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# The influence of C- $\mathfrak{Z}$-permutable subgroups on the structure of finite groups

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**Abstract.** Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group $G$, that is, for each prime $p$ dividing the order of $G$, $\mathfrak{Z}$ contains exactly one and only one Sylow $p$-subgroup of $G$, say $G_p$. Let $C$ be a nonempty subset of $G$. A subgroup $H$ of $G$ is said to be $C$-$\mathfrak{Z}$-permutable (conjugate-$\mathfrak{Z}$-permutable) subgroup of $G$ if there exists some $x \in C$ such that $H^x G_p = G_p H^x$, for all $G_p \in \mathfrak{Z}$. We investigate the structure of the finite group $G$ under the assumption that certain subgroups of prime power orders of $G$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$.

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**Key Words and Phrases**: Sylow subgroup, $\mathfrak{Z}$-permutable subgroup, $C$-$\mathfrak{Z}$-permutable subgroup, $p$-nilpotent group, supersolvable group, Fitting subgroup, generalized Fitting subgroup, saturated formation.

## 1. Introduction

Throughout this article only finite groups are considered. We use conventional notions and notation, as in Doerk and Hawkes [2]. In addtion, $\pi(G)$ denotes the set of distinct primes dividing $|G|$ and $G_p$ is a Sylow $p$-subgroup of the group $G$ for some prime $p \in \pi(G)$. Two subgroups $H$ and $K$ of a group $G$ are said to be permutable if $HK = KH$, that is, $HK$ is a subgroup of $G$. Recall that a subgroup $H$ of a group $G$ is $S$-permutable (or $S$-quasinormal) in $G$ if $H$ permutes with every Sylow subgroup of $G$. This concept was introduced by Kegel [7] in 1962.

Recently, in 2003, Asaad and Heliel [1] introduced the concept of $\mathfrak{Z}$-permutability which generalizes $S$-permutability as follows: Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$. A subgroup $H$ of $G$ is said to be $\mathfrak{Z}$-permutable in $G$ if $H$ permutes with every

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member in $\mathfrak{Z}$.
More recently, in 2013, Heliel and Al-Gafri [4] generalized the concept of $\mathfrak{Z}$-permutability by introducing a new subgroup embedding property, namely, the conjugate-$\mathfrak{Z}$-permutability. Let $C$ be a nonempty subset of a group $G$ and $\mathfrak{Z}$ be a complete set of Sylow subgroups of $G$. A subgroup $H$ of $G$ is said to be $C$-$\mathfrak{Z}$-permutable subgroup of $G$ if there exists some $x \in C$ such that $H^x G_p = G_p H^x$, for all $G_p \in \mathfrak{Z}$. Remark 1.2 and Examples 1.3 and 1.4 in [4] show that $C$-$\mathfrak{Z}$-permutability is a nontrivial generalization of $\mathfrak{Z}$-permutability.
This article may be viewed as a continuation of Heliel and Al-Gafri [4]. In fact, we extend and improve the following theorem:

**Theorem 1.1**. [[4], Theorem 3.11] *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$. Then the following two statements are equivalent:*

(a) $G \in \mathfrak{F}$.

(b) *There is a normal subgroup $H$ in $G$ and a solvable normal subgroup $C$ of $F^*(H)$ such that $G/H \in \mathfrak{F}$, and the maximal subgroups of $G_p \cap F^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$,* where *$F^*(H)$ is the generalized Fitting subgroup of $H$.*

More precisely, we prove the following theorem:

**Theorem 1.2**. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and let $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:*

(a) $G \in \mathfrak{F}$.

(b) *There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$.*

**Remark 1.3**. Let $S(F^*(H))$ denotes the solvable radical of $F^*(H)$, that is, $S(F^*(H))$ is the unique largest solvable normal subgroup of $F^*(H)$. In Theorem 1.1, $C$ is a solvable normal subgroup of $F^*(H)$. Therefore, $C$ is contained in $S(F^*(H))$. Since $S(F^*(H))$ is characteristic in $F^*(H)$ and $F^*(H)$ is normal in $G$, we have that $S(F^*(H))$ is normal in $G$. So, the maximal subgroups of $G_p \cap F^*(H)$ are $S(F^*(H))$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$, where $S(F^*(H))$ is a solvable normal subgroup of $G$. Thus, Theorem 1.1 can be seen as an immediate consequence of Theorem 1.2.

## 2. Basic definitions and preliminaries

In this section, we list some definitions and known results from the literature that will be used in the sequel.

Let $\mathfrak{F}$ be a saturated formation. Then the $\mathfrak{F}$-residual, denoted by $G^{\mathfrak{F}}$, is the unique smallest normal subgroup of $G$ such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$. Throughout, $\mathfrak{U}$ denotes the class of supersolvable groups which is a saturated formations, see [[5], Satz 8.6, p. 713].

A normal subgroup $N$ of a group $G$ is an $\mathfrak{F}$-hypercentral subgroup of $G$ provided $N$ possesses a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \trianglelefteq N_s = N$ such that $N_{i+1}/N_i$ is an $\mathfrak{F}$-central chief factor of $G$, see [[2], p. 387]. The product of all $\mathfrak{F}$-hypercentral subgroups of $G$ is again an $\mathfrak{F}$-hypercentral subgroup, denoted by $Z_{\mathfrak{F}}(G)$, and called the $\mathfrak{F}$-hypercenter of $G$, see [[2], IV, 6.8]. For the formation $\mathfrak{U}$, the $\mathfrak{U}$-hypercenter of a group $G$, denoted by $Z_{\mathfrak{U}}(G)$, is the product of all normal subgroups $N$ of $G$ such that each chief factor of $G$ below $N$ has prime order. For more details about saturated formations, see [[2], IV].

For any group $G$, the generalized Fitting subgroup $F^*(G)$ is the unique maximal normal quasinilpotent subgroup of $G$. In fact, $F^*(G)$ is an important characteristic subgroup of $G$ and it is a natural generalization of $F(G)$. The basic properties of $F^*(G)$ can be found in [[6], X 13]. We define $F_1^*(G) = F^*(G)$ and $F_i^*(G)/F_{i-1}^*(G) = F^*(G/F_{i-1}^*(G))$ for $i > 1$. Since $F^*(G) \neq 1$ when $G \neq 1$, there exists an integer $n$ such that $F_n^*(G) = G$.

Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and let $N$ be a normal subgroup of $G$. We denote the following families of subgroups of $G$, $G/N$ and $N$, respectively:

$$\begin{aligned} \mathfrak{Z}N &= \{G_pN : G_p \in \mathfrak{Z}\}, \\ \mathfrak{Z}N/N &= \{G_pN/N : G_p \in \mathfrak{Z}\}, \\ \mathfrak{Z} \cap N &= \{G_p \cap N : G_p \in \mathfrak{Z}\}. \end{aligned}$$

The following lemmas will be used in the sequel.

**Lemma 2.1**. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$, $C$ be a nonempty subset of $G$ and $N$ be a normal subgroup of $G$. Then:*

(a) *$\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of $N$ and $G/N$, respectively.*

(b) *If $U$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$, then $UN/N$ is $CN/N$-$\mathfrak{Z}N/N$-permutable subgroup of $G/N$.*

(c) *If $U \leq N$, $C \subseteq N$ and $U$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$, then $U$ is $C$-$\mathfrak{Z} \cap N$-permutable subgroup of $N$.*

(d) *Suppose that $N \leq U$. Then $U$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$ if and only if $U/N$ is $CN/N$-$\mathfrak{Z}N/N$-permutable subgroup of $G/N$.*

(e) *If $U$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$, then $U \cap N$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$.*

**Proof**. For (a), see [[1], Lemma 2.1(a)]. For (b), (c) and (d); see [[4], Lemma 2.1].

(e) Let $G_p$ be any member of $\mathfrak{Z}$. By hypothesis, there exists some $x \in C$ such that $U^xG_p$ is a subgroup of $G$, that is, $U^xG_p = G_pU^x$. Let $K = U^x$. It is clear that $G_pKN$ is a subgroup of $G$ as $N$ is normal in $G$. Since $G_p \cap KN$ is a $p$-subgroup of $KN$ and

$|KN : G_p \cap KN| = |G_pKN : G_p|$ is a $p'$-number as $G_p$ is a Sylow $p$-subgroup of $G_pKN$, it follows that $G_p \cap KN$ is a Sylow $p$-subgroup of $KN$. Also, $G_p \cap K$ is a $p$-subgroup of $K$ and $|K : G_p \cap K| = |KG_p : G_p|$ is a $p'$-number. Consequently, $G_p \cap K$ is a Sylow $p$-subgroup of $K$. Therefore, $G_p \cap K \cap N = (G_p \cap K) \cap (K \cap N)$ is a Sylow $p$-subgroup of $K \cap N$ as $K \cap N$ is a normal subgroup of $K$. If $M$ is a subgroup of $G$, let $|M|_p$ denotes the largest power of $p$ dividing the order of $M$. As $|KN| = \frac{|K||N|}{|K \cap N|}$, then $|KN|_p = \frac{|K|_p|N|_p}{|K \cap N|_p}$. Clearly, $(G_p \cap K)(G_p \cap N) \leq G_p$ as $G_p \cap N$ is a normal subgroup of $G_p$ and so $(G_p \cap K)(G_p \cap N) \leq G_p \cap KN$. Now $|(G_p \cap K)(G_p \cap N)| = \frac{|G_p \cap K||G_p \cap N|}{|G_p \cap K \cap N|} = \frac{|K|_p|N|_p}{|K \cap N|_p} = |KN|_p = |G_p \cap KN|$ and hence $(G_p \cap K)(G_p \cap N) = G_p \cap KN$. By [[2], Lemma 1.2, p. 2], $G_p(K \cap N) = G_pK \cap G_pN$ which is a subgroup of $G$ as $G_pK$ and $G_pN$ are subgroups of $G$. So, we have $x \in C$ such that $G_p(U \cap N)^x = G_p(U^x \cap N) = G_p(K \cap N)$ is a subgroup of $G$, for all $G_p \in \mathfrak{Z}$. Thus, $U \cap N$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$.

**Lemma 2.2**. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a nonempty subset of $G$. Assume that $H$ is a normal subgroup of $G$ such that the maximal subgroups of $\mathfrak{Z} \cap H$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$. Then for any nontrivial normal subgroup $N$ of $G$, the maximal subgroups of $(\mathfrak{Z}N/N) \cap (HN/N)$ are $CN/N$-$\mathfrak{Z}N/N$-permutable subgroups of $G/N$.*

**Proof**. See [[4], Lemma 2.2].

**Lemma 2.3**. *Let $G$ be a group. Then:*

(a) $F^*(G) = F(G)E(G)$ *and* $[F(G), E(G)] = 1$, *where* $E(G)$ *is the layer subgroup of* $G$.

(b) $F^*(F^*(G)) = F^*(G) \geq F(G)$; *if* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

(c) $C_G(F^*(G)) \leq F(G)$.

(d) *Suppose that $N$ is a normal subgroup of $G$ contained in $\Phi(G)$, then $F^*(G/N) = F^*(G)/N$.*

**Proof**. (a), (b) and (c) can be found in [[6], X 13]. For (d), see [[10], Lemma 2.3 (8)].

**Lemma 2.4**. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a nonempty subset of $G$. Suppose that $P$ is a normal $p$-subgroup of $G$, where $p$ is a prime, and $N$ is a minimal normal subgroup of $G$ with $N \leq P$. If $N$ is complemented in $P$ and the maximal subgroups of $P$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, then the order of $N$ is $p$.*

**Proof**. By hypothesis, there exists a subgroup $H$ of $P$ such that $P = NH$ and $N \cap H = 1$. Obviously, we have that $N \leq G_p \in \mathfrak{Z}$. Let $N/M$ be a chief factor of $G_p$. Then, the order of $N/M$ is $p$. Clearly, $MH$ is a subgroup of $P$ as $M$ is normal in $P$. Since $M \cap H = M \cap (N \cap H) = 1$, then $|P : MH| = |NH : MH| = |N : M| = p$ and hence $MH$ is a maximal subgroup of $P$. By hypothesis, $MH$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$. Therefore, $M = M(H \cap N) = MH \cap N$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$ by Lemma

2.1(e). So, there exists some $x \in C$ such that $M^x G_q$ is a subgroup of $G$, for all $G_q \in \mathfrak{Z}$. This implies that $MG_q^{x^{-1}}$ is a subgroup of $G$, for all $G_q \in \mathfrak{Z}$. Assume that $q \neq p$. Since $M = M(N \cap G_q^{x^{-1}}) = N \cap MG_q^{x^{-1}}$ and $N \cap MG_q^{x^{-1}}$ is normal in $MG_q^{x^{-1}}$, it follows that $G_q^{x^{-1}} \leq N_G(M)$. If $q = p$, then $M$ is normal in $G_p$ and so $G_p \leq N_G(M)$. Therefore, $N_G(M) = G$ and hence $M$ is normal in $G$. But $N$ is a minimal normal subgroup of $G$ and $M$ is a maximal subgroup of $N$, thus $M = 1$ and the order of $N$ is $p$.

**Lemma 2.5**. *Let $G$ be a group. Then:*

(a) *$E(G)$, the layer subgroup of $G$, is a perfect quasinilpotent characteristic subgroup of $G$.*

(b) *If $M$ is a perfect quasinilpotent subnormal subgroup of $G$, then $M \leq E(G)$.*

(c) *If $M$ is a solvable subgroup of $G$ and $E(G) \leq N_G(M)$, then $[E(G), M] = 1$.*

**Proof**. For (a), see [[6], Definition 13.14, p. 128]. For (b) and (c), see [[6], Theorem 13.15(a), p. 128 and Lemma 13.16(b), p. 128–129], respectively.

**Lemma 2.6**. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. If $p$ is the smallest prime dividing the order of $G$ and the maximal subgroups of $G_p \in \mathfrak{Z}$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, then $G$ is $p$-nilpotent.*

**Proof**. See [[4], Theorem 3.1].

**Lemma 2.7**. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$, $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:*

(a) *$G \in \mathfrak{F}$.*

(b) *There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap H$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$.*

**Proof**. See [[4], Theorem 3.2].

**Lemma 2.8**. *Suppose that $G$ is a finite non-abelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that $G$ has no Hall $\{2, r\}$-subgroup.*

**Proof**. See [[8], Lemma 2.6].

## 3. Results

First, we prove the following lemma:

**Lemma 3.1**. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a nonempty subset of $G$. Suppose that $P$ is a normal $p$-subgroup of $G$. If the maximal subgroups of $P$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, then $P \leq Z_{\mathfrak{U}}(G)$.*

**Proof**. Assume that the result is false and let $G$ be a counterexample of minimal order. If $\Phi(P) \neq 1$, then the maximal subgroups of $P/\Phi(P)$ are $C\Phi(P)/\Phi(P)$-$\mathfrak{Z}\Phi(P)/\Phi(P)$-permutable subgroups of $G/\Phi(P)$ by Lemma 2.1(d). Then, by the minimal choice of $G$, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Hence, by [[11], Theorem 7.19, p. 39], $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus, we may assume that, $\Phi(P) = 1$ and so $P$ is elementary abelian $p$-group. Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Since $N \cap \Phi(P) = 1$ as $\Phi(P) = 1$, it follows, by [[2], Theorem 9.2(f), p. 30], that $N$ is complemented in $P$. The hypothesis and Lemma 2.4 imply that the order of $N$ is $p$. If $N = P$, then $P \leq Z_{\mathfrak{U}}(G)$ by the definition of $Z_{\mathfrak{U}}(G)$, a contradiction. So, we may assume that $N \neq P$. It is easy to see that $\Phi(P/N) = 1$. Let $M/N$ be a maximal subgroup of $P/N$. Then $M$ is a maximal subgroup of $P$ as $|P : M| = |P/N : M/N| = p$. By hypothesis and Lemma 2.1(b), $M/N$ is $CN/N$-$\mathfrak{Z}N/N$-permutable subgroup of $G/N$. So, the maximal subgroups of $P/N$ are $CN/N$-$\mathfrak{Z}N/N$-permutable subgroups of $G/N$. Therefore, $P/N \leq Z_{\mathfrak{U}}(G/N)$ by the minimal choice of $G$. But $Z_{\mathfrak{U}}(G/N) = Z_{\mathfrak{U}}(G)/N$ by [[11], Lemma 7.1(ii), p. 30], then $P \leq Z_{\mathfrak{U}}(G)$, a contradiction completing the proof of the lemma.

Now we can prove:

**Theorem 3.2**. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$, $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:*

(a) $G \in \mathfrak{F}$.

(b) *There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$, $F^*(H) = F(H)$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$.*

**Proof**. We need only to prove (b) $\Rightarrow$ (a).

Let $P$ be any Sylow $p$-subgroup of $F(H)$. Clearly, $P$ is normal in $G$. The hypothesis and Lemma 3.1 imply that $P \leq Z_{\mathfrak{U}}(G)$. Since this is true for any Sylow $p$-subgroup of $F(G)$, we have that $F(H) \leq Z_{\mathfrak{U}}(G)$. Note that $[G^{\mathfrak{U}}, F(H)] \leq [G^{\mathfrak{U}}, Z_{\mathfrak{U}}(G)] = 1$ by [[2], Theorem 6.10, p. 390] and hence $G^{\mathfrak{U}} \leq C_G(F(H))$. Therefore, $G/C_G(F(H))$ is an epimorphic image of $G/G^{\mathfrak{U}} \in \mathfrak{U} \subseteq \mathfrak{F}$ and so $G/C_G(F(H)) \in \mathfrak{U} \subseteq \mathfrak{F}$. Consequently, $G/C_H(F(H)) = G/\left(C_G(F(H)) \cap H\right) \in \mathfrak{F}$ as $G/C_G(F(H)) \in \mathfrak{F}$ and $G/H \in \mathfrak{F}$. But $C_H(F(H)) \leq F(H)$ holds by Lemma 2.3(c) and the fact that $F^*(H) = F(H)$, then

$G/F(H)$ is an epimorphic image of $G/C_H(F(H))$, thus $G/F(H) \in \mathfrak{F}$. Applying Lemma 2.7 yields $G \in \mathfrak{F}$. This comletes the proof of the theorem.

The following lemma is a criterion for the solvability of finite groups:

**Lemma 3.3**. *Let $G$ be a group. Then the following two statements are equivalent:*

(a) *$G$ is solvable.*

(b) *$G$ has a complete set $\mathfrak{Z}$ of Sylow subgroups such that $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable subgroup of $G$, where $p$ is the smallest prime dividing the order of $G$.*

**Proof**. (a) $\Rightarrow$ (b). Since $G$ is solvable, then $G$ has a Sylow basis $S$ by [9, Theorem 9.3.11, p. 229]. Let $p$ be the smallest prime dividing the order of $G$ and let $G_p$ be the Sylow $p$-subgroup of $G$ in $S$. By the definition of the Sylow basis $S$, we have that $G_pG_q$ is a subgroup of $G$, for all $G_q \in S$, where $q$ is a prime. Thus we can take $\mathfrak{Z} = S$ and we have $G_p$ is $\mathfrak{Z}$-permutable subgroup of $G$.

(b) $\Rightarrow$ (a). Assume that the result is false and let $G$ be a counterexample of minimal order. By Feit-Thompson Theorem [3], we may assume that $p = 2$. Since $G_2 \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable subgroup of $G$, it follows that $G_2G_q$ is a subgroup of $G$, for every odd prime $q$ dividing the order of $G$, where $G_q \in \mathfrak{Z}$. Therefore, $G$ is not simple by Lemma 2.8. Let $N$ be a nontrivial proper normal subgroup of $G$. Clearly, $G_2 \cap N$ and $G_2N/N$ are Sylow 2-subgroups of $N$ and $G/N$, respectively. By Lemma 2.1(e), $G_2 \cap N$ is $\mathfrak{Z}$-permutable subgroup of $G$. Therefore, $G_2 \cap N$ is $\mathfrak{Z} \cap N$-permutable subgroup of $N$ by Lemma 2.1(c). Also, $G_2N/N$ is $\mathfrak{Z}N/N$-permutable subgroup of $G/N$ by Lemma 2.1(b). If 2 divides the order of $N$, then $N$ is solvable by the minimal choice of $G$ and if 2 does not divide the order of $N$, then $N$ is solvable by Feit-Thompson Theorem [3]. The same argument holds for $G/N$, thus $G/N$ is solvable. Since $N$ and $G/N$ are solvable, we have that $G$ is solvable, a contradiction completing the proof of the lemma.

**Proof of Theorem 1.2.** We need only to prove (b) $\Rightarrow$ (a) as (a) $\Rightarrow$ (b) is true with $H = 1$.

Let $E(G)$ be the layer subgroup of $G$. Since $C$ is a solvable normal subgroup of $G$, it follows, by Lemma 2.5(c), that $[E(G), C] = 1$ and so $C \leq C_G(E(G))$. By Lemma 2.3(a), we have that $F^*(H) = F(H)E(H)$. Moreover, $E(H)$ is a perfect quasinilpotent characteristic subgroup of $H$ by Lemma 2.5(a). Now $E(H)$ char $H$ and $H$ is normal in $G$, then $E(H)$ is normal in $G$. Note that $E(H) \leq E(G)$ by Lemma 2.5(b), and hence $C \leq C_G(E(H))$. Now we will show that $E(H)$ is solvable. By Feit-Thompson Theorem [3], we may assume that 2 divides the order of $E(H)$. Clearly, $\mathfrak{Z} \cap E(H)$ is a complete set of Sylow subgroups of $E(H)$ by Lemma 2.1(a). Let $U$ be a maximal subgroup of $G_2 \cap F^*(H)$, where $G_2 \in \mathfrak{Z}$. The hypothesis and Lemma 2.1(e) imply that $U \cap E(H)$ is $C$-$\mathfrak{Z}$-permutable subgroup of $G$. So, there exists some $x \in C$ such that $(U \cap E(H))^xG_q$ is a subgroup of $G$, for all $G_q \in \mathfrak{Z}$. But $(U \cap E(H))^x = U \cap E(H)$ as $x \in C \leq C_G(E(H)) \leq C_G(U \cap E(H))$, then $(U \cap E(H))G_q$

is a subgroup of $G$, for all $G_q \in \mathfrak{Z}$. Therefore, $U \cap E(H)$ is $\mathfrak{Z}$-permutable subgroup of $G$. By Lemma 2.1(c), $U \cap E(H)$ is $\mathfrak{Z} \cap E(H)$-permutable subgroup of $E(H)$. Suppose that $G_2 \cap E(H) = G_2 \cap F^*(H)$. By the hypothesis and the previous arguments, the maximal subgroups of $G_2 \cap E(H)$ are $\mathfrak{Z} \cap E(H)$-permutable subgroups of $E(H)$. Consequently, $E(H)$ is 2-nilpotent by Lemma 2.6, where $C = 1$ in this case. So, $E(H) = (G_2 \cap E(H))K$, where $K$ is a normal Hall $2'$-subgroup of $E(H)$. Because $K$ is solvable by Feit-Thompson Theorem [3] and $G/K \cong G_2 \cap E(H)$ is solvable, it follows that $E(H)$ is solvable. Thus, we may assume that $G_2 \cap E(H)$ is a proper subgroup of $G_2 \cap F^*(H)$. Then we can choose $U$ to be a maximal subgroup of $G_2 \cap F^*(H)$ such that $G_2 \cap E(H) \leq U$. Therefore, $G_2 \cap E(H) = U \cap E(H)$ as $G_2 \cap E(H)$ is a Sylow 2-subgroup of $E(H)$. Now we have that $G_2 \cap E(H) = U \cap E(H)$, as we proved in the beginning, is $\mathfrak{Z} \cap E(H)$-permutable subgroup of $E(H)$. This implies that $E(H)$ is solvable by Lemma 3.3. So, in either case, $E(H)$ is solvable. But $E(H)$ is perfect, then $E(H) = 1$ and therefore $F^*(H) = F(H)$. Applying Theorem 3.2 yields $G \in \mathfrak{F}$. This completes the proof of the theorem.

The next theorem is an improvement of Theorem 3.12 in [4]:

**Theorem 3.4**. *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$, $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:*

(a) $G \in \mathfrak{F}$.

(b) *There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F_n^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$, for some positive integer $n$.*

**Proof**. We need only to prove (b) $\Rightarrow$ (a) as (a) $\Rightarrow$ (b) is true with $H = 1$.

If $n = 1$, then $G \in \mathfrak{F}$ by Theorem 1.2. So, we may assume that $n > 1$. Let $K = F_{n-1}^*(H)$. It is clear that $(G/K)/(H/K) \cong G/H \in \mathfrak{F}$. By Lemma 2.2, the maximal subgroups of $(\mathfrak{Z}K/K) \cap (F_n^*(H)/K) = (\mathfrak{Z}K/K) \cap F^*(H/K)$ are $CK/K$-$\mathfrak{Z}K/K$-permutable subgroups of $G/K$. Hence, $G/K \in \mathfrak{F}$ by Theorem 1.2. Thus $G/F_n^*(H) \cong (G/K)/(F_n^*(H)/K) \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F_n^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$. Applying Lemma 2.7 yields $G \in \mathfrak{F}$. This completes the proof of the theorem.

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