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Beta G-Star Relation on Modules

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Abstract. In this work, we say submodules X and Y of M are β_g^* equivalence, $X\beta_g^*Y$, if and only if $Y+K=M$ for every $K \trianglelefteq M$ such that $X+K=M$ and $X+T=M$ for every $T \trianglelefteq M$ such that $Y+T=M$. It is proved that the β_g^* relation is an equivalent relation and has good behaviour with respect to addition of submodules and homomorphisms.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a small submodule of M and denoted by $N \ll M$. Let M be an $R-$module and $N \leq M$. N is called essential submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an $R-$module and K be a submodule of M. K is called a generalized small (briefly, g-small) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, then we write $K \ll_g M$. (in [11], it is called an e-small submodule of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. M is called a (generalized) hollow module if every proper submodule of M is (generalized) small in M. Here it is clear that every hollow module is generalized hollow module. Let M be an $R-$module and $U, V \leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U \cap V \ll V$, then V is called a supplement of U in M. M is called a supplemented module if every submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M=U+V$ and $M=U+T$ with $T \trianglelefteq V$ implies that $T=V$, or equivalently, $M=U+V$ and $U \cap V \ll_g V$, then V is called a g-supplement of U in M. M is called g-supplemented

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if every submodule of M has a g-supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-supplement V' with $V' \leq V$, we say U has ample g-supplements in M. If every submodule of M has ample g-supplements in M, then M is called an amply g-supplemented module. $SocM$ indicates the socle of M (the sum of all simple submodules of M).

Lemma 1. *Let $M = U + V$ and $M = U \cap V + T$. Then $M = U + V \cap T = V + U \cap T$.*

Proof. See [4, Lemma 1.24].

2. The β_g^* Relation

Definition 1. *We define the relation $'\beta_g^*'$ on the set of submodules of an $R-$module M by $X\beta_g^*Y$ if and only if $Y + K = M$ for every $K \trianglelefteq M$ such that $X + K = M$ and $X + T = M$ for every $T \trianglelefteq M$ such that $Y + T = M$.*

Proposition 1. *Let M be an $R-$module and $X, Y \leq M$. If $X\beta^*Y$, then $X\beta_g^*Y$.*

Proof. Clear from definitions. (See [2]).

Lemma 2. *The β_g^* relation is an equivalence relation.*

Proof. The reflective and symmetric properties are clear. For transitive property, assume $X\beta_g^*Y$ and $Y\beta_g^*Z$. Let $K \trianglelefteq M$ and $X + K = M$. Since $X\beta_g^*Y$, then $Y + K = M$, and since $Y\beta_g^*Z$, then $Z + K = M$. Let $T \trianglelefteq M$ and $Z + T = M$. Since $Y\beta_g^*Z$, then $Y + T = M$, and since $X\beta_g^*Y$, then $X + T = M$. Hence $X\beta_g^*Z$.

Lemma 3. *Let $X, Y \leq M$. The following statements are equivalent.*
(*i*) *$X\beta_g^*Y$.*
(*ii*) *For every $T \trianglelefteq M$ such that $X + Y + T = M$, $X + T = M$ and $Y + T = M$.*

Proof. (*i*) $\Longrightarrow$ (*ii*) Let $T \trianglelefteq M$ and $X + Y + T = M$. Since $T \trianglelefteq M$, then $Y + T \trianglelefteq M$ and $X + T \trianglelefteq M$. Then by $X\beta_g^*Y$, $M = X + Y + T = X + X + T = X + T$ and $M = X + Y + T = Y + Y + T = Y + T$.

(*ii*) $\Longrightarrow$ (*i*) Let $K \trianglelefteq M$ and $X + K = M$. Then $X + Y + K = M$ and by hypothesis, $Y + K = M$. Similarly we prove that for every $T \trianglelefteq M$ such that $Y + T = M$, $X + T = M$.

Proposition 2. *Let $X, Y \leq M$. If $X\beta_g^*Y$, then $\frac{X+Y}{X} \ll_g \frac{M}{X}$ and $\frac{X+Y}{Y} \ll_g \frac{M}{Y}$.*

Proof. Let $\frac{X+Y}{X} + \frac{T}{X} = \frac{M}{X}$ for $\frac{T}{X} \trianglelefteq \frac{M}{X}$. Clearly, we can see that $T \trianglelefteq M$. Since $\frac{X+Y}{X} + \frac{T}{X} = \frac{M}{X}$, then $\frac{M}{X} = \frac{X+Y}{X} + \frac{T}{X} = \frac{Y+T}{X}$ and $Y + T = M$. Then by $X\beta_g^*Y$, $X + T = M$, and since $X \leq T$, $T = M$. Hence $\frac{X+Y}{X} \ll_g \frac{M}{X}$. Similarly, we can prove that $\frac{X+Y}{Y} \ll_g \frac{M}{Y}$.

Remark 1. *The converse of the Proposition 2 is not true in general. For example, consider the* $\mathbb{Z}$*-module* ${}_{\mathbb{Z}}\mathbb{Z}$ *and let* p *and* q *be primes with* $p \neq q$*. Since* $\frac{\mathbb{Z}}{\mathbb{Z}p}$ *and* $\frac{\mathbb{Z}}{\mathbb{Z}q}$ *are simple,* $\frac{\mathbb{Z}p+\mathbb{Z}q}{\mathbb{Z}p} = \frac{\mathbb{Z}}{\mathbb{Z}p} \ll_g \frac{\mathbb{Z}}{\mathbb{Z}p}$ *and* $\frac{\mathbb{Z}p+\mathbb{Z}q}{\mathbb{Z}q} = \frac{\mathbb{Z}}{\mathbb{Z}q} \ll_g \frac{\mathbb{Z}}{\mathbb{Z}q}$*. But* $\mathbb{Z}p\beta_g^*\mathbb{Z}q$ *is not true.*

Theorem 1. *Let* $X, Y \leq M$ *such that* $X \leq Y + A$ *and* $Y \leq X + B$*, where* $A, B \ll_g M$*. Then* $X\beta_g^*Y$*.*

Proof. Let $T \trianglelefteq M$ and $X+Y+T = M$. Then $(Y + A)+Y+T = M$ and $A+Y+T = M$. Since $T \trianglelefteq M$, then $Y + T \trianglelefteq M$. Then, by $A \ll_g M$, $Y + T = M$. Similarly, we can see that $X + T = M$.

Lemma 4. *Let* $X \leq M$*.* $X \ll_g M$ *if and only if* $X\beta_g^*0$*.*

Proof. ($\Longrightarrow$) Let $X \ll_g M$ and let $X + 0 + T = X + T = M$ for $T \trianglelefteq M$. Since $X \ll_g M$ and $X + T = M$, then $0 + T = T = M$. Then, by Lemma 3 $X\beta_g^*0$.

($\Longleftarrow$) Let $X\beta_g^*0$. Let $X + T = M$ for $T \trianglelefteq M$. Since $X\beta_g^*0$, then $T = 0 + T = M$. Hence $X \ll_g M$.

Corollary 1. *Let* $X, Y \leq M$ *and* $X\beta_g^*Y$*. If* $X \ll_g M$*, then* $Y \ll_g M$*.*

Proof. Since $X \ll_g M$, then by Lemma 4, $X\beta_g^*0$, and since $X\beta_g^*Y$, then by Lemma 2, $Y\beta_g^*0$. Then, by Lemma 4, $Y \ll_g M$.

Corollary 2. *Let* M *be an* $R-$*module. Then* M *is generalized hollow if and only if* $X\beta_g^*0$ *for every proper submodule* X *of* M*.*

Proof. Clear from Lemma 4.

Corollary 3. *Let* M *be an* $R-$*module. Then* M *is generalized hollow if and only if* $X\beta_g^*Y$ *for every proper submodules* X*,* Y *of* M*.*

Proof. Clear from Lemma 4.

Remark 2. *Let* M *be a nonzero semisimple* $R-$*module. Since* M *have no proper essential submodules,* $M \ll_g M$ *and by Lemma 4,* $M\beta_g^*0$*. But* $M\beta^*0$ *is not true.*

Corollary 4. *Let* M *be an* $R-$*module. Then* $SocM\beta_g^*0$*.*

Lemma 5. *Let* $X_1, X_2, Y_1, Y_2 \leq M$ *such that* $X_1\beta_g^*Y_1$ *and* $X_2\beta_g^*Y_2$*. Then* $(X_1 + X_2)\,\beta_g^*\,(Y_1 + Y_2)$*.*

Proof. Let $X_1 + X_2 + K = M$ for $K \trianglelefteq M$. Since $K \trianglelefteq M$, then $X_2 + K \trianglelefteq M$. Then, by $X_1\beta_g^*Y_1$, $Y_1 + X_2 + K = M$. Since $K \trianglelefteq M$, then $Y_1 + K \trianglelefteq M$. Then, by $X_2\beta_g^*Y_2$, $Y_1 + Y_2 + K = M$. Similarly, we can see that $X_1 + X_2 + T = M$ for every $T \trianglelefteq M$ such that $Y_1 + Y_2 + T = M$.

Corollary 5. *Let* $X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n \leq M$ *and* $X_i\beta_g^*Y_i$ *for every* $i = 1, 2, ..., n$*. Then* $X_1 + X_2 + ... + X_n\beta_g^*Y_1 + Y_2 + ... + Y_n$*.*

Proof. Clear from Lemma 5.

Corollary 6. *Let $X_1, X_2, ..., X_n, Y \leq M$ and $X_i \beta_g^* Y$ for every $i = 1, 2, ..., n$. Then $X_1 + X_2 + ... + X_n \beta_g^* Y$.*

Proof. Clear from Lemma 5.

Lemma 6. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $X, Y \leq M$. If $X\beta_g^* Y$, then $f(X)\beta_g^* f(Y)$.*

Proof. Let $f(X) + f(Y) + T = N$ for $T \trianglelefteq N$. Then $X + Y + f^{-1}(T) = M$. Since $T \trianglelefteq N$, then we can see that $f^{-1}(T) \trianglelefteq M$. Then, by Lemma 3, $X + f^{-1}(T) = M$ and $Y + f^{-1}(T) = M$. Since $X + f^{-1}(T) = M$ and $Y + f^{-1}(T) = M$, then $f(X) + T = N$ and $f(Y) + T = N$. Hence, by Lemma 3, $f(X)\beta_g^* f(Y)$.

Corollary 7. *Let $X, Y, Z \leq M$. If $X\beta_g^* Y$, then $\frac{X+Z}{Z}\beta_g^* \frac{Y+Z}{Z}$.*

Proof. Clear from Lemma 6.

Corollary 8. *Let M be an $R-$module, A be a direct summand of M and $X, Y \leq A$. If $X\beta_g^* Y$ in M, then $X\beta_g^* Y$ in A also holds.*

Proof. Clear from Lemma 6.

Proposition 3. *Let $X, Y \leq M$. If $X\beta_g^* Y$ and Y is an essential maximal submodule of M, then $X \leq Y$.*

Proof. Assume $X \nleq Y$. Then, because Y is an essential maximal submodule of M, $X + Y = M$ and since $X\beta_g^* Y$, $Y = Y + Y = M$. This contradicts maximality of Y.

Definition 2. *Let M be an $R-$module and $U, V \leq M$. If $U + V = M$ and $U \cap V \ll_g M$, then V is called a weak g-supplement of U in M. If every submodule of M has a weak g-supplement in M, then M is called a weakly g-supplemented module. (See [8])*

Proposition 4. *Let $X\beta_g^* Y$ in M.*

(i) *If X has an essential g-supplement V in M, then V is also a g-supplement of Y in M.*

(ii) *If X has an essential weak g-supplement V in M, then V is also a weak g-supplement of Y in M.*

Proof. (i) Since $M = X + V$ and $V \trianglelefteq M$, then by $X\beta_g^* Y$, $Y + V = M$. Let $M = Y + T$ with $T \trianglelefteq V$. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then we can see that $T \trianglelefteq M$. Then by $X\beta_g^* Y$, $X + T = M$. Since $X + T = M$ and $T \trianglelefteq V$, then $T = V$. Hence V is a g-supplement of Y in M.

(ii) Since $M = X + V$ and $V \trianglelefteq M$, then by $X\beta_g^* Y$, $Y + V = M$. Let $Y \cap V + T = M$ with $T \trianglelefteq M$. Since $M = Y + V$ and $M = Y \cap V + T$, then by Lemma 1, $M = Y + V \cap T$.

Since $V \trianglelefteq M$ and $T \trianglelefteq M$, then $V \cap T \trianglelefteq M$. Then by $X\beta_g^*Y$, $X + V \cap T = M$. Since $M = V + T$ and $M = X + V \cap T$, then by Lemma 1, $X \cap V + T = M$. Because $X \cap V + T = M$ and $T \trianglelefteq M$ and $X \cap V \ll_g M$, then $T = M$. Hence $Y \cap V \ll_g M$ and V is a weak g-supplement of Y in M.

Proposition 5. *Let M be an amply g-supplemented module and $X, Y \leq M$. If g-supplements of X and Y in M is the same, then $X\beta_g^*Y$.*

Proof. Let $X + K = M$ with $K \trianglelefteq M$. Since M is amply g-supplemented, there exists a g-supplement K' of X with $K' \leq K$. By hypothesis, K' is a g-supplement of Y in M. Then $Y + K' = M$ and since $K' \leq K$, $Y + K = M$. Similarly, we can see that $X + T = M$ for every $T \trianglelefteq M$ such that $Y + T = M$.

Proposition 6. *Let M be weakly g-supplemented module and $X, Y \leq M$. If weak g-supplements of X and Y in M is the same, then $X\beta_g^*Y$.*

Proof. Let $X + K = M$ with $K \trianglelefteq M$. Since M is weakly g-supplemented, by [8, Proposition 1] there exists a weak g-supplement K' of X with $K' \leq K$. By hypothesis, K' is a weak g-supplement of Y in M. Then $Y + K' = M$ and since $K' \leq K$, $Y + K = M$. Similarly, we can see that $X + T = M$ for every $T \trianglelefteq M$ such that $Y + T = M$.

Proposition 7. *Let M be an $R-$module, $X \leq Y \leq M$ and C be an essential weak g-supplement of X in M. If $X\beta_g^*Y$, then $Y \cap C \ll_g M$.*

Proof. Since $X\beta_g^*Y$ and C is an essential weak g-supplement of X in M, then by Proposition 4, C is also a weak g-supplement of Y in M. Hence $Y \cap C \ll_g M$.

Lemma 7. *Let M be an $R-$module, $X \leq Y \leq M$ and C be a weak g-supplement of X in M. If $Y \cap C \ll_g M$, then $X\beta_g^*Y$.*

Proof. Let $Y + T = M$ with $T \trianglelefteq M$. Since C is a weak g-supplement of X in M, $C + X = M$. Since $X \leq Y$, by Modular Law, $Y = Y \cap M = Y \cap (C + X) = Y \cap C + X$. Then $M = Y + T = Y \cap C + X + T$ and since $Y \cap C \ll_g M$ and $X + T \trianglelefteq M$, $X + T = M$. If $X + K = M$ with $K \trianglelefteq M$, $Y + K = M$ also holds since $X \leq Y$. Hence $X\beta_g^*Y$.

Proposition 8. *Let $M = M_1 \oplus M_2$ and $M_1 \leq X \leq M$. If $X \cap M_2 \ll_g M$, then $X\beta_g^*M_1$.*

Proof. Clear from Lemma 7.

Proposition 9. *Let M be an $R-$module. If every submodule of M equivalent to an essential weak g-supplement in M by β_g^* relation, then M is weakly g-supplemented.*

Proof. Let $X \leq M$. By hypothesis, there exists an essential weak g-supplement V in M such that $X\beta_g^*V$. Let V be a weak g-supplement of U in M. By hypothesis, there exists an essential weak g-supplement Y in M such that $U\beta_g^*Y$. Since V is an essential weak g-supplement of U in M, by Proposition 4, V is a weak g-supplement of Y in M. Then Y is an essential weak g-supplement of V in M and since $X\beta_g^*V$, by Proposition 4, Y is a weak g-supplement of X in M. Hence M is weakly g-supplemented.

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