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LOCALLY β-CLOSED SPACES

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Abstract. In this paper, we generalize the notion of β-closed-ness [7] to arbitrary subsets and in terms of it we introduce the class of locally β-closed spaces and also investigate of its several properties. It is observed that although local β-closedness is independent of local compact T_2-ness but one can be obtained from the other by the help of a new class of functions viz. β-θ-closed functions which are independent not only of closed functions but also of continuous functions.

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1. Introduction

Among various generalized open sets, the notion of β-open sets introduced by Abd. El-Monsef et al. [1] which is equivalent to the notion of semipre-open sets due to Andrijević [4], plays a significant role in General Topology and Real Analysis. Now a days many topologists have focused their research on various topics, using β-open sets. Mention may be made of some of the recent works which are found in [1,2,3, 4,5,6,7,8,9,10,11,12,13,18,19,20,21,23]. Very recently Basu and Ghosh [7] by the help of β-open sets introduced a covering property known as β-closedness and characterized such spaces from different angles. In this paper, we

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generalize the concept of β-closedness to arbitrary subsets and using such sets, we introduce and investigate locally β-closed spaces. Although we have seen that local β-closedness is independent of local compact T_2-ness but our intension is to achieve either of the spaces from the other. In this regard, a new class of functions called β-θ-closed function is introduced, which quite satisfactorily enables to establish our goal. In addition, a sufficient condition for a locally β-closed space to be extremally disconnected is also established.

2. Preliminaries

Throughout the paper, spaces X and Y will always denote topological spaces without any separation axioms and $\psi : X \to Y$ will represent a (single valued) function. Given a set A, its closure and interior are denoted by $cl(A)$ and $int(A)$ respectively. A set A is said to be α-open [17] (resp. preopen [16], semi-open [14], β-open [1] or semi-preopen [4]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$, $A \subset cl(int(A))$, $A \subset cl(int(cl(A)))$). The complement of a β-open (resp. semi-open) set is said to be β-closed [1] or semi-preclosed [4] (resp. semi-closed [14]). The intersection of all β-closed (resp. semi-closed) sets containing A is called the β-closure [1] or semi-preclosure [4] (resp. semiclosure) of A and is denoted by $\beta cl(A)$ or sp-$cl(A)$ (resp. $scl(A)$). A set A is called β-regular [7] (=sp-regular [18]) if its both β-open as well as β-closed. The family of all β-open (resp. β-regular, regular open) sets containing a point $x \in X$ is denoted by $\beta O(X,x)$ (resp. $\beta R(X,x)$, $RO(X,x)$). The family of all β-open (resp. β-regular, regular open) sets in X is denoted by $\beta O(X)$ (resp. $\beta R(X)$, $RO(X)$). A point x of X is in the β-θ-closure [7] (=sp-θ-closure [18]) of A, denoted by $x \in \beta$-θ-$cl(A)$ (resp. $x \in$sp-θ-$cl(S)$) if $A \cap \beta cl(U) \neq \emptyset$ for each $U \in \beta O(X,x)$. A subset A is said to be β-θ-closed [7] (or sp-θ-closed [18]) if $A = \beta$-θ-$cl(A)$ or A=sp-θ-$cl(A)$. The complement of a β-θ-closed or sp-θ-closed set is said to be β-θ-open [7] or sp-θ-open [18]. The family of all β-θ-open sets of X is denoted by β-θ-$O(X)$ and that containing a point x of X is denoted by β-θ-$O(X,x)$.

A filter base $\mathscr{F}$ is said to β-θ-adhere at some point x of X if $x \in \beta$-θ-$cl(F)$ for each $F \in \mathscr{F}$ and is said to β-θ-converge to a point x of X if for each $U \in \beta O(X,x)$, there is an $F \in \mathscr{F}$ such that $F \subset \beta cl(U)$. A subset is said to be an NC-set [22] if every cover of A by regular open sets of X has a finite subcover.

We state the following results which will be frequently used in the sequel.

Lemma 2.1 (**T. Noiri [18], Basu and Ghosh [7]).** *The following hold for a subset A of a space X:*

1. $A \in \beta O(X)$ if and only if $\beta cl(A) \in \beta R(X)$.

2. β-θ-$cl(A) = \cap\{V : A \subset V : and\ V \in \beta R(X)\}$.

3. $x \in \beta$-θ-$cl(A)$ if and only if $A \cap V \neq \emptyset$ for each $V \in \beta R(X, x)$.

4. If $A \subset B$, then β-θ-$cl(A) \subset \beta$-θ-$cl(B)$.

5. β-θ-$cl(\beta$-θ-$cl(A)) = \beta$-θ-$cl(A)$.

6. $A \in \beta$-θ-$O(X)$ if and only if for each $x \in A$, there exists a $V \in \beta R(X, x)$ such that $x \in V \subset A$.

7. β-θ-$cl(A)$ is a β-θ-closed set and union of even two β-θ-closed sets is not necessarily a β-θ-closed set.

8. If $A \in \beta O(X)$, then $\beta cl(A) = \beta$-θ-$cl(A)$.

9. $A \in \beta R(X)$ if and only if A is β-θ-open and β-θ-closed.

10. β-regular $\Rightarrow$ β-θ-open $\Rightarrow$ β-open. But the converses are not necessarily true.

3. β-Closed Subsets and β-θ-Closed Functions

Definition 3.1. *A subset S of a topological space X is said to be β-closed relative to X (β-set, for short) if for every cover $\{V_\alpha : \alpha \in I\}$ of S by β-open sets in X, there exists a finite subset I_0 of I such that $S \subset \cup\{\beta cl(V_\alpha) : \alpha \in I_0\}$. If in particular, if $S = X$ and S is β-closed relative to X then X is β-closed [7].*

It is not hard to prove the theorem 3.2 that gives several characterizations of a subset of a space X which is β-closed relative to X and will be utilized in establishing several results.

Theorem 3.1. *For a non-void subset S of a space, the following are equivalent:*

(a) S is β-closed relative to X.

(b) Every filter base on X which meets S, β-θ-adheres at some point of S.

(c) *Every maximal filter base on X which meets S, β-θ-converges to some point of S.*

(d) *Every cover of S by β-θ-open sets of X has a finite subcover.*

(e) *Every cover of S by β-regular sets of X has a finite subcover.*

(f) *For every family $\{U_\alpha : \alpha \in I\}$ of β-regular sets of X with $[\cap_{\alpha\in I} U_\alpha] \cap S = \emptyset$, there is a finite subset I_0 of I such that $[\cap_{\alpha\in I_0} U_\alpha] \cap S = \emptyset$.*

(g) *Every filter base on S, β-θ-adheres to some point of S.*

(h) *Every maximal filter base on S, β-θ-converges to some point of S.*

Theorem 3.2. *For a space X, the following are equivalent:*

(a) *X is β-closed.*

(b) *Every proper β-θ-closed set is β-closed relative to X.*

(c) *Every proper β-regular set is β-closed relative to X.*

Proof : $(a) \Rightarrow (b)$: Let $\{U_\alpha : \alpha \in I\}$ be a cover of a proper β-θ-closed set S by β-regular sets of X. Since $X - S$ is β-θ-open, for each $x \in X - S$, there exists a $V_x \in \beta R(X, x)$ such that $x \in V_x \subset X - S$. Hence the family $\{V_x : x \in X - S\} \cup \{U_\alpha : \alpha \in I\}$ is a cover of X by β-regular sets of X. Since X is β-closed, there is a finite subset I_0 of I such that $S \subset \cup\{U_\alpha : \alpha \in I_0\}$. Therefore by theorem 3.2, S is β-closed relative to X.

$(b) \Rightarrow (c)$: Since every β-regular set is β-θ-closed, the proof is obvious.

$(c) \Rightarrow (a)$: Let $S \neq \emptyset$, X be a β-regular set. Since $X = S \cup (X - S)$ and S and $X - S$ are both β-regular, the proof is obvious.

Theorem 3.3. *If S is β-closed relative to X where X is T_2 then S is a β-θ-closed set.*

Proof : Let $x \notin S$. Then for each $y \in S$, there exists an open set U_y containing y such that $x \notin cl(U_y) = V_y$ (say). Since each V_y is regular closed and hence is β-regular and $S \subset \cup_{y\in S} V_y$, then by theorem 3.2, there exists a finite subset S_0 of S such that $S \subset \cup_{y\in S_0} V_y = V$ (say). Now the set V is being a regular closed set and hence its complement $X - V \in RO(X, x)$. $X - V$ is therefore β-regular set containing x. So S is β-θ-closed.

Lemma 3.1. *(Abd. El-Monsef et al. [1]) Let A and Y be subsets of a space X. Then*

(i) *If $A \in \beta O(X)$ and Y is α-open in X, then $A \cap Y \in \beta O(Y)$.*

(ii) *If $A \in \beta O(Y)$ and $Y \in \beta O(X)$, then $A \in \beta O(X)$.*

Lemma 3.2. *[13] Let X be a space and A, Y be subsets of X such that $A \subset Y \subset X$ and Y is α-open in X. Then the following properties hold:*
(i) $A \in \beta O(Y)$ if and only if $A \in \beta O(X)$.
(ii) $\beta cl_X(A) \cap Y = \beta cl_Y(A)$, where $\beta cl_Y(A)$ denotes the β-closure of A in the subspace Y.

Theorem 3.4. *Let (X, τ) be a space and Y is α-open in X, then $\beta R(Y, \tau_Y) = \beta R(X, \tau) \cap Y$.*

Proof : Let $A \in \beta R(X, \tau) \cap Y$. Then $A = U \cap Y$ for some $U \in \beta R(X, \tau)$. Then by lemma 3.5, $A \in \beta O(Y, \tau_Y)$. Now by lemma 3.6, we have $\beta cl_Y(A) = \beta cl_X(A) \cap Y \subset \beta cl_X(U) \cap Y = U \cap Y = A$. So, A is β-closed in (Y, τ_Y) and hence $A \in \beta R(Y, \tau_Y)$. Therefore, $\beta R(X, \tau) \cap Y \subset \beta R(Y, \tau_Y)$. Conversely, let $W \in \beta R(Y, \tau_Y)$. Then by lemma 3.6(i), $W \in \beta O(X, \tau)$. Clearly $W = \beta cl_Y(W) = \beta cl_X(W) \cap Y$ (by lemma 3.6(ii)) and hence $W \in \beta R(X, \tau) \cap Y$.

Theorem 3.5. *Let Y be an α-open set in a space (X, τ). Then (Y, τ_Y) is β-closed if and only if Y is β-closed relative to X.*

Proof : Let (Y, τ_Y) be β-closed. If $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a cover of Y by β-regular sets of X. Then by above theorem 3.7, $\mathscr{U}_Y = \{U_\alpha \cap Y : \alpha \in \Lambda\}$ is a cover of Y by β-regular sets of (Y, τ_Y). Since (Y, τ_Y) is β-closed then Y is covered by finite number of sets of $\mathscr{U}$ say, $U_1, ..., U_n$ and hence Y is β-closed relative to X.

Conversely, let Y be β-closed relative to X. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of Y, where each $U_\alpha \in \beta R(Y, \tau_Y)$. Then by above theorem 3.7, for each $\alpha \in \Lambda$, there exists a β-regular set $V_\alpha \in \beta R(X)$ such that $U_\alpha = V_\alpha \cap Y$. Therefore Y is covered by the family $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$. Since Y is β-closed relative to X, there exits $V_{\alpha_1}, ..., V_{\alpha_n} \in \mathscr{V}$ such that $Y \subset \cup_{i=1}^{n} V_{\alpha_i}$. Now as $Y = \cup_{i=1}^{n}(V_{\alpha_i} \cap Y)$, $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ is a finite subfamily of $\mathscr{U}$ which covers Y. So (Y, τ_Y) is β-closed.

Theorem 3.6. *Let A, Y be subsets of a space (X, τ) such that $A \subset Y \subset X$ and Y be α-open in (X, τ). Then A is β-closed relative to (Y, τ_Y) if and only if A is β-closed relative to (X, τ).*

Proof : The proof is obvious because of theorem 3.7.

Corollary 3.1. *If Y, Z are open subsets of a space X such that $Z \subset Y \subset X$ then Z is a β-closed subspace of Y if and only if Z is a β-closed subspace of X.*

Proof : The proof follows from theorem 3.8 and theorem 3.9.

Definition 3.2. *A function $\psi : X \to Y$ is said to be β-θ-closed if image of each β-θ-closed set is closed in Y.*

It is not hard to prove the following characterizations for β-θ-closed function.

Theorem 3.7. *For a function $\psi : X \to Y$, the following are equivalent:*
(i) ψ is β-θ-closed.
(ii) $cl(\psi(B)) \subset \psi(\beta\text{-}\theta\text{-}cl(B))$, for each $B \subset X$.
(iii) For each $y \in Y$ and each β-θ-open set V containing $\psi^{-1}(y)$, there is an open set U containing y satisfying $\psi^{-1}(U) \subset V$.
(iv) For each subset A of Y and each β-θ-open set V containing $\psi^{-1}(A)$ there is an open set U containing A such that $\psi^{-1}(U) \subset V$.
(v) $\{y \in Y : \psi^{-1}(y) \subset V\}$ is open in Y whenever U is β-θ-open in X.
(vi) $\{y \in Y : \psi^{-1}(y) \cap B \neq \emptyset\}$ is closed in Y whenever B is β-θ-closed in X.

The concepts of closed functions and β-θ-closed functions are independent to each other. In addition, the notions of continuity and β-θ-closedness are also independent. To validitate these we establish the following examples:

Example 3.1. *Let $X = \mathbb{R}$, the set of reals with the topology τ_X in which the non-void open sets are subsets of X which contain the point 1. Clearly every non-void β-open set must contains the point 1 and β-θ-closed sets are $\emptyset$ and X only. Let $Y = \{a, b, c\}$ with the topology $\tau_Y = \{\emptyset, Y, \{a\}, \{a, c\}\}$. Let $\psi : (X, \tau_X) \to (Y, \tau_Y)$ be defined as $\psi(x) = a$ for all $x \in X$. Clearly ψ is continuous but as $\psi(X) = \{a\}$, where $\{a\}$ is not closed in Y, ψ is not a β-θ-closed function.*

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\emptyset\ , X, \{a\}, \{a, b\}\}$. Consider the identity function $\psi_1 : (X, \tau) \to (X, \sigma)$. Clearly ψ_1 is not a closed function but ψ_1 is β-θ-closed since the only β-θ-closed sets of (X, τ) are ϕ and X only. Again the identity function $\psi_2 : (X, \sigma) \to (X, \tau)$ is clearly a closed function which is not continuous. Since the family of* *all β-open sets of (X,σ) is $\{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$ then $\beta R(X,\sigma) = \beta$-$\theta$-$O(X,\sigma) = \{\emptyset, X\}$. Therefore ψ_2 is a β-θ-closed function.*

Example 3.3. *The identity function $\psi : (\mathbb{R}, \mathscr{U}) \to (\mathbb{R}, \mathscr{U})$, where $(\mathbb{R}, \mathscr{U})$ is the set of reals with the usual topology $\mathscr{U}$ is a closed function which is not a β-θ-closed function. In fact each closed rays $(-\infty, a]$, $[b, \infty)$ are β-open sets in $(\mathbb{R}, \mathscr{U})$. So for $a < b$, the interval (a, b) is β-θ-closed. But its image $\psi((a,b)) = (a,b)$ is not closed in $(\mathbb{R}, \mathscr{U})$.*

Theorem 3.8. *Let $\psi : X \to Y$ be a surjective β-θ-closed function having β-closed relative to X (i.e. β-set) point inverses. Then $\psi^{-1}(A)$ is β-closed relative to X whenever A is compact in Y.*

Proof : Let $\mathscr{V} = \{V_\lambda : \lambda \in \Lambda\}$ be a β-θ-open cover of $\psi^{-1}(A)$. By hypothesis and by theorem 3.2, for each $y \in A$ there exist $\lambda_1, \ldots\ldots, \lambda_n$ such that $\psi^{-1}(y) \subset \cup_{i=1}^{n} V_{\lambda_i} = V_y$ (say). Since V_y is β-θ-open and ψ is β-θ-closed, by theorem 3.12, there exists an open set U_y containing y such that $\psi^{-1}(U_y) \subset V_y$. Since A is compact, there exist $y_1, \ldots\ldots, y_n \in A$ such that $\psi^{-1}(A) \subset \cup_{i=1}^{k} \psi^{-1}(U_{y_i})$. Hence $\psi^{-1}(A) \subset \cup_{i=1}^{k} V_{y_i}$, where each V_{y_i} is a union of finite number of members of $\mathscr{V}$. Therefore $\psi^{-1}(A)$ is β-closed relative to X.

Theorem 3.9. *Let $\psi : X \to Y$ be a surjective β-θ-closed function having β-closed relative to X (i.e. β-set) point inverses. If Y is compact and X is T_2 then ψ is continuous.*

Proof : Since Y is compact then by the above theorem 3.16, $\psi^{-1}(A)$ is β-closed relative to X whenever A is closed in Y. Obviously $\psi^{-1}(A)$ is an NC-set and since X is T_2 then $\psi^{-1}(A)$ is closed as well in X. Therefore ψ is continuous.

4. Locally β-Closed Spaces

Definition 4.1. *A space X is called locally β-closed if for each $x \in X$, there exists a regular open neighbourhood of which is a β-closed subspace of X.*

Remark 4.1. *Every β-closed space is locally β-closed. But the converse is not true, in general. Any infinite set with the discrete topology is an example of a locally β-closed space which is not β-closed.*

Theorem 4.1. *A space X is locally β-closed if and only if for each $x \in X$, there is a $V \in RO(X,x)$ such that V is locally β-closed.*

Proof : The necessity part is obvious.

For the sufficiency part, let $W \in RO(X)$. We shall prove that if $A \in RO(W)$ then $A \in RO(X)$. Indeed, $A = int_W(cl_W(A)) = int_W(W \cap cl_X(A)) = int_X(W \cap cl_X(A)) = int_X(W) \cap int_X(cl_X(A)) = W \cap int_X(cl_X(A)) = int_X(cl_X(A))$ (as $A \subset W$). By hypothesis, for each $x \in X$, there is a $W \in RO(X,x)$ such that (W, τ_W) is locally β-closed. Then by definition for each $x \in W$, there is an $U \in RO(W)$ such that $x \in U$ and U is a β-closed subspace of W. Then by the argument given above $U \in RO(X,x)$ and hence by the corollary 3.10, U is a β-closed subspace of X. So, (X,τ) is locally β-closed.

Theorem 4.2. *For a topological space (X,τ), the following are equivalent:*
(a) *(X,τ) is locally β-closed.*
(b) *For each point x of X, there is a $V \in RO(X,x)$ which is β-closed relative to X.*
(c) *Each point x of X has an open neighbourhood V of x such that $int(cl(V))$ is β-closed relative to X.*
(d) *For each point x of X, there is an open neighbourhood U of x such that $scl(U)$ is β-closed relative to X.*
(e) *For each point x of X, there is an open neighbourhood U of x such that $\beta cl(U)$ is β-closed relative to X.*
(f) *For each point of X, there is an α-open set V containing x such that $int(cl(V))$ is a β-closed subspace of X.*

Proof : The proof is followed from the facts that a set A is pre-open if and only if $scl(A) = int(cl(A))$ and for an open set U, $\beta cl(U) = int\ cl(U)$ and from theorem 3.8.

Theorem 4.3. *A space (X,τ) is locally β-closed if and only if (X,τ_α) is locally β-closed.*

Proof : Since $\beta O(X,\tau) = \beta O(X,\tau_\alpha)$, the proof is immediate.

The following examples show that local β-closedness and local compact T_2-ness are independent to each other.

Example 4.1. ***Example of a locally β-closed space which is not locally compact T_2***
Let $X = \mathbb{R}$, the set of reals with the countable complement topology τ. Clearly $PO(X) =$ {uncountable infinite subset of X or ϕ} $= \beta O(X)$. Since for a subset S of X, $\beta cl(S) = S \cup int(cl(int(S))$ [6], then $\beta cl(V) = X$ for any non-empty $V \in \beta O(X)$. Therefore (X, τ) is β-closed and hence is locally β-closed. Clearly (X, τ) is not locally compact T_2.

Example 4.2. ***Example of a locally compact T_2 space which is not locally β-closed***
Let $X = \mathbb{R}$, the set of reals with the usual topology $\mathscr{U}$. Clearly $(X, \mathscr{U})$ is locally compact T_2 but is not a locally β-closed space.

Although we have seen from above examples that local β-closedness and local compact T_2-ness are independent concepts but the next two theorems are two of our main results and relate locally compact T_2 spaces to locally β-closed spaces

Theorem 4.4. *Let $\psi : X \to Y$ be continuous β-θ-closed surjection with point inverses are β-closed sets relative to X (i.e. β-set). Then X is locally β-closed whenever Y is locally compact T_2.*

Proof : Since Y is being a locally compact T_2 space, for each $x \in X$, there exists an open neighbourhood V of x such that $cl(V)$ is compact in Y. As ψ is β-θ-closed, by theorem 3.16, $\psi^{-1}(cl(V))$ is a β-closed set relative to X. As ψ is continuous it is obvious that $int(cl(\psi^{-1}(V)) \subset \psi^{-1}(cl(V))$. But $int(cl(\psi^{-1}(V))$ is obviously β-regular set containing x and hence by theorem 3.3, $int(cl(\psi^{-1}(V))$ is β-closed relative to X. Therefore by theorem 4.4, X is locally β-closed.

Remark 4.2. *In the above theorem X is not necessarily T_2.*

Example 4.3. *Let $X =$ any finite set and $\tau_X = \{\emptyset, X\}$. Let $Y = \{a\}$ with the discrete topology. Let $\psi : X \to Y$ be the constant function. Then Y is locally compact T_2 and ψ is continuous β-θ-closed surjection with β-closed set relative to X (i.e. β-set) point inverses. However X is not T_2.*

Definition 4.2. *[7] A function $\psi : X \to Y$ is said to be (θ,β)-continuous if for each $x \in X$ and each $V \in \beta R(Y,\psi(x))$ there is an open set U containing x such that $\psi(U) \subset V$.*

Theorem 4.5. *If $\psi : X \to Y$ is a β-θ-closed, (θ,β)-continuous surjection with point inverses are β-closed sets relative to X (i.e. β-sets). Then Y is locally β-closed if X is locally compact T_2.*

Proof : First of all we claim that Y is T_2. Indeed, by the hypothesis, for distinct points y_1 and y_2 in Y, $\psi^{-1}(y_i)$ for $i = 1,2$ are disjoint β-closed sets relative to X and hence they are NC-sets. Since Y is T_2, one can check easily that, there exist disjoint W_1, $W_2 \in RO(X)$ satisfying $\psi^{-1}(y_i) \subset U_i$ for $i = 1,2$. As every regular open set is β-regular and hence is β-θ-open and as ψ is β-θ-closed, then by theorem 3.12, there exist open sets U_i containing y_i such that $\psi^{-1}(U_i) \subset W_i$ for $i = 1,2$. Therefore Y is T_2. Let $y \in Y$. Since X is a locally compact T_2 space, then for each $x \in \psi^{-1}(y)$, there exists a closed compact neighbourhood V_x of x in X. Now as the family $\{int(V_x) : x \in \psi^{-1}(y)\}$ is being a β-regular cover (argument given above) of the set $\psi^{-1}(y)$ which is β-closed relative to X, then by theorem 3.2, there exist $x_1, x_2, ..., x_k \in \psi^{-1}(y)$ such that $\psi^{-1}(y) \subset \cup_{i=1}^{k} int(V_{x_i})$. Hence $\psi^{-1}(y) \subset \cup_{i=1}^{k} int(V_{x_i}) \subset int(\cup_{i=1}^{k} V_{x_i})$. Since the latter subset is β-θ-open and ψ is β-θ-closed then by theorem 3.12 there exists an open set W_y containing y such that $\psi^{-1}(W_y) \subset int(\cup_{i=1}^{k} V_{x_i})$. Hence $y \in W_y \subset \psi(int(\cup_{i=1}^{k} V_{x_i})) \subset \psi(\cup_{i=1}^{k} V_{x_i})$. As it is obvious that (θ,β)-continuous image of a compact set is a β-closed set and Y is T_2, $\psi(\cup_{i=1}^{k} V_{x_i})$ is obviously closed. Since $y \in W_y \subset int(cl(W_y)) \subset \psi(\cup_{i=1}^{k} V_{x_i})$ and $int(cl(W_y))$ is β-regular then $int(cl(W_y))$ is β-closed relative to X. Therefore by theorem 4.4, Y is locally β-closed.

Definition 4.3. *A topological space X is called γ-perfect if for each $U \in RO(X)$ and each $x \notin U$, there exists a family of open sets $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ such that $U \subset \cup_{\alpha \in I} cl(V_\alpha)$ with $x \notin \cup_{\alpha \in I} cl(V_\alpha)$.*

Theorem 4.6. *Every locally β-closed γ-perfect space is extremally disconnected.*

Proof : Let $U \in RO(X)$, where X is a locally β-closed γ-perfect space. Let $x \notin U$. Since X is locally β-closed, there is a $V \in RO(X,x)$ such that V is β-closed relative to X. Let $R = U \cap V$.

Case-I: Suppose $R = \emptyset$. Then U will be obviously closed.

Case-II: Suppose $R \neq \emptyset$. Then as $R \in RO(X) \subset \beta R(X)$ and $R \subset V$, where V is β-closed relative to X, R is clearly β-closed relative to X. Since X is γ-perfect and $R \in RO(X)$ with

$x \notin R$ the there exists a family of open sets $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ such that $R \subset \cup_{\alpha \in I} cl(V_\alpha)$ with $x \notin \cup_{\alpha \in I} cl(V_\alpha)$. Since every regular closed set is being β-regular, the family $\{cl(V_\alpha) : \alpha \in I\}$ is a β-regular cover of the set R which is β-closed relative to X. So there exist $\alpha_1, ..., \alpha_n \in I$ such that $R \subset \cup_{i=1}^{n} cl(V_{\alpha_i})$. Let $W = X - \cup_{i=1}^{n} cl(V_{\alpha_i})$. Clearly $W \cap V$ is an open set containing x disjoint from U. Hence U is closed. Therefore X is extremally disconnected.

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