frac{1}{\gamma}}, \quad q > 0. \tag{10}$$

The median of the $TLIW$ distribution can be defined at $q = 0.5$. We can easily generate the random sample from (14) using q as uniform random number.

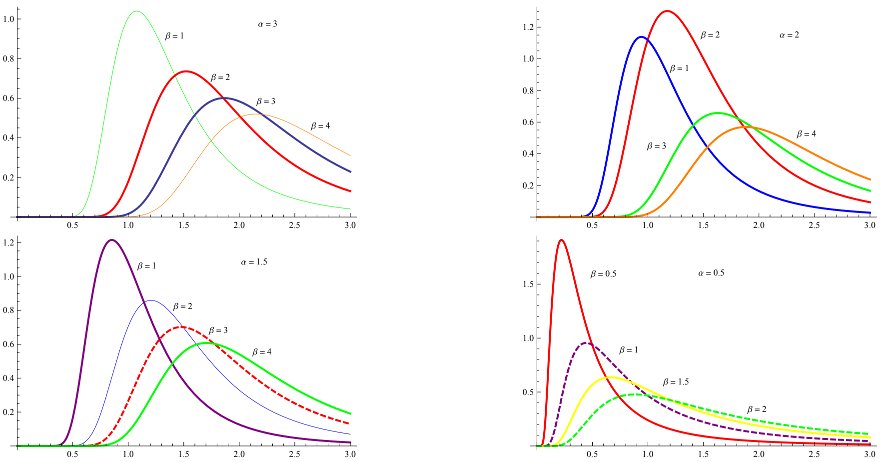

Figure 4: Graph for pdf of $TLIW$ for $\alpha = 0.5, 1.5, 2, 3$ and $\beta = 1, 2, 3, 4$ and $\gamma = 2$

3.2. Moments

The moments of Topp-Leone Inverse Weibull distribution is computed using following expression

$$\mu_r' = \int_0^\infty y^r F(y)dy = \int_0^\infty y^r \sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}(-1)^{j+m}\frac{2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)}\binom{2j+1}{m}\frac{\beta\gamma}{y^{\gamma+1}}\{e^{-\frac{\beta}{y^\gamma}}\}^{m+1}dy. \tag{11}$$

Making transformation as $z = \frac{(m+1)\beta}{y^r}$ in above expression to solve the moment of Topp-Leone Inverse Weibull distribution and result are given as follows

$$\mu_r' = \sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}(-1)^{j+m}\frac{2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)}\binom{2j+1}{m}\left(\frac{1}{\beta(m+1)}\right)^{1-\frac{r}{\gamma}}\Gamma(1-\frac{r}{\gamma}). \tag{12}$$

These moments are existing for $r < \gamma$. The coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) $TLIW$ distribution are obtained as follows

$$CV = \sqrt{\frac{\mu_2}{\mu_1} - 1}.$$

$$CS = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{(\mu_2-\mu_1)^{\frac{3}{2}}}.$$

$$CK = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2}{(\mu_2-\mu_1^2)^2}.$$

3.3. Reliability Analysis

The $TLIW$ distribution is used for describing a random lifetime in reliability analysis. The reliability analysis of the $TLIW$ distribution is denoted by $R(y)$, also known as survival function and obtained as follows

$$R(y) = 1 - F(y). \tag{13}$$

The survival function of $TLIW$ distribution is obtained by inserting (5) in to above expression (15) to attain the following results

$$R(y) = 1 - [1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^\alpha. \tag{14}$$

Note that $R(y) + F(y) = 1$. Hazard rate function is another characteristics in reliability

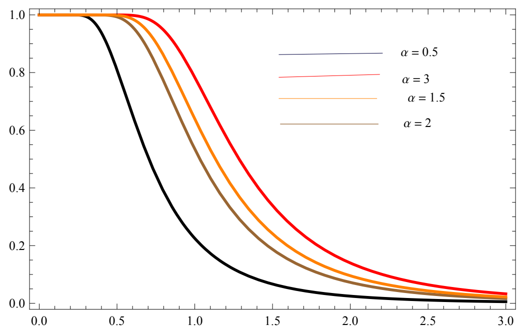

Figure 5: Graph for Survival function of $TLIW$ for $\beta = 1$ & $\gamma = 2$ and various values of α

analysis. It is denoted by $h(y)$ and define a follows

$$h(y) = \frac{\frac{2\alpha\beta\gamma}{y^{\gamma+1}} e^{-\frac{\beta}{y^\gamma}} (1 - e^{-\frac{\beta}{y^\gamma}})[1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^{\alpha-1}}{1 - [1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^\alpha} \tag{15}$$

The units for $h(y)$ is the probability of failure per unit of distance and time. We define these failure rates at the different values of parameters. The cumulative hazard rate function of $TLIW$ distribution is represented as $H(y)$ and the result obtained are given as follows

$$h(y) = -log|1 - [1 - \{1 - e^{-\frac{\beta}{y^\gamma}}\}^2]^\alpha|. \tag{16}$$

$h(y)$ unite is the cumulative probability of failure per unit of time, distance or cycles. The distribution has decreasing cumulative instantaneous failure rate for all choices of parameters.

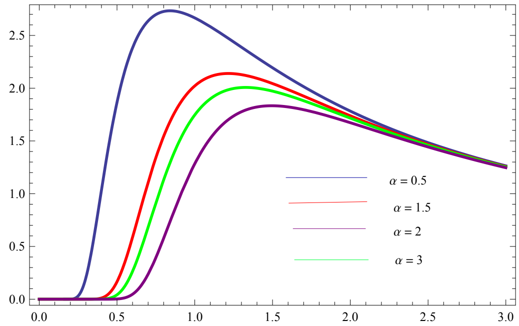

Figure 6: Graph for Hazard rate function of $TLIW$ for $\beta = 1$ & $\gamma = 2$ and various values of α

3.4. Mode

We consider the density function of $TLIW$ distribution given in (6) and solve $\frac{\partial \ln f(y)}{\partial y} = 0$ for y, to obtain the mode of Topp-Leone Inverse Weibull distribution as follows

$$\frac{\partial \ln f(y)}{\partial y} = \frac{\beta\gamma}{y} - \frac{e^{-\beta y^{-\gamma}} y^{-(1+\gamma)}\beta\gamma}{1 - e^{-\beta y^{-\gamma}}} + \frac{2e^{-\beta y^{-\gamma}}(1 - e^{-\beta y^{-\gamma}})y^{-(1+\gamma)}\beta\gamma}{1(1 - e^{-\beta y^{-\gamma}})^2} + \frac{1+\gamma}{y}.$$

By putting $\frac{\partial \ln f(y)}{\partial y} = 0$, we have:

$$\frac{\beta\gamma}{y} - \frac{e^{-\beta y^{-\gamma}} y^{-(1+\gamma)}\beta\gamma}{1 - e^{-\beta y^{-\gamma}}} + \frac{2e^{-\beta y^{-\gamma}}(1 - e^{-\beta y^{-\gamma}})y^{-(1+\gamma)}\beta\gamma}{1(1 - e^{-\beta y^{-\gamma}})^2} + \frac{1+\gamma}{y} = 0 \qquad (17)$$

The maxima can be obtained by solving (17) iteratively.

4. Order Statistics

Order statistics is used in the field of reliability and life testing widely. Let $X_1, X_2, ..., X_n$ be a simple random sample from $TLIW(\alpha, \beta, \gamma)$ with distribution and density functions given in (5) and (6). Let $X_{(1:n)} \leq X_{(2:n)}l........ \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(j:n)}$ is used to model the lifetime of an $(ni+1)$-out-of-n system which consists of n independent and identically distributed components. The density function of $X_{(i:n)}$, $1 \leq k \leq n$ is given as follows:

$$f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!}[F_{EGWE}(x)]^{i-1}[1 - F_{EGWE}(x)]^{n-i} f_{EGWE}(x)$$

The first order statistic is given by $X_{(1)} = min(X_1, X_2, ...X_n)$ and the last order statistics is given by $X_{(n)} = max(X_1, X_2, ...X_n)$. The distribution of first order statistics is given by

$$f_{1:n}(y) = 2n\alpha\beta\gamma \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(n)}{j!\Gamma(n-j)} [1 - (1 - e^{-\frac{\beta}{y^\gamma}})^2]^{\alpha j + \alpha - 1} y^{-(\gamma+1)} e^{-\frac{\beta}{y^\gamma}} (1 - e^{-\frac{\beta}{y^\gamma}}). \qquad (18)$$

The distribution of the nth order statistics is given by

$$f_{n:n}(y) = 2n\alpha\beta\gamma y^{-(\gamma+1)} e^{-\frac{\beta}{y^\gamma}}(1-e^{-\frac{\beta}{y^\gamma}})[1-(1-e^{-\frac{\beta}{y^\gamma}})^2]^{n+\alpha-2} \tag{19}$$

5. Parameters Estimation and Fisher Information Matrix

In this section, we derive the maximum likelihood estimates (MLE) and inference for unknown parameters of Topp-Leone Inverse Weibull distribution. Let $y_1, y_2........y_n$ be a realization of a random sample of size n from $TLIW$ distribution than the likelihood function is written as follows

$$LF = L(\alpha, \beta, \gamma|y_i) = \prod_{i=0}^{n} F(y_i),$$

the log-likelihood function is given as follows

$$\begin{aligned}\ln(LF) =& n\ln(2) + n\ln(\alpha) + n\ln(\beta) + n\ln(\gamma) - (\gamma+1)\sum_{j=1}^{n}\ln(y_j) - \beta\sum_{j=1}^{n}\ln(y_j^{-\gamma}) + \\ &\sum_{j=1}^{n}\ln(1-e^{-\frac{\beta}{y_j^\gamma}}) + (\alpha-1)\sum_{j=1}^{n}\ln(1-\{1-e^{-\frac{\beta}{y_j^\gamma}}\}^2),\end{aligned} \tag{20}$$

differentiating (22) w.r.t α, β, γ, and equating them 0, we have

$$\frac{n}{\alpha} + \sum_{j=1}^{n}\ln\left(1-w^2\right) = 0, \tag{21}$$

$$\frac{n}{\beta} + \sum_{j=1}^{n}\frac{y_j^{-\gamma}e^{-\frac{\beta}{y_j^\gamma}}}{w} - \sum_{j=1}^{n}\ln(y_j^{-\gamma}) - (\alpha+1)\sum_{j=1}^{n}\frac{2y_j^{-\gamma}(w)e^{-\frac{\beta}{y_j^\gamma}}}{1-w^2} = 0, \tag{22}$$

$$\begin{aligned}&\frac{n}{\gamma} + \beta\sum_{j=1}^{n}\ln(y_j) - \sum_{j=1}^{n}\ln(y_j) - \sum_{j=1}^{n}\frac{y_j^{-\gamma}\beta\ln(y_j)e^{-\frac{\beta}{y_j^\gamma}}}{w} + \\ &(\alpha+1)\sum_{j=1}^{n}\frac{2y_j^{-\gamma}\beta\ln(y_j)we^{-\frac{\beta}{y_j^\gamma}}}{1-w^2} = 0,\end{aligned} \tag{23}$$

where $(1-e^{-\frac{\beta}{y_j^\gamma}}) = w$. The maximum likelihood estimate of α, β, and γ are obtained iteratively solving (21), (22), and (23), simultaneously. The fisher information matrix for the parameters of the $TLIW$ distribution is obtained as follows

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\gamma}\end{pmatrix} \sim N\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.\begin{pmatrix}\hat{J}_{\alpha\alpha} & \hat{J}_{\alpha\beta} & \hat{J}_{\alpha\gamma}\\ & \hat{J}_{\beta\beta} & \hat{J}_{\beta\gamma}\\ & & \hat{J}_{\gamma\gamma}\end{pmatrix}\right),$$

$$\frac{1}{J} = -E \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\gamma} \\ & J_{\beta\beta} & J_{\beta\gamma} \\ & & J_{\gamma\gamma} \end{pmatrix}.$$

By determining the inverse dispersion matrix, the asymptotic variances and covariances of the ML estimators for α, β, and γ may be obtained. Using above, approximate $100(1-\lambda)\%$ confidence intervals for α, β, and γ are determined respectively as follows

$$\hat{\alpha} \pm Z_{\frac{\lambda}{2}} \sqrt{\hat{J}_{\alpha\alpha}}, \quad \hat{\beta} \pm Z_{\frac{\lambda}{2}} \sqrt{\hat{J}_{\beta\beta}}, \quad \hat{\gamma} \pm Z_{\frac{\lambda}{2}} \sqrt{\hat{J}_{\gamma\gamma}}, \tag{24}$$

where Z_{γ} is demonstrated the upper $100_{\gamma th}$ quantile of the standard normal distribution.

6. Simulation Study

In this section of article, we discuss some simulations for different sample size to determine the efficiency of MLEs. The different methods have been derived for simulating a random variable like the inversion method, the rejection, acceptance sampling techniques, and many more from different probability distributions in the field of computational statistics. The Inversion method is considered the most powerful technique. We can simulate random variable Y given by

$$y = -\left(\frac{\beta}{\ln[1 - \sqrt{1 - U^{\frac{1}{\alpha}}}]} \right)^{\frac{1}{\gamma}},$$

where U is uniform random number in (0,1). We generate sample of size $n = 50, 100, 200, 500,$ 1000 from $TLIW$ distribution for some selected combination of parameters. This process is repeated $N = 1000$ time to calculate mean estimate and means squared error. Obtained results are given in following tables.

It is observed that when sample size increases the mean squared error decreases. Therefore, the maximum likelihood method works very well to estimate the parameters of $TLIW$ distribution.

7. Application

In this section, we provide the application with real data sets to assess the flexibility of $TLIW$ distribution. The parameters are estimated using maximum likelihood method and R software is used for computation. We describe data sets to find the MLEs of the parameters. To assess the fitness of the real data for proposed distribution, we apply goodness-of-fit tests. The log-likelihood values for different models are obtained to decide the best model.

Table 1: Estimated Mean and MSEs of TLIW distribution

	α=2.5	β=2.0	γ=1.5
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
50	2.5073	2.0288	1.5741
	0.6507	0.4327	0.3810
100	2.5033	2.0118	1.8671
	0.3684	0.4463	0.3544
200	2.5018	2.1605	1.8339
	0.3582	0.3758	0.3673
500	2.5014	2.3003	1.7115
	0.2661	0.3688	0.3637
1000	2.5009	2.0421	1.9818
	0.2373	0.2884	0.2906

Table 2: Estimated Mean and MSEs of TLIW distribution

	α=1.5	β=2.5	γ=2.0
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
50	1.5111	2.8992	2.2547
	0.5547	0.4081	0.3601
100	1.5017	2.9065	2.1330
	0.4977	0.4028	0.4072
200	1.5033	2.8129	2.1991
	0.4018	0.4181	0.3456
500	1.5006	2.7396	2.1248
	0.3475	0.2794	0.3011
1000	1.5006	2.6807	2.1041
	0.2536	0.2679	0.2137

Data Set 1:

For getting the performance of the proposed model a data related to influence of physiographic and historical factors on species richness of native and non-native vascular plants on 22 coastal islands is selected. Different variables are effecting on the richness. We select variable area (hectares) having values 3, 4, 4, 8, 10, 34, 40, 46, 47, 61, 128, 140, 350, 1190, 1350, 1900, 2300, 2707, 10900, 13600, 13600 and 26668.
It is depicted from the results of Table 7 that our proposed model provide best fit than recent developed models. It is be more reliable with these types of data.
From Figure. 7, we see that the data provides best fitting for proposed distribution.

Data Set 2:

This data set consists of the waiting times(in seconds), between 65 successive eruptions of the Kiama Blowhole. These values were recorded with the aid of digital watch on July

Table 3: Parameter Estimation for Various Distributions

Model	parameters			LL
	α	β	γ	
$TLIW$	4.597689	0.4233699	0.20622405	-73.86899
EE	0.328589	0.02146033		-76.14519
EW	2.07528	1.533359	0.2841065	-96.15165
IE	4.597689	0.42333699		-249.7032

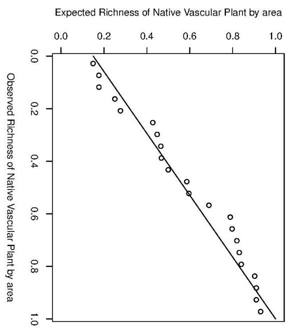

Figure 7: Goodness of Fit

12, 1998 by Jim Irish and has been referenced by several authors including da Silva, Thiago, Maciel, Campos and Cordeiro [8] and Pinho, Cordeiro and Nobre [9]. The actual data are: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 4: Parameter Estimation for Various Distributions

Model	parameters			LL
	α	β	γ	
$TLIW$	4.597689	0.4233699	0.2062405	-54.82303
EW	2.075585	1.532594	0.291697	-166.880

In Table 8, the value of log-likelihood of $TLIW$ distribution is minimum than other existing distributions, which indicates that new model is better.
The data of waiting time of customers are also provides better fit to follow the curve.

8. Conclusion

We derive a three parameter Topp-Leone Inverse Weibull distribution. Some of desirable properties are computed. We study the distributional properties of order statistics. The parameters are estimated by method of maximum likelihood. Performance of MLE's are tested through simulation study. Finally, two real data applications are analysed to assess the flexibility of new model over existing distribution. It is significantly observed that the proposed model provides better result than derived models.

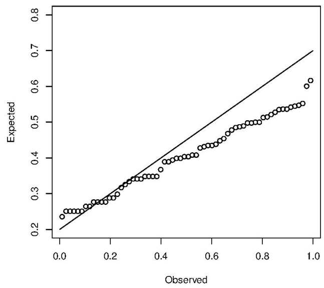

Figure 8: Goodness of Fit

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Appendix

Table 5: Mean of Toop-Leone Inverse Weibull distribution for different values of parameters

		β					
α	γ	4	5	6	7	8	9
1	1	0.125	0.221556731	0.281270921	0.320461669	0.347920893	0.368165662
1	2	0.08	0.158533092	0.208887035	0.242459148	0.266188013	0.283779878
1	3	0.055555556	0.120600209	0.163808521	0.193046537	0.213880406	0.229405316
1	4	0.040816327	0.095703512	0.133374899	0.159213001	0.177760331	0.191645616
1	5	0.03125	0.078332134	0.111622439	0.134737534	0.151441365	0.163999158
1	6	0.024691358	0.065646439	0.095399949	0.116291492	0.131480543	0.142943258
2	1	0.097222222	0.228051309	0.316474846	0.376617996	0.419571752	0.45161973
2	2	0.062222222	0.163180233	0.235031378	0.284946648	0.321006795	0.348105772
2	3	0.043209877	0.124135409	0.184310829	0.226875184	0.257926955	0.281405839
2	4	0.031746032	0.098508906	0.150068128	0.187112804	0.214368309	0.235086947
2	5	0.024305556	0.080628313	0.125593126	0.158348361	0.182629212	0.201173719
2	6	0.01920439	0.067570758	0.107340226	0.136669914	0.158557657	0.175344966
3	1	0.087916667	0.238778166	0.347213665	0.422774371	0.477437645	0.518548753
3	2	0.056266667	0.170855748	0.257859691	0.319868251	0.365278948	0.399694261
3	3	0.039074074	0.129974371	0.202212717	0.254679846	0.293499353	0.323109548
3	4	0.028707483	0.103142473	0.164644064	0.210044392	0.243933249	0.269926302
3	5	0.021979167	0.08442083	0.137791834	0.177754726	0.207816806	0.23098721
3	5	0.017366255	0.070749086	0.117766053	0.153419479	0.180425384	0.201330694
4	1	0.083129252	0.248977245	0.373566076	0.461917367	0.526433444	0.575236864
4	2	0.053202721	0.178153614	0.27743042	0.349483579	0.402764752	0.443389116
4	3	0.036946334	0.135526046	0.217560017	0.278259639	0.323618963	0.358432109
4	4	0.027144245	0.107548061	0.177140024	0.229491566	0.268966266	0.299434833
4	5	0.020782313	0.088026749	0.148249795	0.194212329	0.229143466	0.256238892
4	6	0.016420593	0.073771036	0.126704121	0.167623978	0.198941071	0.223340305
5	1	0.080157943	0.258247777	0.39666873	0.496177192	0.569372114	0.624992261
5	2	0.051301083	0.184787067	0.294587704	0.375404333	0.435616356	0.481740277
5	3	0.035625752	0.140572285	0.231014702	0.298897803	0.350015021	0.389434872
5	4	0.026174022	0.111552554	0.188094993	0.246512665	0.290904564	0.325334597
5	5	0.020039486	0.091304377	0.15741809	0.208616811	0.247833607	0.278402402
5	6	0.015833668	0.07651786	0.134539954	0.180056436	0.215167746	0.242658235
6	1	0.078108153	0.26666609	0.417325949	0.526846126	0.607887428	0.669699166
6	2	0.049989218	0.190810721	0.309928874	0.398608242	0.465083729	0.51620009
6	3	0.034714735	0.145154634	0.243045198	0.31737281	0.373691871	0.417291901
6	4	0.025504703	0.115188924	0.197890369	0.261749722	0.310582873	0.34860641
6	5	0.019527038	0.0942807	0.165615913	0.221511509	0.264598371	0.298317064
6	6	0.015428771	0.079012175	0.141546358	0.191185805	0.229722821	0.260016048

Table 6: Variance of Toop-Leone Inverse Weibull distribution for different values of parameters

		β					
α	γ	4	5	6	7	8	9
1	1	0.001953125	0.004661601	0.005216654	0.004891093	0.004335786	0.003775348
1	2	0.001152	0.003083723	0.003646001	0.003521791	0.003180363	0.002804469
1	3	0.000685871	0.002115403	0.002640853	0.002624235	0.002411365	0.00215147
1	4	0.000424993	0.001506302	0.001977551	0.002016728	0.001882668	0.001697738
1	5	0.000274658	0.001108364	0.001523567	0.001590785	0.001506476	0.00137166
1	6	0.000184404	0.000838659	0.001202209	0.001282461	0.00123043	0.001130183
2	1	0.002492311	0.008558125	0.010981446	0.01103603	0.010200011	0.00913207
2	2	0.00118062	0.00509626	0.007082722	0.00741353	0.007023159	0.006392909
2	3	0.000623835	0.003274471	0.004870687	0.005277731	0.00510592	0.00471372
2	4	0.000358556	0.002228641	0.003515642	0.003925325	0.003867254	0.0036141
2	5	0.000220038	0.001586127	0.002634618	0.003020203	0.003023556	0.002856213
2	6	0.000142254	0.001169653	0.002034192	0.002387347	0.00242439	0.002312322
3	1	0.002475099	0.011379596	0.016066296	0.016927302	0.016091374	0.014677856
3	2	0.001117531	0.006565329	0.010091408	0.011102363	0.010834666	0.010058523
3	3	0.000573636	0.004131432	0.006816692	0.007775728	0.007756672	0.007308022
3	4	0.000323372	0.002770284	0.004856429	0.00571401	0.005808386	0.005542175
3	5	0.000195709	0.001949481	0.003603018	0.004355608	0.004501103	0.004342689
3	6	0.000125211	0.001424863	0.002759653	0.003417197	0.003583404	0.003491525
4	1	0.002426513	0.013752778	0.02079772	0.022671748	0.021994874	0.020335553
4	2	0.001072231	0.007806363	0.012879095	0.014676978	0.014627679	0.013771144
4	3	0.000543067	0.00485898	0.008615332	0.010186439	0.010382128	0.009922473
4	4	0.000303361	0.003232387	0.00609377	0.007435109	0.007724326	0.00747801
4	5	0.000182388	0.002260907	0.004495791	0.005637689	0.00595556	0.005830823
4	6	0.000116105	0.001644537	0.003427988	0.004404187	0.00472185	0.004669276
5	1	0.002383433	0.015857995	0.02528833	0.028305203	0.027897746	0.026065817
5	2	0.001040135	0.008909254	0.015516512	0.018167471	0.018402466	0.017513043
5	3	0.000522688	0.005506887	0.010313572	0.012533485	0.012986384	0.012548475
5	4	0.000290405	0.003644747	0.007260406	0.009107116	0.00962012	0.009417354
5	5	0.000173912	0.002539342	0.005336683	0.006881104	0.007391947	0.007318602
5	6	0.000110377	0.001841286	0.004056991	0.005360118	0.00584439	0.005844781
6	1	0.0023482	0.017777048	0.029595667	0.033846826	0.033792679	0.03184612
6	2	0.001016344	0.009915348	0.018039936	0.021589905	0.022158922	0.021273907
6	3	0.000508067	0.006098486	0.011935698	0.01482955	0.015571548	0.015180957
6	4	0.000281262	0.00402163	0.008373418	0.010740034	0.011498479	0.011357662
6	5	0.000167991	0.002794055	0.006138195	0.008093839	0.008813009	0.008804798
6	6	0.000106404	0.002021425	0.004656108	0.006291462	0.006953607	0.007017534

Table 7: Coefficient Skewness Table of Toop-Leone Inverse Weibull distribution for different values of parameters

		β					
α	γ	4	5	6	7	8	9
1	1	0.595170064	0.44020366	0.355459867	0.296998208	0.254493126	0.222361967
1	2	0.582377652	0.458810019	0.373560513	0.312841916	0.26825751	0.234420311
1	3	0.595170064	0.47859545	0.391051237	0.327786601	0.281104898	0.245608469
1	4	0.611011148	0.496551293	0.406805639	0.341251432	0.292686152	0.25569677
1	5	0.625259785	0.512279767	0.420789994	0.353286558	0.303076239	0.264766849
1	6	0.63718253	0.525966374	0.433203744	0.364068159	0.312429005	0.272954236
2	1	0.61181318	0.491730312	0.399990748	0.334114492	0.285803679	0.249242975
2	2	0.636540479	0.519239851	0.424383364	0.355018962	0.303788457	0.264901376
2	3	0.653449639	0.540633294	0.444183098	0.372308778	0.318809681	0.278054474
2	4	0.665160849	0.557618288	0.460579618	0.386891791	0.331601108	0.289318072
2	5	0.673507548	0.571399058	0.474418518	0.399413524	0.342683707	0.299128606
2	6	0.679630911	0.582797364	0.48629096	0.410328964	0.352425876	0.307795142
3	1	0.64455463	0.525932188	0.42920009	0.358551554	0.306481958	0.267030449
3	2	0.663187691	0.551736925	0.453570335	0.379982246	0.32515735	0.283407843
3	3	0.674746479	0.570998667	0.472801624	0.397306103	0.340444782	0.296913327
3	4	0.682333473	0.585910543	0.488451014	0.41171273	0.353303161	0.308348995
3	5	0.687557619	0.597799548	0.501499074	0.423962495	0.364349609	0.318232934
3	6	0.691298849	0.607505692	0.512591088	0.434563263	0.373999243	0.326915003
4	1	0.661850384	0.547910128	0.449100662	0.375623929	0.321119419	0.279720092
4	2	0.6760347	0.571771624	0.472867786	0.396987389	0.339943143	0.296332247
4	3	0.684527968	0.589268286	0.491389767	0.414082734	0.355216045	0.309920793
4	4	0.689984409	0.602654923	0.506337079	0.428204846	0.367988327	0.321366816
4	5	0.693687521	0.613237061	0.518724385	0.440154911	0.37891557	0.331223044
4	6	0.696311928	0.621819793	0.529205403	0.450458034	0.388431041	0.339856478
5	1	0.672058453	0.563383103	0.463827394	0.388529415	0.332308138	0.2894842
5	2	0.683344122	0.585604157	0.486944539	0.409687845	0.351123296	0.306176831
5	3	0.689975891	0.601727657	0.504826258	0.426517997	0.366310149	0.31976694
5	4	0.694186506	0.613975473	0.519183129	0.440364459	0.378966181	0.331178398
5	5	0.697021075	0.623606163	0.531035751	0.452045938	0.389766266	0.340982479
5	6	0.699018113	0.631385074	0.541034307	0.462093722	0.399152327	0.349555166
6	1	0.678684577	0.575005194	0.475356676	0.398813552	0.341307364	0.297380969
6	2	0.687993054	0.595871432	0.497868403	0.419734508	0.360056827	0.31409033
6	3	0.69339867	0.610904142	0.51519317	0.436308022	0.375137626	0.327651048
6	4	0.696805022	0.622266808	0.529053126	0.449904702	0.387674997	0.339013353
6	5	0.699086216	0.631168414	0.540464558	0.461351002	0.39835462	0.34875972
6	6	0.700687231	0.638337579	0.55007028	0.471179891	0.407622908	0.357271411

Table 8: Coefficient Kurtosis Table of Toop-Leone Inverse Weibull distribution for different values of parameters

		β					
α	γ	4	5	6	7	8	9
1	1	2.21799308	1.966578604	1.862381523	1.80009056	1.760174038	1.73327196
1	2	2.16901906	1.989904972	1.883432354	1.816402864	1.772737797	1.74310779
1	3	2.181110946	2.016869168	1.90481036	1.832527085	1.785022917	1.752667989
1	4	2.202955207	2.042152468	1.924734817	1.847602602	1.796534824	1.761637457
1	5	2.223602742	2.064704401	1.942880501	1.861487726	1.807202265	1.769977724
1	6	2.24096953	2.084540107	1.959311759	1.874237491	1.817070246	1.777725965
2	1	2.204489811	2.036104942	1.916580368	1.839858533	1.78981241	1.755946971
2	2	2.240606729	2.075608857	1.94820648	1.863880822	1.808153183	1.770216939
2	3	2.264802891	2.10672518	1.974688289	1.884564185	1.824177496	1.782790477
2	4	2.281082808	2.131525557	1.997085128	1.902538595	1.838305036	1.79396941
2	5	2.29236257	2.151632244	2.016268719	1.918330963	1.850887744	1.804006191
2	6	2.300429861	2.168208738	2.032900341	1.932349445	1.862200357	1.813098363
3	1	2.252468424	2.085733006	1.954876814	1.868222783	1.811079219	1.772270322
3	2	2.278579891	2.123307968	1.987756083	1.894151879	1.831246757	1.788122103
3	3	2.294145059	2.151374944	2.014300699	1.915844774	1.848442191	1.801788923
3	4	2.304018064	2.173008331	2.036215819	1.934338592	1.863357444	1.81376549
3	5	2.310626055	2.190142298	2.054660701	1.95036039	1.876483991	1.824405257
3	6	2.315248396	2.204025169	2.070437485	1.964427808	1.888176302	1.833964757
4	1	2.276880262	2.11795912	1.981834288	1.888905862	1.826872113	1.784520163
4	2	2.295923215	2.152696196	2.014564254	1.91555608	1.847940813	1.801233494
4	3	2.306864739	2.178027155	2.040520928	1.937535073	1.865686459	1.815487319
4	4	2.313668839	2.197244192	2.061686867	1.956085192	1.88094743	1.827883598
4	5	2.318168782	2.212292909	2.079337196	1.972033938	1.894291198	1.838833043
4	6	2.321292188	2.224382876	2.094325414	1.985952959	1.906115698	1.848625904
5	1	2.290740331	2.140641649	2.00214852	1.904969444	1.83933291	1.794274901
5	2	2.305401566	2.172858021	2.034373807	1.931912495	1.86092724	1.811540645
5	3	2.313681404	2.196018321	2.059645978	1.953931371	1.878974432	1.826163993
5	4	2.318782032	2.213421623	2.080091359	1.972393941	1.894407978	1.838818611
5	5	2.322136141	2.226956821	2.097039316	1.988188021	1.907844608	1.849953668
5	6	2.324455544	2.237774744	2.111363531	2.001916881	1.919710249	1.859882055
6	1	2.299464409	2.157619068	2.018233649	1.918014776	1.849588667	1.802366837
6	2	2.311255591	2.187713111	2.049862918	1.945056629	1.871517707	1.82001981
6	3	2.317850054	2.209140615	2.074472692	1.967012738	1.889742551	1.834897154
6	4	2.321890778	2.22513783	2.094270298	1.985335727	1.905264668	1.847725164
6	5	2.324539269	2.237521501	2.110611256	2.000953224	1.918735791	1.858981232
6	6	2.326366829	2.247384254	2.124375815	2.014488774	1.930601569	1.868994819