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On the Irreducibility of Perron Representations of Degrees 4 and 5

Malak M. Dally[1], Mohammad N. Abdulrahim[1,*]

[1] *Department of Mathematics, Faculty of Science, Beirut Arab University, P.O. Box 11-5020, Beirut, Lebanon*

Abstract. We consider the graph $E_{n+1,1}$ with (n+1) generators $\sigma_1, ..., \sigma_n$, and δ, where σ_i has an edge with σ_{i+1} for $i = 1, ..., n+1$, and σ_1 has an edge with δ. We then define the Artin group of the graph $E_{n+1,1}$ for $n = 3$ and $n = 4$ and consider its reduced Perron's representation of degrees four and five respectively. After we specialize the indeterminates used in defining the representation to non-zero complex numbers, we obtain necessary and sufficient conditions that guarantee the irreducibility of the representations for $n = 3$ and 4 .

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1. Introduction

Let Γ be an undirected simple graph. The Artin group A is defined as an abstract group whose generators are the vertices of Γ that satisfy the two relations: $xy = yx$ for vertices x and y that have no edge in common and $xyx = yxy$ if the vertices x and y have a common edge.

Having defined A, we consider the graph A_n having n vertices σ_i's $(1 \leq i \leq n)$ in which σ_i and σ_{i+1} share a comon edge, where $i = 1, 2, ..., n-1$. Indeed, the Artin group of A_n, denoted by $A(A_n)$, is the braid group on $n+1$ strands, B_{n+1}. That is, $A(A_n) = B_{n+1}$.

From the graph A_n, we obtain the graph $E_{n+1,p}$ by adding a vertex δ and an edge connecting σ_p and δ. Here $1 \leq p \leq n$. Clearly, the graph A_n embeds in the graph $E_{n+1,p}$. Consequently, $A(A_n) \subset A(E_{n+1,p})$. As a result, a representation of $A(E_{n+1,p})$ yields a representation of B_{n+1}.

Perron's strategy is to begin with the reduced Burau representation of B_{n+1} of degree n and extend it to a representation of B_{n+1} of degree $2n$. The representation obtained is referred to as Burau bis representation. Next, Perron constructs for each $\lambda = (\lambda_1, \ldots, \lambda_n)$

*Corresponding author.

Email addresses: malakdally@hotmail.com (M. Dally), mna@bau.edu.lb (M. Abdulrahim)

a representation $\psi_\lambda : A(E_{n+1,p}) \to GL_{2n}(Q(t, d_1, \ldots, d_n))$, where $t, d_1, \ldots, d_n$ $\lambda_1, \ldots, \lambda_n$ are indeterminates.

In [3], we determined necessary and sufficient condition that guarantees the irreducibility of the representation ψ_λ for $n = 2$. In our work, we extend our work to $n = 3$ and $n = 4$. We reduce the complex specialization of the representation ψ_λ to representations of $A(E_{4,1})$ and $A(E_{5,1})$ of degrees 4 and 5 respectively. In each case, a necessary and sufficient condition which guarantees the irreducibility of the considered representation is obtained. The obtained conditions are similar to the condition obtained in the case $n = 2$, which was studied in [3].

2. Burau bis Representation

The Burau Bis representation is a representation of B_{n+1} of degree $2n$.
It is defined as follows:

$$\psi : B_{n+1} \to Gl_{2n}(\mathbb{Z}[t, t^{-1}])$$

$$\psi(\sigma_i) = \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix}, \quad 1 \le i \le n$$

Here, R_i denotes an $n \times n$ block of zeros with a t placed in the (i, i) th position and I_n denotes the $n \times n$ identity matrix.

This representation was constructed by Perron by extending the reduced Burau representation of degree n to a representation of B_{n+1} of degree $2n$.

The reduced Barau representation $B_{n+1} \to GL_n(\mathbb{Z}[t, t^{-1}])$ is defined as follows:

$$\sigma_i \to J_i = \left(\begin{array}{c|ccc|c} I_{i-2} & & 0 & & 0 \\ \hline & 1 & 0 & 0 & \\ 0 & t & -t & 1 & 0 \\ & 0 & 0 & 1 & \\ \hline 0 & & 0 & & I_{n-i-1} \end{array}\right),$$

where I_k stands for the $k \times k$ identity matrix. Here, $i = 2, \ldots, n-1$.

$$\sigma_1 \to J_1 = \left(\begin{array}{cc|c} -t & 1 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$$

$$\sigma_n \to J_n = \left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 1 & 0 \\ & t & -t \end{array}\right)$$

For more details, see [2] and [5].

3. Perron Representation

The Burau bis representation extends to $A(E_{n+1,p})$ for all possible values of n and p in the following way.
We define the following $n \times n$ matrices:

$$A = (\lambda_1 b, \lambda_2 b, \ldots, \lambda_n b)$$
$$B = (0, \ldots, 0, b, 0, \ldots, 0)$$
$$C = (\lambda_1 d, \lambda_2 d, \ldots, \lambda_n d)$$
$$D = (0, \ldots, 0, d, 0, \ldots, 0),$$

where 0 denotes a column of n zeros, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$.

For each $i = 1, \ldots, n$, we have that b_i satisfies the following conditions

$$tb_i = -td_{i-1} + (1+t)d_i - d_{i+1}, \;\; i \neq p,$$
$$tb_p = -td_{p-1} + (1+t)d_p - d_{p+1} + t,$$
$$\sum_{i=1}^{n} \lambda_i b_i = -(1 + d_p + t),$$

setting any undefined d_j equal zero.

For any choice $\lambda = (\lambda_1, \ldots, \lambda_n)$, we get a linear representation

$$\psi_\lambda : A(E_{n+1,p}) \to Gl_{2n}(R),$$

where R is the field of rational fractions in n+1 indeterminates $\mathbb{Q}(t, d_1, ..., d_n)$.

$$\psi_\lambda(\sigma_i) \to \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix},$$

$$\psi_\lambda(\delta) \to \begin{pmatrix} I_n + A & B \\ C & I_n + D \end{pmatrix}.$$

For more details, see [2].

4. Reducibility of $\psi_\lambda : A(E_{4,1}) \to GL_6(\mathbb{C})$

Having defined Perron's representation, we set $n = 3$ and $p = 1$ to get the following vectors. $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

After we specialize the indeterminate d_3 to $\frac{-t(1+t+t^2)}{1+t}$, we get the following 3×3 matrices:

$$A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 & \lambda_3 b_2 \\ \lambda_1 b_3 & \lambda_2 b_3 & \lambda_3 b_3 \end{pmatrix},$$

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 \\ \lambda_1 d_2 & \lambda_2 d_2 & \lambda_3 d_2 \\ \frac{-t(1+t+t^2)}{1+t}\lambda_1 & \frac{-t(1+t+t^2)}{1+t}\lambda_2 & \frac{-t(1+t+t^2)}{1+t}\lambda_3 \end{pmatrix},$$

and

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ d_2 & 0 & 0 \\ \frac{-t(1+t+t^2)}{1+t} & 0 & 0 \end{pmatrix}.$$

Simple computations show that the parameters satisfy the following equations:

- $tb_2 = -td_1 + (1+t)d_2 + \frac{t(1+t+t^2)}{1+t}$
- $tb_3 = -td_2 - t(1+t+t^2)$
- $tb_1 = (1+t)d_1 - d_2 + t$
- $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = -(1 + t + d_1)$

Having defined the 3×3 matrices A, B, C and D, we obtain the multiparameter representation $A(E_{4,1})$. This representation is of degree 6. We specialize the parameters $\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3, d_1, d_2, t$ to values in $\mathbb{C} - \{0\}$. We further assume that $t \neq -1$. The representation $\psi_\lambda : A(E_{4,1}) \to GL_6(\mathbb{C})$ is defined as follows:

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ t & 0 & 0 & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & t & 0 & t & -t & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & t & -t \end{pmatrix},$$

and

$$\psi_\lambda(\delta) = \begin{pmatrix} 1+\lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & b_1 & 0 & 0 \\ \lambda_1 b_2 & 1+\lambda_2 b_2 & \lambda_3 b_2 & b_2 & 0 & 0 \\ \lambda_1 b_3 & \lambda_2 b_3 & 1+\lambda_3 b_3 & b_3 & 0 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & 1+d_1 & 0 & 0 \\ \lambda_1 d_2 & \lambda_2 d_2 & \lambda_3 d_2 & d_2 & 1 & 0 \\ \frac{-t(1+t+t^2)}{1+t}\lambda_1 & \frac{-t(1+t+t^2)}{1+t}\lambda_2 & \frac{-t(1+t+t^2)}{1+t}\lambda_3 & \frac{-t(1+t+t^2)}{1+t} & 0 & 1 \end{pmatrix}.$$

The graph $E_{4,1}$ has 4 vertices $\sigma_1, \sigma_2, \sigma_3$ and δ. Since $p = 1$, it follows that the vertex δ has a common edge with $\sigma_p = \sigma_1$. Therefore, the following relations are satisfied.

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \quad (4.1)$$

$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \quad (4.2)$$

$$\sigma_1\sigma_3 = \sigma_3\sigma_1 \quad (4.3)$$

$$\sigma_2\delta = \delta\sigma_2 \quad (4.4)$$

$$\sigma_3\delta = \delta\sigma_3 \qquad (4.5)$$

$$\sigma_1\delta\sigma_1 = \delta\sigma_1\delta \qquad (4.6)$$

We note that relations (4.1),(4.2), and (4.3) are actually Artin's braid relation of the classical braid group, B_4 having σ_1, σ_2, and σ_3 as standard generators. This assures that a representation of $A(E_{4,1})$ yields a representation of B_4.
For more details, see [1] and [4].

Lemma 1. *The representation $\psi_\lambda : A(E_{4,1}) \to GL_6(\mathbb{C})$ is reducible.*

Proof. For simplicity, we write σ_i instead of $\psi_\lambda(\sigma_i)$ .The subspace
$S = \left\langle e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3,\ e_4,\ e_5,\ e_6 \right\rangle$ is an invariant subspace of dimension 4.
To see this:

(i) $\sigma_1(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + te_4 \in S$

(ii) $\sigma_2(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + t\frac{b_2}{b_1}e_5 \in S$

(iii) $\sigma_3(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + t\frac{b_3}{b_1}e_6 \in S$

(iv) $\delta(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) = (1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) + \frac{d_1}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_4 + \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_5 + \frac{-(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)e_6 \in S$

(v) $\sigma_1 e_4 = -te_4 \in S$

(vi) $\sigma_2 e_4 = e_4 + te_5 \in S$

(vii) $\sigma_3 e_4 = e_4 \in S$

(viii) $\delta e_4 = b_1(e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3) + (1 + d_1)e_4 + d_2 e_5 \frac{-t(1+t+t^2)}{1+t}e_6 \in S$

(ix) $\sigma_1 e_5 = e_4 + e_5 \in S$

(x) $\sigma_2 e_5 = -te_5 \in S$

(xi) $\sigma_3 e_5 = e_5 + te_6 \in S$

(xii) $\delta e_5 = e_5 \in S$

(xiii) $\sigma_1 e_6 = e_6 \in S$

(xiv) $\sigma_2 e_6 = e_5 + e_6 \in S$

(xv) $\sigma_3 e_6 = -te_6 \in S$

(xvi) $\delta e_6 = e_6 \in S$

5. On the Irreducibility of $\psi'_\lambda : A(E_{4,1}) \to GL_4(\mathbb{C})$

We consider the representation $\psi_\lambda : A(E_{4,1}) \to GL_6(\mathbb{C})$ restricted to the basis e_1, e_2, $e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3, e_4, e_5$, and e_6. The matrix of σ_1 becomes

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We reduce our representation to a 4-dimensional one by considering the sub-basis $e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3, e_4, e_5$, and e_6 to get $\psi'_\lambda : A(E_{4,1}) \to GL_4(\mathbb{C})$. The representation is defined as follows:

$$\psi'_\lambda(\sigma_1) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{tb_2}{b_1} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & \frac{tb_3}{b_1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & -t \end{pmatrix},$$

and

$\psi'_\lambda(\delta) =$

$$\begin{pmatrix} 1+\sum_{i=1}^{3}\lambda_i b_i & \frac{d_1}{b_1}(\sum_{i=1}^{3}\lambda_i b_i) & \frac{d_2}{b_1}(\sum_{i=1}^{3}\lambda_i b_i) & \frac{-t(1+t+t^2)}{b_1(1+t)}(\sum_{i=1}^{3}\lambda_i b_i) \\ b_1 & 1+d_1 & d_2 & -t(1+t+t^2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then diagonalize the matrix corresponding to $\psi'_\lambda(\sigma_1)$ by an invertible matrix, say T, and conjugate the matrices of $\psi'_\lambda(\sigma_2)$, $\psi'_\lambda(\sigma_3)$, and $\psi'_\lambda(\delta)$ by the same matrix T. The invertible matrix T is given by

$$T = \begin{pmatrix} 0 & 0 & 1 & t \\ 0 & 0 & 0 & -1-t \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, a computation shows that

$$T^{-1}\psi'_\lambda(\sigma_1)T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\psi'_\lambda(\sigma_2)T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-t^2}{1+t} & 0 & \frac{-(1+t+t^2)}{1+t} \\ 0 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} & 1 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ 0 & \frac{-t}{1+t} & 0 & \frac{1}{1+t} \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_3)T = \begin{pmatrix} -t & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{tb_3}{b_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$T^{-1}\psi'_\lambda(\delta)T =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-t(1+t+t^2)}{(1+t)^2} & 1+\frac{d_2}{1+t} & \frac{b_1}{1+t} & \frac{t}{1+t} \\ \frac{-t(1+t+t^2)}{b_1(1+t)}k & \frac{d_2}{b_1}k & 1+k & \frac{(-d_1(1+t)+d_2+b_1t)((\sum_{i=1}^3\lambda_ib_i)(1+t)+b_1t)}{b_1(1+t)} \\ \frac{t(1+t+t^2)}{1+t} & \frac{-d_2}{1+t} & \frac{-b_1}{1+t} & \frac{1}{1+t} \end{pmatrix}.$$

where $k = \frac{b_1t}{1+t} + \sum_{i=1}^3 \lambda_ib_i$.

The entries of the matrices $T^{-1}\psi'_\lambda(\sigma_2)T$ and $T^{-1}\psi'_\lambda(\delta)T$ are well-defined since we assume in our work that $t \neq -1$. For simplicity, we denote $T^{-1}\psi'_\lambda(\sigma_1)T$ by $\psi'_\lambda(\sigma_1)$, $T^{-1}\psi'_\lambda(\sigma_2)T$ by $\psi'_\lambda(\sigma_2)$, $T^{-1}\psi'_\lambda(\sigma_3)T$ by $\psi'_\lambda(\sigma_3)$, and $T^{-1}\psi'_\lambda(\delta)T$ by $\psi'_\lambda(\delta)$.

We now prove some lemmas and propositions to determine a sufficient and necessary condition for irreducibility of $\psi'_\lambda : A(E_{4,1}) \to GL_4(\mathbb{C})$.

Lemma 2. *The proper subspace* $S = \left\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \right\rangle$ *is not invariant if and only if* $t^4 + t^3 + t^2 + t + 1 \neq 0$.

Proof. First, we prove that proper subspace $S = \left\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \right\rangle$ is not invariant if $t^4 + t^3 + t^2 + t + 1 \neq 0$.
Assume, for contradiction, that S is invariant.

We have $\psi'_\lambda(\sigma_2)(e_4) = \begin{pmatrix} 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} \in S.$

This implies that $(1 + t + t^2)b_3 = -t(b_2 + tb_2 + tb_1)$.

By using the equations: $tb_2 = -td_1 + (1+t)d_2 + \frac{t(1+t+t^2)}{1+t}, tb_3 = -td_2 - t(1+t+t^2)$, and $tb_1 = (1 + t)d_1 - d_2 + t$, simple computations give $t^4 + t^3 + t^2 + t + 1 = 0$, a contradiction.

On the other hand, we assume that $t^4 + t^3 + t^2 + t + 1 = 0$. We prove that the proper subspace $S = \left\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \right\rangle$ is invariant as follows:

(i) $\psi'_\lambda\sigma_1(e_1) = e_1 \in S$.

(ii) $\psi'_\lambda\sigma_2(e_1) = e_1 \in S$.

(iii) $\psi'_\lambda\sigma_3(e_1) = \begin{pmatrix} -t \\ t \\ \frac{tb_3}{b_1} \\ 0 \end{pmatrix} \in S$.

(iv) $\psi'_\lambda\delta(e_1) = \begin{pmatrix} 1 \\ \frac{-t(1+t+t^2)}{(1+t)^2} \\ \frac{-t^2(1+t+t^2)}{(1+t)^2} + \frac{-t(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \\ \frac{-t(1+t+t^2)}{(1+t)^2} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3)$.

Here, we have $a = 1$, $b = \frac{-d_3}{1+t}$, $c = \frac{d_3}{1+t}$, and
$\frac{cb_3}{b_1} = \frac{-t^2(1+t+t^2)}{(1+t)^2} + \frac{-t(1+t+t^2)}{b_1(1+t)}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)$.

Thus, $\frac{b_3}{b_1} = \frac{b_1 t + (1+t)(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)}{b_1}$. (5.1)

(v) $\psi'_\lambda\sigma_1(e_4) = -te_4 \in S$.

(vi) $\psi'_\lambda\sigma_2(e_4) = \begin{pmatrix} 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3)$.

Here, we have $a = 1$, $b = \frac{1}{1+t}$, $c = \frac{-(1+t+t^2)}{1+t}$, and $\frac{cb_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)}$.

Thus, $-(1+t+t^2)\frac{b_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1}$. (5.2)

(vii) $\psi'_\lambda\sigma_3(e_4) = e_4 \in S$.

(viii) $$\psi'_\lambda\delta(e_4) = \begin{pmatrix} 0 \\ \frac{t}{1+t} \\ \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1b_1+\lambda_2b_2+\lambda_3b_3)(1+t)+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have $a = 0$, $b = \frac{1}{1+t}$, $c = \frac{t}{1+t}$, and $\frac{cb_3}{b_1} = \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1b_1+\lambda_2b_2+\lambda_3b_3)(1+t)+b_1t)}{b_1(1+t)}$.

Thus, $\frac{b_3}{b_1} = \frac{(-d_1(1+t)+d_2+b_1t)((\lambda_1b_1+\lambda_2b_2+\lambda_3b_3)(1+t)+b_1t)}{b_1t}$. (5.3)

(ix) $\psi'_\lambda\sigma_1(e_2 + \frac{b_3}{b_1}e_3) = e_2 + \frac{b_3}{b_1}e_3 \in S$.

(x) $$\psi'_\lambda\sigma_2(e_2 + \frac{b_3}{b_1}e_3) = \begin{pmatrix} 0 \\ \frac{-t^2}{1+t} \\ \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} + \frac{b_3}{b_1} \\ \frac{-t}{1+t} \end{pmatrix} = ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3).$$

Here, we have $a = 1$, $b = \frac{-t}{1+t}$, $c = \frac{-t^2}{1+t}$, and $\frac{cb_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1(1+t)} + \frac{b_3}{b_1}$.

Thus, $-(1 + t + t^2)\frac{b_3}{b_1} = \frac{t(b_2+b_2t+b_1t)}{b_1}$. (5.4)

(xi) $\psi'_\lambda\sigma_3(e_2 + \frac{b_3}{b_1}e_3) = e_2 + \frac{b_3}{b_1}e_3 \in S$.

(xii) $\psi'_\lambda\delta(e_2 + \frac{b_3}{b_1}e_3) =$

$$\begin{pmatrix} 0 \\ 1 + \frac{d_2}{1+t} + \frac{b_3}{1+t} \\ \frac{d_2}{b_1}(\lambda_1b_1 + \lambda_2b_2 + \lambda_3b_3 + \frac{b_1t}{1+t}) + \frac{b_3}{b_1}(1 + \lambda_1b_1 + \lambda_2b_2 + \lambda_3b_3 + \frac{b_1t}{1+t}) \\ \frac{-d_2}{1+t} - \frac{b_3}{1+t} \end{pmatrix}$$

$= ae_1 + be_4 + c(e_2 + \frac{b_3}{b_1}e_3)$.

Here, we have $a = 0$, $b = \frac{-d_2}{1+t} - \frac{b_3}{1+t}$, $c = 1 + \frac{d_2}{1+t} + \frac{b_3}{1+t}$, and $c\frac{b_3}{b_1} = \frac{d_2}{b_1}(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t}) + \frac{b_3}{b_1}(1 + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \frac{b_1 t}{1+t})$.

Thus, $(1+\frac{d_2}{1+t}+\frac{b_3}{1+t})\frac{b_3}{b_1} = \frac{d_2}{b_1}(\lambda_1 b_1+\lambda_2 b_2+\lambda_3 b_3+\frac{b_1 t}{1+t})+\frac{b_3}{b_1}(1+\lambda_1 b_1+\lambda_2 b_2+\lambda_3 b_3+\frac{b_1 t}{1+t})$. (5.5)

By simple computations, we can verify that equations (5.1), (5.2), (5.3) , and (5.5) are clearly satisfied without any assumption of the indeterminates whereas equation (5.4) is satisfied only if $t^4 + t^3 + t^2 + t + 1 = 0$.

Lemma 3. *Any proper subspace S containing the vector $e_i + ue_j + ve_k$, where $i, j, k \in \{1, 2, 3, 4\}$, except possibly the subspace having the form $\left\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \right\rangle$, is not invariant.*

Proof. We consider all the subspaces containing the vector $e_i + ue_j + ve_k$, where $i, j, k \in \{1, 2, 3, 4\}$ except possibly the subspace of the form $\left\langle e_1, e_4, e_2 + \frac{b_3}{b_1}e_3 \right\rangle$.
We then assume, for contradiction, that each considered subspace is invariant.
In each case, simple computations give a contradiction.

Thus, we have determined a necessary and sufficient condition for irreducibility.

Theorem 1. *Assume all the indeterminates used in defining Perron representation of degree 4 are non zero complex numbers. Let $d_3 = \frac{-t(1+t+t^2)}{1+t}$ and $t \neq -1$. The representation $\psi'_\lambda : A(E_{4,1}) \to GL_4(\mathbb{C})$ is irreducible if and only if $t^4 + t^3 + t^2 + t + 1 \neq 0$.*

In the following sections, we set $n = 4$ and $p = 1$ and we study the irreducibility of the reduced representation of $\psi_\lambda : A(E_{5,1}) \to GL_8(\mathbb{C})$. Indeed, we obtain a sufficient and necessary condition that gauarantees the irreducibility of $\psi'_\lambda : A(E_{5,1}) \to GL_5(\mathbb{C})$.

6. Reducibility of $\psi_\lambda : A(E_{5,1}) \to GL_8(\mathbb{C})$

Having defined Perron's representation, we set $n = 4$ and $p = 1$ to get the following vectors. $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$, and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

After we specialize the indeterminates d_2 and d_3 to $-(1+t+t^2)$ and $-t(1+t)$ respectively, we get the following 4×4 matrices:

$$A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & \lambda_4 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 & \lambda_3 b_2 & \lambda_4 b_2 \\ \lambda_1 b_3 & \lambda_2 b_3 & \lambda_3 b_3 & \lambda_4 b_3 \\ \lambda_1 b_4 & \lambda_2 b_4 & \lambda_3 b_4 & \lambda_4 b_4 \end{pmatrix},$$

$$B = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & \lambda_4 d_1 \\ -(1+t+t^2)\lambda_1 & -(1+t+t^2)\lambda_2 & -(1+t+t^2)\lambda_3 & -(1+t+t^2)\lambda_4 \\ -t(1+t)\lambda_1 & -t(1+t)\lambda_2 & -t(1+t)\lambda_3 & -t(1+t)\lambda_4 \\ \lambda_1 d_4 & \lambda_2 d_4 & \lambda_3 d_4 & \lambda_4 d_4 \end{pmatrix},$$

and

$$D = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ -(1+t+t^2) & 0 & 0 & 0 \\ -t(1+t) & 0 & 0 & 0 \\ d_4 & 0 & 0 & 0 \end{pmatrix}.$$

Simple computations show that the parameters satisfy the following equations:

- $tb_2 = -td_1 - (1+t)(1+t+t^2) + t(1+t) = -td_1 - (1+t)(1+t^2)$
- $tb_3 = t(1+t+t^2) - t(1+t)^2 - d_4 = -t(2t^2+3t+2) - d_4$
- $tb_4 = t^2(1+t) + (1+t)d_4$
- $tb_1 = (1+t)d_1 + 1 + 2t + t^2$
- $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 = -(1+t+d_1)$

Having defined the 4×4 matrices A, B, C and D, we obtain the multiparameter representation $A(E_{5,1})$. This representation is of degree 8. We specialize the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4, d_1, d_4, t$ to values in $\mathbb{C} - \{0\}$. We further assume that $t \neq -1$. The representation $\psi_\lambda : A(E_{5,1}) \rightarrow GL_8(\mathbb{C})$ is defined as follows:

$$\psi_\lambda(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & -t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & t & -t & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_\lambda(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & t & -t \end{pmatrix},$$

and

$\psi_\lambda(\delta)$=

$$\begin{pmatrix} 1+\lambda_1 b_1 & \lambda_2 b_1 & \lambda_3 b_1 & \lambda_4 b_1 & b_1 & 0 & 0 & 0 \\ \lambda_1 b_2 & 1+\lambda_2 b_2 & \lambda_3 b_2 & \lambda_4 b_2 & b_2 & 0 & 0 & 0 \\ \lambda_1 b_3 & \lambda_2 b_3 & 1+\lambda_3 b_3 & \lambda_4 b_3 & b_3 & 0 & 0 & 0 \\ \lambda_1 b_4 & \lambda_2 b_4 & \lambda_3 b_4 & 1+\lambda_4 b_4 & b_4 & 0 & 0 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & \lambda_3 d_1 & \lambda_4 d_1 & 1+d_1 & 0 & 0 & 0 \\ k\lambda_1 & k\lambda_2 & k\lambda_3 & k\lambda_4 & k & 1 & 0 & 0 \\ -t(1+t)\lambda_1 & -t(1+t)\lambda_2 & -t(1+t)\lambda_3 & -t(1+t)\lambda_4 & -t(1+t) & 0 & 1 & 0 \\ \lambda_1 d_4 & \lambda_2 d_4 & \lambda_3 d_4 & \lambda_4 d_4 & d_4 & 0 & 0 & 1 \end{pmatrix},$$

where $k = -(1+t+t^2)$.

The graph $E_{5,1}$ has 5 vertices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and δ. Since $p = 1$, it follows that the vertex δ has a common edge with $\sigma_p = \sigma_1$. Therefore, the following relations are satisfied.

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \qquad (6.1)$$

$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \qquad (6.2)$$

$$\sigma_3\sigma_4\sigma_3 = \sigma_4\sigma_3\sigma_4 \qquad (6.3)$$

$$\sigma_1\sigma_3 = \sigma_3\sigma_1 \qquad (6.4)$$

$$\sigma_1\sigma_4 = \sigma_4\sigma_1 \qquad (6.5)$$

$$\sigma_2\sigma_4 = \sigma_4\sigma_2 \qquad (6.6)$$

$$\sigma_2\delta = \delta\sigma_2 \qquad (6.7)$$

$$\sigma_3\delta = \delta\sigma_3 \qquad (6.8)$$

$$\sigma_4\delta = \delta\sigma_4 \qquad (6.9)$$

$$\sigma_1\delta\sigma_1 = \delta\sigma_1\delta \qquad (6.10)$$

We note that relations (6.1),(6.2), (6.3), (6.4), (6.5) and (6.6) are actually Artin's braid relation of the classical braid group, B_5 having σ_1, σ_2, σ_3, and σ_4 as standard generators. This assures that a representation of $A(E_{5,1})$ yields a representation of B_5.
For more details, see [1] and [4].

Lemma 4. *The representation $\psi_\lambda : A(E_{5,1}) \to GL_8(\mathbb{C})$ is reducible.*

Proof. For simplicity, we write σ_i instead of $\psi_\lambda(\sigma_i)$ .The subspace
$S = \left\langle e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + \frac{b_4}{b_1}e_4, e_5, e_6, e_7, e_8 \right\rangle$ is an invariant subspace of dimension 5.

7. On the Irreducibility of $\psi'_\lambda : A(E_{5,1}) \to GL_5(\mathbb{C})$

We consider the representation $\psi_\lambda : A(E_{5,1}) \to GL_8(\mathbb{C})$ restricted to the basis e_1, e_2, e_3, $e_1+\frac{b_2}{b_1}e_2+\frac{b_3}{b_1}e_3+\frac{b_4}{b_1}e_4, e_5, e_6, e_7$, and e_8 to get the subrepresentation $\psi'_\lambda : A(E_{5,1}) \to GL_5(\mathbb{C})$ which is the representation restricted to the sub-basis $e_1 + \frac{b_2}{b_1}e_2 + \frac{b_3}{b_1}e_3 + \frac{b_4}{b_1}e_4, e_5, e_6, e_7$. This representation is defined as follows:

$$\psi'_\lambda(\sigma_1) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{tb_2}{b_1} & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & \frac{tb_3}{b_1} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\psi'_\lambda(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{tb_4}{b_1} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & -t \end{pmatrix},$$

and
$\psi'_\lambda(\delta) =$

$$\begin{pmatrix} 1+r & \frac{d_1}{b_1}r & \frac{-(1+t+t^2)}{b_1}r & \frac{-t(1+t)}{b_1}r & \frac{d_4}{b_1}r \\ b_1 & 1+d_1 & -(1+t+t^2) & -t(1+t) & d_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $r = \sum_{i=1}^{4} \lambda_i b_i$.

We then diagonalize the matrix corresponding to $\psi'_\lambda(\sigma_1)$ by an invertible matrix, say T, and conjugate the matrices of $\psi'_\lambda(\sigma_2)$, $\psi'_\lambda(\sigma_3)$,$\psi'_\lambda(\sigma_4)$ and $\psi'_\lambda(\delta)$ by the same matrix T. The invertible matrix T is given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & -1-t \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, a computation shows that

$$T^{-1}\psi'_\lambda(\sigma_1)T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix}.$$

After conjugation, we get

$$T^{-1}\psi'_\lambda(\sigma_2)T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & \frac{-t^2}{1+t} & 0 & \frac{-(1+t+t^2)}{1+t} \\ 0 & 0 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} & 1 & \frac{t(b_2+b_1t+b_2t)}{b_1(1+t)} \\ 0 & 0 & \frac{-t}{1+t} & 0 & \frac{1}{1+t} \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_3)T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & t & 1 & 0 & 0 \\ 0 & \frac{tb_3}{b_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}\psi'_\lambda(\sigma_4)T = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{tb_4}{b_1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and
$T^{-1}\psi'_\lambda(\delta)T =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{d_4}{1+t} & -t & 1+\frac{-(1+t+t^2)}{1+t} & \frac{b_1}{1+t} & \frac{t}{1+t} \\ \frac{d_4}{b_1}w & \frac{-t(1+t)}{b_1}w & \frac{-(1+t+t^2)}{b_1}w & 1+w & \frac{(-d_1(1+t)-(1+t+t^2)+b_1t)}{b_1}w \\ \frac{-d_4}{1+t} & t & \frac{1+t+t^2}{1+t} & \frac{-b_1}{1+t} & \frac{1}{1+t} \end{pmatrix},$$

where $w = \sum_{i=1}^{4} \lambda_i b_i + \frac{b_1 t}{1+t}$.

The entries of the matrices $T^{-1}\psi'_\lambda(\sigma_2)T$,$T^{-1}\psi'_\lambda(\sigma_3)T$, $T^{-1}\psi'_\lambda(\sigma_4)T$ and $T^{-1}\psi'_\lambda(\delta)T$ are well-defined since we assume in our work that $t \neq -1$. For simplicity, we denote $T^{-1}\psi'_\lambda(\sigma_1)T$ by $\psi'_\lambda(\sigma_1)$, $T^{-1}\psi'_\lambda(\sigma_2)T$ by $\psi'_\lambda(\sigma_2)$, $T^{-1}\psi'_\lambda(\sigma_3)T$ by $\psi'_\lambda(\sigma_3)$, $T^{-1}\psi'_\lambda(\sigma_4)T$ by $\psi'_\lambda(\sigma_4)$, and $T^{-1}\psi'_\lambda(\delta)T$ by $\psi'_\lambda(\delta)$.

We now prove some lemmas and propositions to determine a sufficient and necessary condition for irreducibility of $\psi'_\lambda : A(E_{5,1}) \to GL_5(\mathbb{C})$.

Lemma 5. *Except possibly the subspaces having the forms $\langle e_1, e_3, e_5, e_2 + ue_4\rangle$ and $\langle e_2, e_3, e_5, e_1 + ue_4\rangle$, where $u \in \mathbb{C}^*$, every proper subspace is not invariant.*

Proof. We assume, for contradiction, that every subspace, except those having the forms $\langle e_1, e_3, e_5, e_2 + ue_4\rangle$ and $\langle e_2, e_3, e_5, e_1 + ue_4\rangle$, is invariant.
We then study each possible form. In each case, simple computations give a contradiction.

Lemma 6. *If $t^3 \neq -1$, then the subspaces $\langle e_1, e_3, e_5, e_2 + ue_4\rangle$ and $\langle e_2, e_3, e_5, e_1 + ue_4\rangle$ are not invariant.*

Proof. First, we assume, for contradiction, that $S = \langle e_1, e_3, e_5, e_2 + ue_4\rangle$ is invariant.

- $\psi'_\lambda \sigma_4(e_1) = \begin{pmatrix} -t \\ t \\ 0 \\ \frac{tb_4}{b_1} \\ 0 \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4)$ which implies that $u = \frac{b_4}{b_1}$.
 (7.1)

- $\psi'_\lambda \sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1 t + b_2 + b_2 t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4)$ which implies that $u = \frac{t(b_1 t + b_2 + b_2 t)}{b_1(1+t)}$. (7.2)

- $\psi'_\lambda \sigma_3(e_2 + ue_4) = \begin{pmatrix} 1 \\ -t \\ t \\ u + \frac{tb_3}{b_1} \\ 0 \end{pmatrix} = ae_1 + be_3 + ce_5 + d(e_2 + ue_4)$ which implies that $u = \frac{-tb_3}{b_1(1+t)}$. (7.3)

Since equations (7.1) and (7.3) are equal, we have $(1 + t + t^2)d_4 = -t^2(1 + t + t^2)$. Thus, $d_4 = -t^2$.

Moreover, equations (7.2) and (7.3) are equal. This implies that $d_4 = -(1+t^2)^2 + t^2$. By substituting $d_4 = -t^2$, we get $t^4 + t^3 + t + 1 = (t+1)(t^3+1) = 0$, a contradiction.

Now, we assume, for contradiction, that $S = \langle e_2, e_3, e_5, e_1 + ue_4\rangle$ is invariant.

- $\psi'_\lambda \sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4).$

  This implies that $t(b_1t + b_2 + b_2t) = 0$.

  Simple computations give $-(1+t+t^2)^2 + t(1+t)^2 + t^2 = 0$. Thus, $t^4 + t^3 + t + 1 = 0$, a contradiction.

We now determine conditions under which one of the subspaces mentioned in Lemma 7 is invariant. But first we write down the following lemma.

Lemma 7. *The proper subspaces* $S_1 = \langle e_1, e_3, e_5, e_2 + ue_4 \rangle$ *and* $S_2 = \langle e_2, e_3, e_5, e_1 + ue_4 \rangle$ *cannot be both invariant.*

Proof. Assume that S_1 is invariant. This implies that $\psi'_\lambda \sigma_2(e_3)$ and $\psi'_\lambda \sigma_3(e_2 + ue_4) \in S_1$. Simple computations give $b_1t + b_2 + b_2t = -b_3 \neq 0$.

Assume, for contradiction, that S_2 is invariant. This implies that $\psi'_\lambda \sigma_2(e_3) \in S_2$. Simple computations give $b_1t + b_2 + b_2t = 0$, a contradiction.

Lemma 8. *If* $t^3 = -1$*, then the subspace* $S = \langle e_2, e_3, e_5, e_1 + ue_4 \rangle$ *is invariant.*

Proof.

- $\psi'_\lambda \sigma_1(e_2) = \psi'_\lambda \sigma_2(e_2) = \psi'_\lambda \sigma_4(e_2) = e_2 \in S$.

- $\psi'_\lambda \sigma_3(e_2) = \begin{pmatrix} 1 \\ -t \\ t \\ \frac{tb_3}{b_1} \\ 0 \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$

  if $u = \frac{tb_3}{b_1}$. (7.4)

- $\psi'_\lambda\delta(e_2) = \begin{pmatrix} 0 \\ 1 \\ \frac{d_3}{1+t} \\ \frac{d_3 t}{1+t} + \frac{d_3}{b_1}(\sum_{i=1}^4 \lambda_i b_i) \\ \frac{-d_3}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$
  if $\frac{t}{1+t} = \frac{-\sum_{i=1}^4 \lambda_i b_i}{b_1}$. (7.5)

- $\psi'_\lambda\sigma_1(e_3)=\psi'_\lambda\sigma_3(e_3)=\psi'_\lambda\sigma_4(e_3)=e_3 \in S.$

- $\psi'_\lambda\sigma_2(e_3) = \begin{pmatrix} 0 \\ 1 \\ \frac{-t^2}{1+t} \\ \frac{t(b_1 t+b_2+b_2 t)}{b_1(1+t)} \\ \frac{-t}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$
  if $t(b_1 t + b_2 + b_2 t) = 0.$ (7.6)

- $\psi'_\lambda\delta(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 + \frac{-(1+t+t^2)}{1+t} \\ \frac{-t^2(1+t+t^2)}{1+t} + \frac{-(1+t+t^2)}{b_1}(\sum_{i=1}^4 \lambda_i b_i) \\ \frac{1+t+t^2}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$
  if $\sum_{i=1}^4 \lambda_i b_i + \frac{b_1 t}{1+t} = 0.$ (7.7)

- $\psi'_\lambda\sigma_1(e_5) = -te_5 \in S.$

- $\psi'_\lambda\sigma_3(e_5)=\psi'_\lambda\sigma_4(e_5)=e_5 \in S.$

- $\psi'_\lambda \sigma_2(e_5) = \begin{pmatrix} 0 \\ 1 \\ \frac{-(1+t+t^2)}{1+t} \\ \frac{t(b_1t+b_2+b_2t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$

  if $t(b_1t + b_2 + b_2t) = 0.$ (7.8)

- $\psi'_\lambda \delta(e_5) = \begin{pmatrix} 0 \\ 0 \\ \frac{t}{1+t} \\ \frac{(-d_1(1+t)-t(1+t+t^2)+b_2t)(\sum_{i=1}^4 \lambda_i b_i(1+t)+b_1t)}{b_1(1+t)} \\ \frac{1}{1+t} \end{pmatrix} = ae_2+be_3+ce_5+d(e_1+ue_4),$

  if $(-d_1(1+t) + d_2 + b_2t)(\sum_{i=1}^4 \lambda_i b_i(1+t) + b_1t) = 0.$ (7.9)

- $\psi'_\lambda \sigma_1(e_1 + ue_4)=\psi'_\lambda \sigma_2(e_1 + ue_4)=\psi'_\lambda \sigma_3(e_1 + ue_4)=e_1 + ue_4 \in S.$

- $\psi'_\lambda \sigma_4(e_1 + ue_4) = \begin{pmatrix} -t \\ t \\ 0 \\ u + \frac{tb_4}{b_1} \\ 0 \end{pmatrix} = ae_2 + be_3 + ce_5 + d(e_1 + ue_4),$

  if $u = \frac{-tb_4}{b_1(1+t)}.$ (7.10)

- $\psi'_\lambda \delta(e_1 + ue_4) = \begin{pmatrix} 1 \\ 0 \\ \frac{d_4}{1+t} + \frac{b_1u}{1+t} \\ \frac{d_4}{b_1}(\sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t}) + u(1 + \sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t}) \\ \frac{-d_4}{1+t} - \frac{b_1u}{1+t} \end{pmatrix} = ae_2 + be_3 +$
  $ce_5 + d(e_1 + ue_4),$

  if $(\sum_{i=1}^4 \lambda_i b_i + \frac{b_1t}{1+t})(\frac{d_4}{b_1} + u) = 0.$ (7.11)

Using the relations, we prove that equations (7.5), (7.7), (7.9), and (7.11) are clearly satisfied.
Also, we verify that equations (7.4), (7.6), (7.8) and (7.10) are satisfied if $-t(1+t)^2 = -(1+t+t^2)(1+t^2)+t^2$ which implies that $t^3 = -1$.

Thus, we have determined a necessary and sufficient condition for irreducibility.

Theorem 2. *Assume all the indeterminates used in defining Perron representation of degree 5 are non zero complex numbers. Let* $d_2 = -(1+t+t^2)$, $d_3 = -t(1+t)$, *and* $t \neq -1$. *The representation* $\psi'_\lambda : A(E_{5,1}) \to GL_5(\mathbb{C})$ *is irreducible if and only if* $t^3 \neq -1$.

Remark 1.

- *For n=2 and for* $t \neq -1$, *we proved that a complex specialization of the representation* $\psi'_\lambda : A(E_{3,1}) \to Gl_3(\mathbb{C})$ *is irreducible if and only if* $t^2 \neq -1$ *which is equivalent to* $t^3+t^2+t+1 \neq 0$.
- *For n=3 and for* $t \neq -1$, *we have proved that a complex specialization of the representation* $\psi'_\lambda : A(E_{4,1}) \to Gl_4(\mathbb{C})$ *is irreducible if and only if* $t^4+t^3+t^2+t+1 \neq 0$.
- *For n=4 and for* $t \neq -1$, *we have proved that a complex specialization of the representation* $\psi'_\lambda : A(E_{5,1}) \to Gl_5(\mathbb{C})$ *is irreducible if and only if* $t^3 \neq -1$ *which is equivalent to* $t^5+t^4+t^3+t^2+t+1 \neq 0$.

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