EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 2, No. 3, 2009, (325-337)
ISSN 1307-5543 – www.ejpam.com

# A Note On Pairwise Continuous Mappings and Bitopological Spaces

Adem Kılıçman[1*] and Zabidin Salleh[2]

[1] *Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*

[2] *Department of Mathematics, Faculty of Science and Technology, University Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia*

---

**Abstract.** We shall continue the study of bitopological separation axioms that was begun by Kelly and obtained some results. Furthermore, we introduce a concept of pairwise Lindelöf bitopological spaces, namely, $p_2$-Lindelöf spaces and their properties are established. We also show that a $p_2$-Lindelöf space is not a hereditary property. Finally, we show that a $p_2$-Lindelöf space is a $p_2$-topological property.

**2000 Mathematics Subject Classifications**: 54E55

**Key Words and Phrases**: Bitopological space, $p_2$-compact, $p_2$-Lindelöf, $p_1$-regular, $p_2$-normal, $p_2$-continuity, $p_2$-homeomorphism.

---

*Corresponding author.

*Email addresses:* akilicman@putra.upm.edu.my (A. Kılıçman), zabidin@umt.edu.my (Z. Salleh)

 © 2009 EJPAM All rights reserved.

## 1. Introduction

A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is known as a set $X$ together with two arbitrary topologies $\mathscr{P}$ and $\mathscr{Q}$ which are defined on $X$, for the detail definitions, terminologies and notations we refer to [1]. In [3] and [4], the authors introduced and studied the idea of pairwise continuity and pairwise Lindelöfness in bitopological spaces and give some results concerning these ideas. Recently the authors in [5], [6] introduced and studied the notion of pairwise almost Lindelöf and pairwise weakly Lindelöf spaces in bitopological spaces. In this paper, we are concerned with another concept of pairwise regular Lindelöfness and pairwise continuity in bitopological spaces.

In section 3, we shall introduce another concept of pairwise regular and pairwise normal bitopological spaces, i.e., $p_1$-regular spaces and $p_2$-normal spaces and obtain a result. Furthermore, some examples will be given to describe its properties.

In section 4, we shall define another concept of pairwise Lindelöf spaces, i.e., $p_2$-Lindelöf spaces. We obtain a result about subset of such spaces. The main result we are obtain here is every $p_1$-regular and $p_2$-Lindelöf bitopological space is $p_2$-normal.

In section 5, we extend idea of continuity to a bitopological space, namely, $p_2$-continuity and study their properties where the purpose is to study the effect of mapping and $p_2$-continuity on $p_2$-Lindelöf bitopological spaces. We also show that this mapping preserve $p_2$-Lindelöf property. The main result here is that the image of a $p_2$-Lindelöf space under a $p_2$-continuous function is $p_2$-Lindelöf.

## 2. Preliminaries

Throughout this paper, all spaces $(X,\mathscr{P})$ and $(X,\mathscr{P},\mathscr{Q})$ (or simply $X$) are always meant topological spaces and bitopological spaces, respectively. In this paper, we shall use $p$- to denote pairwise. For instance, $p$-Lindelöf stands for pairwise Lindelöf. While $p_1$- and $p_2$- are used to denote other concepts of pairwise. Sometimes the authors write the term "pairwise Lindelöf spaces" which means that pairwise Lindelöf bitopological spaces.

Kelly [1] was the first one who introduced the idea of $p$-regular spaces and $p$-normal spaces. Later these spaces will be generalized to $p_1$-regular spaces and $p_2$-normal spaces respectively.

**Definition 2.1** (Kelly). *In a space $(X,\mathscr{P},\mathscr{Q})$, $\mathscr{P}$ is said to be regular with respect to $\mathscr{Q}$ if, for each point $x\in X$, there is a $\mathscr{P}$-neighbourhood base of $\mathscr{Q}$-closed sets, or, as is easily seen to be equivalent, if, for each point $x\in X$ and each $\mathscr{P}$-closed set $P$ such that $x\notin P$, there are a $\mathscr{P}$-open set $U$ and a $\mathscr{Q}$-open set $V$ such that*

$$x\in U,\ P\subseteq V,\ \text{and}\ U\cap V=\emptyset.$$

*Similarly, $(X,\mathscr{P},\mathscr{Q})$ is, or $\mathscr{P}$ and $\mathscr{Q}$ are, p-regular if $\mathscr{P}$ is regular with respect to $\mathscr{Q}$ and vice versa.*

**Definition 2.2** (Kelly). *A bitopological space $(X,\mathscr{P},\mathscr{Q})$ is said to be p-normal if, given a $\mathscr{P}$-closed set $A$ and a $\mathscr{Q}$-closed set $B$ with $A\cap B=\emptyset$, there exist a $\mathscr{Q}$-open set $U$ and a $\mathscr{P}$-open set $V$ such that $A\subseteq U$, $B\subseteq V$, and $U\cap V=\emptyset$.*

Further, in [3] the authors introduced the concept of $p_1$-normal spaces. Moreover in the same paper, the authors also introduced the concept of $p$-Lindelöf spaces and $p_1$-Lindelöf spaces as the following.

**Definition 2.3** (see [3] ). *A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-Lindelöf if the topological space $(X, \mathscr{P})$ and $(X, \mathscr{Q})$ are both Lindelöf. Equivalently, $(X, \mathscr{P}, \mathscr{Q})$ is p-Lindelöf if every $\mathscr{P}$-open cover of $X$ can be reduced to a countable $\mathscr{P}$-open cover and every $\mathscr{Q}$-open cover of $X$ can be reduced to a countable $\mathscr{Q}$-open cover.*

**Definition 2.4** (see [3] ). *In a bitopological space $(X, \mathscr{P}, \mathscr{Q})$, $\mathscr{P}$ is said to be Lindelöf with respect to $\mathscr{Q}$ if, every $\mathscr{P}$-open cover of $X$ can be reduced to a countable $\mathscr{Q}$-open cover and similarly, $(X, \mathscr{P}, \mathscr{Q})$ is, or $\mathscr{P}$ and $\mathscr{Q}$ are, $p_1$-Lindelöf if $\mathscr{P}$ is Lindelöf with respect to $\mathscr{Q}$ and vice versa.*

## 3. Bitopological Separation Axioms

In this section, we shall introduce the concept of $p_1$-regular spaces and $p_2$-normal spaces. Before that we need the following definition.

**Definition 3.1.** *Let $(X, \mathscr{P}, \mathscr{Q})$ be a bitopological space.*

*(i) A set $H$ is said to be $p_1$-open if $H$ is $(\mathscr{P} \cup \mathscr{Q})$-open in $X$.*

*(ii) A set $M$ is said to be $p_1$-closed if $M$ is $(\mathscr{P} \cup \mathscr{Q})$-closed in $X$.*

**Definition 3.2.** *A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is said to be $p_1$-regular if for each point $x \in X$, there is a $p_1$-neighbourhood base of $p_1$-closed sets, or, as is easily seen to be equivalent, if, for each point $x \in X$ and each $p_1$-closed set $P$ such that $x \notin P$, there are $p_1$-open sets $U$ and $V$ such that*

$$x \in U, \ P \subseteq V, \text{ and } U \cap V = \emptyset.$$

Note that, below we shall use the notation $p_1\text{-cl}(U)$, which means that, the closure of $U$ with respect to $\mathscr{P} \cup \mathscr{Q}$, or, in other words $(\mathscr{P} \cup \mathscr{Q})\text{-cl}(U)$.

**Theorem 3.1.** *A bitopological space $(X,\mathscr{P},\mathscr{Q})$ is $p_1$-regular if and only if for each point $x \in X$ and $p_1$-open set $H$ containing $x$, there exists a $p_1$-open set $U$ such that*

$$x \in U \subseteq p_1\text{-}\operatorname{cl}(U) \subseteq H.$$

*Proof.* $(\Rightarrow)$ : Suppose $(X,\mathscr{P},\mathscr{Q})$ is $p_1$-regular. Let $x \in X$ and $H$ is a $p_1$-open set containing $x$. Then $G = X \diagdown H$ is a $p_1$-closed set which $x \notin G$. Since $(X,\mathscr{P},\mathscr{Q})$ is $p_1$-regular, then there are $p_1$-open sets $U$ and $V$ such that $x \in U$, $G \subseteq V$, and $U \cap V = \emptyset$. Since $U \subseteq X \diagdown V$, then $p_1$-$\operatorname{cl}(U) \subseteq p_1$-$\operatorname{cl}(X \diagdown V) = X \diagdown V \subseteq X \diagdown G = H$. Thus, $x \in U \subseteq p_1$-$\operatorname{cl}(U) \subseteq H$ as desired.

$(\Leftarrow)$ : Suppose the condition holds. Let $x \in X$ and $P$ is a $p_1$-closed set such that $x \notin P$. Then $x \in X \diagdown P$, and by hypothesis there exists a $p_1$-open set $U$ such that $x \in U \subseteq p_1$-$\operatorname{cl}(U) \subseteq X \diagdown P$. It follows that $x \in U$, $P \subseteq X \diagdown p_1$-$\operatorname{cl}(U)$ and $U \cap \big(X \diagdown p_1\text{-}\operatorname{cl}(U)\big) = \emptyset$. This completes the proof.

**Definition 3.3.** *A bitopological space $(X,\mathscr{P},\mathscr{Q})$ is said to be $p_2$-normal if given $p_1$-closed sets $A$ and $B$ with $A \cap B = \emptyset$, there exist $p_1$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.*

**Theorem 3.2.** *A space $(X,\mathscr{P},\mathscr{Q})$ is $p_2$-normal if and only if given a $p_1$-closed set $C$ and a $p_1$-open set $D$ such that $C \subseteq D$, there are a $p_1$-open set $G$ and a $p_1$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.*

*Proof.* $(\Rightarrow)$ : Suppose $(X,\mathscr{P},\mathscr{Q})$ is $p_2$-normal. Let $C$ be a $p_1$-closed set and $D$ a $p_1$-open set such that $C \subseteq D$. Then $K = X \diagdown D$ is a $p_1$-closed set with $K \cap C = \emptyset$. Since $(X,\mathscr{P},\mathscr{Q})$ is $p_2$-normal, there exists $p_1$-open sets $U$ and $G$ such that $K \subseteq U$, $C \subseteq G$, and $U \cap G = \emptyset$. Hence $G \subseteq X \diagdown U \subseteq X \diagdown K = D$. Thus $C \subseteq G \subseteq X \diagdown U \subseteq D$ and the result follows by taking $X \setminus U = F$.

$(\Leftarrow)$ : Suppose the condition holds. Let $A$ and $B$ are $p_1$-closed sets with $A \cap B = \emptyset$. Then $D = X \setminus A$ is a $p_1$-open set with $B \subseteq D$. By hypothesis, there are a $p_1$-open set $G$ and a $p_1$-closed set $F$ such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X \setminus D \subseteq X \setminus F$, $B \subseteq G$ and $(X \setminus F) \cap G = \emptyset$ where $X \setminus F$ and $G$ are $p_1$-open sets. This completes the proof.

**Example 3.1.** *Consider $X = \{a, b, c\}$ with topologies $\mathscr{P} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathscr{Q} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ defined on $X$. Then*

$$\mathscr{P} \cup \mathscr{Q} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

*Observe that $\mathscr{P} \cup \mathscr{Q}$ is a discrete topology and $p_1$-closed subsets of $X$ are $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ and $X$. It follows that $(X, \mathscr{P}, \mathscr{Q})$ does satisfy the condition in definition of $p_1$-regular and $p_2$-normal. Hence $(X, \mathscr{P}, \mathscr{Q})$ is $p_1$-regular and $p_2$-normal space.*

**Example 3.2.** *Consider $X = \{a, b, c, d\}$ with topologies $\mathscr{P} = \{\emptyset, \{a, b\}, X\}$ and $\mathscr{Q} = \{\emptyset, \{a\}, \{b, c, d\}, X\}$ defined on $X$. Then*

$$\mathscr{P} \cup \mathscr{Q} = \{\emptyset, \{a\}, \{a, b\}, \{b, c, d\}, X\}.$$

*Observe that $p_1$-closed subsets of $X$ are $\emptyset, \{a\}, \{b, c, d\}, \{c, d\}$ and $X$. Hence $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-normal as we can checks. However $(X, \mathscr{P}, \mathscr{Q})$ is not $p_1$-regular since the $p_1$-closed set $P = \{c, d\}$ satisfy $b \notin P$, but do not exist the $p_1$-open sets $U$ and $V$ such that $b \in U$, $P \subseteq V$ and $U \cap V = \emptyset$.*

*Note that $\mathscr{P} \cup \mathscr{Q}$ is not a topology of $X$ since $\{a, b\}, \{b, c, d\} \in \mathscr{P} \cup \mathscr{Q}$, but $\{a, b\} \cap \{b, c, d\} = \{b\} \notin \mathscr{P} \cup \mathscr{Q}$.*

## 4. On $p_2$-Lindelöf Spaces

In this section, we shall introduce a new concept of pairwise compact spaces and pairwise Lindelöf spaces as the following.

**Definition 4.1.** *A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is said to be $p_2$-compact if every $p_1$-open cover of $X$ has a finite subcover.*

**Definition 4.2.** *A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is said to be $p_2$-Lindelöf if every $p_1$-open cover of $X$ has a countable subcover.*

It is very clear that, every $p_2$-compact space is $p_2$-Lindelöf but not the converse by the following counter-example.

**Example 4.1.** *Let $\mathscr{B}$ be the collection of open-closed intervals in the real line $\mathbb{R}$*

$$\mathscr{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}.$$

*Hence $\mathscr{B}$ is a base for the upper limit topology $\mathscr{P}$ on $\mathbb{R}$. Similarly, the collection of closed-open intervals,*

$$\mathscr{B}^* = \{[c, d) : c, d \in \mathbb{R}, c < d\}$$

*is a base for the lower limit topology $\mathscr{Q}$ on $\mathbb{R}$. Observe that $\mathscr{B} \cup \mathscr{B}^*$ is the set of the form $(a, b], [a, b), (a, d), [c, b]$ and $(a, b] \cup [c, d)$, i.e., $\mathscr{B} \cup \mathscr{B}^*$ is a base for the $\mathscr{P} \cup \mathscr{Q}$. Thus $(\mathbb{R}, \mathscr{P}, \mathscr{Q})$ is a $p_2$-Lindelöf space. But $(\mathbb{R}, \mathscr{P}, \mathscr{Q})$ is not $p_2$-compact since for example $\{(n, n+1] : n \in \mathbb{Z}\}$ is a $p_1$-open cover of $\mathbb{R}$ contains no finite subcover.*

**Lemma 4.1.** *Every $p_1$-closed subset of a $p_2$-Lindelöf bitopological space is $p_2$-Lindelöf.*

*Proof.* Let $(X, \mathscr{P}, \mathscr{Q})$ be a $p_2$-Lindelöf bitopological space and let $F$ is a $p_1$-closed subset of $X$. If $\{U_\alpha : \alpha \in \Delta\}$ is a $p_1$-open cover of $F$, then $X = \left(\bigcup_{\alpha \in \Delta} U_\alpha\right) \bigcup (X \setminus F)$. Hence the collection $\{U_\alpha : \alpha \in \Delta\}$ and $X \setminus F$ forms a $p_1$-open cover of $X$. Since $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-Lindelöf, there will be a countable subcover $\{X \setminus F, U_{\alpha_1}, U_{\alpha_2}, \ldots\}$. But

$F$ and $X \setminus F$ are disjoint; hence the subcollection of $p_1$-open set $\left\{U_{\alpha_i} : i \in \mathbb{N}\right\}$ also cover $F$, and so $\left\{U_\alpha : \alpha \in \Delta\right\}$ has a countable subcover. This completes the proof.

**Theorem 4.1.** *Every $p_1$-regular and $p_2$-Lindelöf bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-normal.*

*Proof.* Let $A$ and $B$ are $p_1$-closed sets in $X$ with $A \cap B = \emptyset$. Since $(X, \mathscr{P}, \mathscr{Q})$ is $p_1$-regular, then by Theorem 3.1, for each $x$ in $B$ and $p_1$-open set $X \setminus A$ containing $x$, there is a $p_1$-open set $P_x$ such that

$$x \in P_x \subseteq p_1\text{-}\operatorname{cl}(P_x) \subseteq X \setminus A,$$

i.e., $p_1\text{-cl}\left(P_x\right) \cap A = \emptyset$. The collection $\{P_x : x \in B\}$ forms a $p_1$-open cover of $B$. Since $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-Lindelöf, then $B$ is also $p_2$-Lindelöf by Lemma 4.1. Hence we obtain a countable $p_1$-open cover of $B$, which we denote by $\left\{P_i : i \in \mathbb{N}\right\}$.

Similarly, for each $y$ in $A$ and $p_1$-open set $X \setminus B$ containing $y$, there is a $p_1$-open set $Q_y$ such that

$$y \in Q_y \subseteq p_1\text{-}\operatorname{cl}\left(Q_y\right) \subseteq X \setminus B,$$

i.e., $p_1\text{-cl}\left(Q_y\right) \cap B = \emptyset$. The collection $\left\{Q_y : y \in A\right\}$ forms a $p_1$-open cover of $A$. Since $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-Lindelöf, then $A$ is also $p_2$-Lindelöf by Lemma 4.1. Hence we obtain a countable $p_1$-open cover of $A$, which we denote by $\left\{Q_i : i \in \mathbb{N}\right\}$. Let

$$U_n = Q_n \setminus \bigcup \left\{p_1\text{-}\operatorname{cl}\left(V_i\right) : i \leq n\right\}$$

and

$$V_n = P_n \setminus \bigcup \left\{p_1\text{-}\operatorname{cl}\left(U_i\right) : i \leq n\right\}.$$

Since $U_n \cap p_1\text{-cl}\left(V_m\right) = \emptyset$ for $m \leq n$, it follows that $U_n \cap V_m = \emptyset$ for $m \leq n$.

Similarly, $V_m \cap p_1\text{-cl}\left(U_n\right) = \emptyset$ for $n \leq m$, it follows that $V_m \cap U_n = \emptyset$ for $n \leq m$. Thus $U_n \cap V_m = \emptyset$ for all $m$ and $n$, and consequently $U = \bigcup \left\{U_n : n \in \mathbb{N}\right\}$ is disjoint from $V = \bigcup \{V_n : n \in \mathbb{N}\}$. Finally, $p_1$-cl$\left(V_i\right) \cap A$ and $p_1$-cl$\left(U_i\right) \cap B$ are empty set for all $i$ and hence the set $U$ contains $A$ and is $p_1$-open, whilst the set $V$ contains $B$ and is $p_1$-open. The proof is complete.

## 5. On $p_2$-continuous Functions

Now we shall give another concept of pairwise continuous functions and pairwise homeomorphism in the sense of A. Tallafha [5] and study its properties.

**Definition 5.1.** *A function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is said to be $p_2$-continuous if the inverse image $f^{-1}(U) \in \mathscr{P} \cup \mathscr{Q}$ for every $U \in \mathscr{R} \cup \mathscr{T}$, or equivalently, $f^{-1}(U)$ is $p_1$-open in $X$ for every $U$ is $p_1$-open in $Y$.*

**Definition 5.2.** *A function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is said to be $p_2$-homeomorphism if $f$ is bijection, $p_2$-continuous and $f^{-1} : (Y, \mathscr{R}, \mathscr{T}) \to (X, \mathscr{P}, \mathscr{Q})$ is $p_2$-continuous. The bitopological spaces $(X, \mathscr{P}, \mathscr{Q})$ and $(Y, \mathscr{R}, \mathscr{T})$ are then called $p_2$-homeomorphic.*

A. Tallafha et. al. [5] called $p_2$-continuous function in Definition 5.1 as $p$-continuous function and $p_2$-homeomorphism in Definition 5.2 as $p$-homeomorphism.

**Theorem 5.1.** *If $(X, \mathscr{P}, \mathscr{Q})$ and $(Y, \mathscr{R}, \mathscr{T})$ are bitopological spaces and $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ be a function, then the following statements are equivalent:*

*(i)* *$f$ is $p_2$-continuous,*

*(ii)* *for each $x \in X$ and each $p_1$-open set $V$ in $Y$ containing $f(x)$, there exists a $p_1$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.*

*(iii)* *$f^{-1}(V)$ is $p_1$-closed in $X$ for every $p_1$-closed set $V$ in $Y$,*

*(iv)* *for every $A \subseteq X$, $f\left(p_1\text{-}\operatorname{cl}(A)\right) \subseteq p_1\text{-}\operatorname{cl}\left(f(A)\right)$,*

*(v) for every* $B \subseteq Y$, $p_1\text{-cl}\left(f^{-1}(B)\right) \subseteq f^{-1}\left(p_1\text{-}\operatorname{cl}(B)\right)$.

*Proof.* $(i) \Leftrightarrow (ii)$ : Let $x \in X$ and $V$ is a $p_1$-open set in $Y$ containing $f(x)$. By $(i)$, $f^{-1}(V)$ is a $p_1$-open set in $X$ containing $x$. Take $U = f^{-1}(V)$, and then $f(U) = f\left(f^{-1}(V)\right) \subseteq V$.

Conversely, let $U$ be a $p_1$-open set in $Y$ and let $x \in f^{-1}(U)$. Then $f(x) \in U$ and by $(ii)$, there exists a $p_1$-open set $V$ in $X$ containing $x$ such that $f(V) \subseteq U$. Hence $x \in V \subseteq f^{-1}(U)$ and $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V$. This shows that $f^{-1}(U)$ is $p_1$-open set in $X$. Thus $f$ is $p_2$-continuous.

$(i) \Leftrightarrow (iii)$ : Let $V$ is $p_1$-closed set in $Y$. Then $Y \setminus V$ is $p_1$-open set in $Y$. Hence $f^{-1}(Y \setminus V)$ is $p_1$-open set in $X$ by $(i)$. Since $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$, it follows that $f^{-1}(V)$ is $p_1$-closed in $X$.

The converse can be proved similarly.

$(iii) \Rightarrow (iv)$ : Let $V$ be any $p_1$-closed set in $Y$ containing $f(A)$. Then $f^{-1}(V)$ is a $p_1$-closed set in $X$ containing $A$ by $(iii)$. Hence, $p_1\text{-cl}(A) \subseteq f^{-1}(V)$, and it follows that $f\left(p_1\text{-}\operatorname{cl}(A)\right) \subseteq f\left(f^{-1}(V)\right) \subseteq V$. Since this is true for any $p_1$-closed set $V$ in $Y$ containing $f(A)$, we have $f\left(p_1\text{-}\operatorname{cl}(A)\right) \subseteq p_1\text{-cl}\left(f(A)\right)$.

$(iv) \Rightarrow (v)$ : Let $B \subseteq Y$ and $A = f^{-1}(B)$. Then $f(A) \subseteq B$, and by $(iv)$, $f\left(p_1\text{-}\operatorname{cl}(A)\right) \subseteq p_1\text{-cl}\left(f(A)\right) \subseteq p_1\text{-cl}(B)$. This means $p_1\text{-cl}(A) \subseteq f^{-1}\left(f\left(p_1\text{-}\operatorname{cl}(A)\right)\right) \subseteq f^{-1}\left(p_1\text{-}\operatorname{cl}(B)\right)$. Thus we have

$$p_1 - \operatorname{cl}\left(f^{-1}(B)\right) \subseteq f^{-1}\left(p_1\text{-}\operatorname{cl}(B)\right)$$

by substituting $A = f^{-1}(B)$.

$(v) \Rightarrow (iii)$ : Let $V$ be a $p_1$-closed set in $Y$. Then $V = p_1\text{-cl}(V)$. By $(v)$, $p_1$-cl$\left(f^{-1}(V)\right) \subseteq f^{-1}\left(p_1\text{-}\mathrm{cl}(V)\right) = f^{-1}(V)$. Since $f^{-1}(V) \subseteq p_1\text{-cl}\left(f^{-1}(V)\right)$, we have $f^{-1}(V) = p_1\text{-cl}\left(f^{-1}(V)\right)$. Therefore $f^{-1}(V)$ is $p_1$-closed in $X$.

**Definition 5.3.** *A function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is said to be $p_2$-open if $f(U)$ is $p_1$-open in $Y$ for every $U$ is $p_1$-open in $X$, and $p_2$-closed if $f(V)$ is $p_1$-closed in $Y$ for every $V$ is $p_1$-closed in $X$.*

**Theorem 5.2.** *Let $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ be a $p_2$-continuous, surjective and $p_2$-open function. If $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-Lindelöf, then $(Y, \mathscr{R}, \mathscr{T})$ is $p_2$-Lindelöf.*

*Proof.* Let $(X, \mathscr{P}, \mathscr{Q})$ is a $p_2$-Lindelöf space. Suppose $\{G_i : i \in \Delta\}$ is a $p_1$-open cover of $Y$, i.e., $Y \subseteq \bigcup_{i\in\Delta} G_i$ with $G_i \in \mathscr{R} \cup \mathscr{T}$. Since $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is $p_2$-continuous and surjective, then $f^{-1}\left(G_i\right) \in \mathscr{P} \cup \mathscr{Q}$ and

$$X = f^{-1}(Y) \subseteq f^{-1}\left(\bigcup_{i\in\Delta} G_i\right) = \bigcup_{i\in\Delta} f^{-1}\left(G_i\right).$$

Hence $\{f^{-1}\left(G_i\right) : i \in \Delta\}$ is a $p_1$-open cover of $X$. But $(X, \mathscr{P}, \mathscr{Q})$ is $p_2$-Lindelöf, so there exists a countable subcover of $X$, say $\left\{f^{-1}\left(G_{i_n}\right) : n \in \mathbb{N}\right\}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} f^{-1}\left(G_{i_n}\right)$. Accordingly,

$$Y = f(X) \subseteq f\left(\bigcup_{n\in\mathbb{N}} f^{-1}\left(G_{i_n}\right)\right) = \bigcup_{n\in\mathbb{N}} f\left(f^{-1}\left(G_{i_n}\right)\right) \subseteq \bigcup_{n\in\mathbb{N}} G_{i_n}.$$

Thus we obtain $\left\{G_{i_n} : n \in \mathbb{N}\right\}$ is a countable subcover of $Y$ since $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is a $p_2$-open function. This shows that $(Y, \mathscr{R}, \mathscr{T})$ is $p_2$-Lindelöf.

**Example 5.1.** *Consider $X = \{a, b, c, d\}$ with $\mathscr{P}$ is discrete topology and topology $\mathscr{Q} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ on $X$, and $Y = \{x, y, z, w\}$ with topologies $\mathscr{R} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}, Y\}$ and $\mathscr{T} = \{\emptyset, \{x\}, \{y, z, w\}, Y\}$ on $Y$. Observe that $\mathscr{P} \cup \mathscr{Q}$ is a discrete topology and*

$$\mathscr{R} \cup \mathscr{T} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}, Y\}$$

*on $Y$. Define a function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ by $f(a) = y, f(b) = f(d) = z$ and $f(c) = w$. Thus the function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is $p_2$-continuous since the inverse image of each member of $\mathscr{R} \cup \mathscr{T}$ on $Y$ is a member of $\mathscr{P} \cup \mathscr{Q}$ on $X$.*

**Example 5.2.** *Consider $X = \{a, b, c, d\}$ with $\mathscr{P} = \{\emptyset, \{a\}, X\}$ and topology $\mathscr{Q} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ on $X$, and $Y = \{x, y, z, w\}$ with topologies $\mathscr{R} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}, Y\}$ and $\mathscr{T} = \{\emptyset, \{x\}, \{y, z, w\}, Y\}$ on $Y$. Observe that $\mathscr{P} \cup \mathscr{Q} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mathscr{R} \cup \mathscr{T} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}, Y\}$ on $Y$. Define a function $g : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ by $g(a) = g(b) = x, g(c) = z$ and $g(d) = w$. Thus the function $g : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is not $p_2$-continuous since $\{y, z, w\} \in \mathscr{R} \cup \mathscr{T}$ but its inverse image $g^{-1}(\{y, z, w\}) = \{c, d\} \notin \mathscr{P} \cup \mathscr{Q}$.*

**Example 5.3.** *Consider a function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ as in Example 5.1. Observe that the function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is not $p_2$-open since $\{b\} \in \mathscr{P} \cup \mathscr{Q}$ but $f(\{b\}) = \{z\} \notin \mathscr{R} \cup \mathscr{T}$.*

**Example 5.4.** *Consider a function $g : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ as in Example 5.2. Observe that the function $g : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ is not $p_2$-open since $\{a, b, c\} \in \mathscr{P} \cup \mathscr{Q}$ but $g(\{a, b, c\}) = \{x, z\} \notin \mathscr{R} \cup \mathscr{T}$.*

**Example 5.5.** *The function $f : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ in Example 5.1 is not $p_2$-homeomorphism since $f^{-1} : (Y, \mathscr{R}, \mathscr{T}) \to (X, \mathscr{P}, \mathscr{Q})$ is not $p_2$-continuous, and the function $g : (X, \mathscr{P}, \mathscr{Q}) \to (Y, \mathscr{R}, \mathscr{T})$ in Example 5.2 is not $p_2$-homeomorphism since it is not $p_2$-continuous.*

Recall that, a property $P$ of sets is called topological property if whenever a topological space $(X, \tau)$ has property $P$, then every space homeomorphic to $(X, \tau)$ also has property $P$. In the case of bitopological space $(X, \mathscr{P}, \mathscr{Q})$, there are three types of topological properties since now we have three types of pairwise homeomorphism.

The first two types, the reader is suggested to refer [3] for the detail. Now if $p_2$-homeomorphism considered, we shall call such property as $p_2$-topological property. It is very clear that, Theorem 5.2 yields the following corollary.

**Corollary 5.1.** *A $p_2$-Lindelöf property is $p_2$-topological property.*

## References


[1] J. C. Kelly, Bitopological Spaces, *Proc. London Math. Soc.*, (3) 13 (1963), 71-89.

[2] J. L. Kelly, General Topology, *Springer-Verlag*, New York, 1955.

[3] A. Kılıçman and Z. Salleh, On pairwise Lindelöf Bitopological spaces, Topology & Its Appl., **154** (8) (2007), 1600-1607.

[4] A. Kılıçman and Z. Salleh, Mappings and Pairwise Continuity on Pairwise Lindelöf Bitopological Spaces, Albanian J. Math., **1**(2) (2007), 115–120.

[5] A. Kılıçman and Z. Salleh, Pairwise almost Lindelöf bitopological spaces, *Journal of Malaysian Mathematical Sciences*, **1**(2)(2007), 227–238.

[6] A. Kılıçman and Z. Salleh, Pairwise Weakly Lindelöf Bitopological Spaces, *Abstract and Applied Analysis*, Volume 2008, Article ID 184243, 13 pages doi:10.1155/2008/184243.

[7] W. J. Pervin, Foundations of General Topology, *Academic Press, Inc.*, London, 1964.

[8] A. Tallafha, A. Al-Bsoul and A. Fora, Countable dense homogeneous bitopological spaces, *Tr. J. Math.*, 23 (1999), 233-242 © TÜBİTAK.

[9] S. Willard, General Topology, *Addison-Wesley*, Canada, 1970.