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On supra b−open sets and supra b-continuity on topological spaces

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Abstract. In this paper, we introduce and investigate a new class of sets and maps between topological spaces called supra $b-$open sets and supra $b-$continuous maps, respectively. Furthermore, we introduce the concepts of supra b-open maps and supra b-closed maps and investigate several properties of them.

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1. Introduction

In 1983, A. S. Mashhour et al. [6] introduced the supra topological spaces and studied s-continuous maps and $s^{\star}-$continuous maps. In 1996, D. Andrijevic' [2] introduced and studied a class of generalized open sets in a topological space called b-open sets. This class of sets contained in the class of β-open sets [1] and contains all semi-open sets [4] and all pre-open sets [5]. In 2008, R. Devi et al. [3] introduced and studied a class of sets and maps between topological spaces called supra $\alpha-$open sets and supra $\alpha-$continuous maps, respectively. Now, we introduce the concept of supra $b-$open sets and study some basic properties of it. Also, we introduce the concepts of supra $b-$continuous maps, supra b-open maps and supra b-closed maps and investigate several properties for these classes of maps. In particular, we study the relation between supra b-continuous maps and supra b-open maps (supra b-closed maps).

Throughout this paper, (X,τ) , (Y,σ) and (Z,υ) (or simply, X , Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset

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A of (X,τ), the closure and the interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. The complement of A is denoted by $X - A$. In the space (X,τ), a subset A is said to be β-open (resp. b-open, semi-open, pre-open, α-open [7]) if $A \subseteq Cl(Int(Cl(A)))$ (resp. $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$, $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(A))$, $A \subseteq Int(Cl(Int(A)))$. The family of all β-open (resp. b-open, semi-open, preopen, α-open) sets of (X,τ) is denoted by $\beta(X)$ (resp. $B(X), SO(X), PO(X), \alpha(X)$). A subcollection $\mu \subset 2^X$ is called a supra topology [6] on X if $X \in \mu$ and μ is closed under arbitrary union. (X,μ) is called a supra topological space. The elements of μ are said to be supra open in (X,μ) and the complement of a supra open set is called a supra closed set. The supra closure of a set A, denoted by $Cl^{\mu}(A)$, is the intersection of supra closed sets including A. The supra interior of a set A,denoted by $Int^{\mu}(A)$, is the union of supra open sets included in A. The supra topology μ on X is associated with the topology τ if $\tau \subset \mu$. A set A is called a supra α-open set [3] (resp. supra semi-open set [6]) if $A \subseteq Int^{\mu}(Cl^{\mu}(Int^{\mu}(A)))$ (resp. $A \subseteq Cl^{\mu}(Int^{\mu}(A))$).

Before we study the basic properties of supra b-open sets we have the following correction in [3].

(1) Definition 6 should be written as we stated before.

(2) Example 3.2 is not correct and we will state instead of it.

(3) The proof of Theorem 3.3 (ii) is not correct as τ^* is not supa-topology as well as Theorem 3.4 (ii).

2. Supra b-open sets

In this section, we introduce a new class of generalized open sets called supra b-open sets and study some of their properties.

Definition 1. *Let (X,μ) be a supra topological space. A set A is called a supra b -open set if $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \bigcup Int^{\mu}(Cl^{\mu}(A))$. The complement of a supra b-open set is called a supra b-closed set.*

Theorem 1. *Every supra semi-open set is supra b-open.*

Proof. Let A be a supra semi-open set in (X,μ). Then $A \subseteq Cl^{\mu}(Int^{\mu}(A))$. Hence, $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \bigcup Int^{\mu}(Cl^{\mu}(A))$ and A is supra b-open in (X,μ).

The converse of the above theorem need not be true as shown by the following example.

Example 1. *Let (X,μ) be a supra topological space, where $X = \{a,b,c\}$ and $\mu = \{X,\phi,\{a\},\{a,b\},\{b,c\}\}$. Here $\{a,c\}$ is a supra b-open set, but it is not supra semi-open.*

In [3], the author proved that every supra α-open set is supra semi-open. The following example (Instead of Example 3.2 [3]) shows the converse need not be true.

Example 2. *Let (X,μ) be a supra topological space, where $X = \{a,b,c,d\}$ and $\mu = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. Here $\{b,c\}$ is a supra semi-open set, but it is not supra α-open.*

From Theorems 3.1 and 3.2 in [3], the above theorem, Example 3.1 [3], and the above two examples, we have the following diagram in which the converses of the implications need not be true:

(DIAGRAM 1)

$$supra-open \rightarrow supra\ \alpha-open \rightarrow supra\ semi-open\ \rightarrow supra\ b-open$$

Theorem 2.

(i) Arbitrary union of supra b-open sets is always supra b-open.

(ii) Finite intersection of supra b-open sets may fail to be supra b-open.

(iii) X is a supra b-open set.

Proof.

(i) Let A and B be two supra b-open sets. Then, $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \bigcup Int^{\mu}(Cl^{\mu}(A))$ and $B \subseteq Cl^{\mu}(Int^{\mu}(B)) \bigcup Int^{\mu}(Cl^{\mu}(B))$. Then, $A \cup B \subseteq Cl^{\mu}(Int^{\mu}(A \cup B)) \bigcup Int^{\mu}(Cl^{\mu}(A \cup B))$. Therefore, $A \cup B$ is supra b-open set.

(ii) In Example 1, both $\{a, c\}$ and $\{b, c\}$ are supra b-open sets, but their intersection $\{c\}$ is not supra b-open.

Theorem 3.

(i) Arbitrary intersection of supra b-closed sets is always supra b-closed.

(ii) Finite union of supra b-closed sets may fail to be supra b-closed.

Proof.

(i) This follows immediately from Theorem 2.

(ii) In Example 1, both $\{a\}$ and $\{b\}$ are supra b-closed sets, but their union $\{a, b\}$ is not supra b-closed.

Definition 2. *The supra $b-$closure of a set A, denoted by $Cl^{\mu}_{b}(A)$, is the intersection of supra $b-$closed sets including A. The supra $b-$interior of a set A, denoted by $Int^{\mu}_{b}(A)$, is the union of supra $b-$open sets included in A.*

Remark 1. *It is clear that $Int^{\mu}_{b}(A)$ is a supra b-open set and $Cl^{\mu}_{b}(A)$, is a supra $b-$closed set.*

Theorem 4.

(i) $A \subseteq Cl^{\mu}_{b}(A)$; and $A = Cl^{\mu}_{b}(A)$ iff A is a supra b-closed set;

(ii) $Int^{\mu}_{b}(A) \subseteq A$; and $Int^{\mu}_{b}(A) = A$ iff A is a supra b-open set;

(iii) $X - Int_b^{\mu}(A) = Cl_b^{\mu}(X - A)$;

(iv) $X - Cl_b^{\mu}(A) = Int_b^{\mu}(X - A)$.

Proof. Obvious.

Theorem 5.

(a) $Int_b^{\mu}(A) \cup Int_b^{\mu}(B) \subseteq Int_b^{\mu}(A \cup B)$;

(b) $Cl_b^{\mu}(A \cap B) \subseteq Cl_b^{\mu}(A) \cap Cl_b^{\mu}(B)$.

Proof. obvious.

Proposition 1. *The intersection of a supra α-open set and a supra b-open set is a supra b-open set.*

3. Supra b-continuous maps

In this section, we introduce a new type of continuous maps called a supra b-continuous map and obtain some of their properties and characterizations.

Definition 3. *Let (X, τ) and (Y, σ) be two topological spaces and μ be an associated supra topology with τ. A map $f : (X, \tau) \to (Y, \sigma)$ is called a supra b-continuous map if the inverse image of each open set in Y is a supra b-open set in X.*

Theorem 6. *Every continuous map is supra b-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous map and A is open in Y. Then $f^{-1}(A)$ is an open set in X. Since μ is associated with τ, then $\tau \subseteq \mu$. Therefore, $f^{-1}(A)$ is supra open in X and it is supra b-open in X . Hence f is supra b-continuous.

The converse of the above theorem is not true as shown in the following example.

Example 3. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$ be a topology on X. The supra topology μ is defined as follows: $\mu = \{X, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a map defined as follows: $f(a) = a, f(b) = c, f(c) = b$. The inverse image of the open set $\{a, b\}$ is $\{a, c\}$ which is not an open set but it is a supra b-open.Then f is supra b-continuous but it is not continuous.*

The following example shows that supra b-continuous maps need not be supra semi-continuous.

Example 4. *Consider the set $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a, c\}, \{b, d\}\}$ and the supra topology $\mu = \{X, \phi, \{a, c\}, \{b, d\}, \{a, c, d\}\}$. Also, suppose $Y = \{x, y, z\}$ with the topology $\sigma = \{Y, \phi, \{z\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by: $f(a) = y$, $f(b) = f(c) = z$, $f(d) = x$. The inverse image of the open set $\{z\}$ is $\{b, c\}$ which is a supra b-open set but it is not a supra semi-open set.Then f is supra b-continuous but it is not supra semi-continuous map.*

Therefore, from diagram 1 we have the following diagram in which the converses of the implications need not be true by the above discussion.

(DIAGRAM 2) *supra-continuity* → *supra α-continuity* → *supra semi-continuity* → *supra b-continuity*

Theorem 7. *Let (X,τ) and (Y,σ) be two topological spaces and μ be an associated supra topology with τ. Let f be a map from X into Y. Then the following are equivalent:*

(1) f is a supra b-continuous map;

(2) The inverse image of a closed set in Y is a supra b-closed set in X;

(3) $Cl_b^\mu(f^{-1}(A)) \subseteq f^{-1}(Cl(A))$ for every set A in Y;

(4) $f(Cl_b^\mu(A)) \subseteq Cl(f(A))$ for every set A in X;

(5) $f^{-1}(Int(B)) \subseteq Int_b^\mu(f^{-1}(B))$ for every B in Y.

Proof.

- (1)⇒(2): Let A be a closed set in Y, then $Y-A$ is an open set in Y. Then $f^{-1}(Y-A) = X - f^{-1}(A)$ is a supra b-open set in X. It follows that $f^{-1}(A)$ is a supra b-closed subset of X.
- (2)⇒(3): Let A be any subset of Y. Since $Cl(A)$ is closed in Y, then $f^{-1}(Cl(A))$ is supra b-closed in X. Therefore, $Cl_b^\mu(f^{-1}(A)) \subseteq Cl_b^\mu(f^{-1}(Cl(A))) = f^{-1}(Cl(A))$.
- (3)⇒(4): Let A be any subset of X. By (3) we have $f^{-1}(Cl(f(A))) \supseteq Cl_b^\mu(f^{-1}(f(A))) \supseteq Cl_b^\mu(A)$.Therefore, $f(Cl_b^\mu(A)) \subseteq Cl(f(A))$.
- (4) ⇒ (5):Let B be any subset of Y. By (4), $f(Cl_b^\mu(X - f^{-1}(B))) \subset Cl(f(X - f^{-1}(B)))$ and $f(X - Int_b^\mu(f^{-1}(B))) \subset Cl(Y-B) = Y - Int(B)$. Therefore,we have $X - Int_b^\mu(f^{-1}(B)) \subset f^{-1}(Y - Int(B))$ and $f^{-1}(Int(B)) \subset Int_b^\mu(f^{-1}(B))$.
- (5)⇒(1): Let B be an open set in Y and $f^{-1}(Int(B)) \subseteq Int_b^\mu(f^{-1}(B))$. Then, $f^{-1}(B) \subseteq Int_b^\mu(f^{-1}(B))$. But, $Int_b^\mu(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence, $f^{-1}(B) = Int_b^\mu(f^{-1}(B))$. Therefore, $f^{-1}(B)$ is supra b-open in X.

Theorem 8. *Let (X,τ),(Y,σ) and (Z,υ) be three topological spaces. If a map $f:(X,\tau)\to(Y,\sigma)$ is supra b-continuous and $g:(Y,\sigma)\to(Z,\upsilon)$ is a continuous map, then $g\circ f:(X,\tau)\to(Z,\upsilon)$ is supra b-continuous.*

Proof. Obvious.

Theorem 9. *Let (X,τ) and (Y,σ) be two topological spaces and μ and ν be the associated supra topologies with τ and σ, respectively. Then $f:(X,\tau)\to(Y,\sigma)$ is a supra b-continuous map, if one of the following holds:*

(1) $f^{-1}(Int_b^{\nu}(B)) \subseteq Int(f^{-1}(B))$ *for every set B in Y.*

(2) $Cl(f^{-1}(B)) \subseteq f^{-1}(Cl_b^{\nu}(B))$ *for every set B in Y.*

(3) $f(Cl(A)) \subseteq Cl_b^{\mu}(f(A))$ *for every set A in X.*

Proof. Let B be any open set of Y. If condition (1) is satisfied, then $f^{-1}(Int_b^{\nu}(B)) \subseteq Int(f^{-1}(B))$. We get $f^{-1}(B) \subseteq Int(f^{-1}(B))$. Therefore, $f^{-1}(B)$ is an open set. Every open set is supra b-open. Hence, f is a supra b-continuous map.
If condition (2) is satisfied, then we can easily prove that f is a supra b-continuous map.
Let condition (3) be satisfied and B be any open set of Y. Then $f^{-1}(B)$ is a set in X and $f(Cl(f^{-1}(B))) \subseteq Cl_b^{\mu}(f(f^{-1}(B)))$. This implies $f(Cl(f^{-1}(B))) \subseteq Cl_b^{\mu}(B)$. This is nothing but condition (2). Hence f is a supra b-continuous map.

4. Supra b-open maps and supra b-closed maps

Definition 4. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is called a supra b-open (resp. supra b-closed) if the image of each open (resp. closed) set in X is supra b-open (resp. supra b-closed) in* (Y, ν)*.*

Theorem 10. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is supra b-open if and only if* $f(Int(A)) \subseteq Int_b^{\nu}(f(A))$ *for each set A in X.*

Proof. Suppose that f is a supra b-open map. Since $Int(A) \subseteq A$, then $f(Int(A)) \subseteq f(A)$. By hypothesis, $f(Int(A))$ is a supra b-open set and $Int_b^{\nu}(f(A))$ is the largest supra b-open set contained in $f(A)$. Hence $f(Int(A)) \subseteq Int_b^{\nu}(f(A))$.

Conversely, suppose A is an open set in X. Then, $f(Int(A)) \subseteq Int_b^{\nu}(f(A))$. Since $Int(A) = A$, then $f(A) \subseteq Int_b^{\nu}(f(A))$. Therefore $f(A)$ is a supra b-open set in (Y, ν) and f is a supra b-open map.

Theorem 11. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is supra b-closed if and only if* $Cl_b^{\nu}(f(A)) \subseteq f(Cl(A))$ *for each set A in X.*

Proof. Suppose f is a supra b-closed map. Since for each set A in X, $Cl(A)$ is closed set in X, then $f(Cl(A))$ is a supra b-closed set in Y. Also, since $f(A) \subseteq f(Cl(A))$, then $Cl_b^{\nu}(f(A)) \subseteq f(Cl(A))$.

Conversely, Let A be a closed set in X. Since $Cl_b^{\nu}(f(A))$ is the smallest supra b-closed set containing $f(A)$, then $f(A) \subseteq Cl_b^{\nu}(f(A)) \subseteq f(Cl(A)) = f(A)$. Thus, $f(A) = Cl_b^{\nu}(f(A))$. Hence, $f(A)$ is a supra b-closed set in Y. Therefore, f is a supra b-closed map.

Theorem 12. *Let* $(X, \tau), (Y, \sigma)$ *and* (Z, υ) *be three topological spaces and* $f : (X, \tau) \to (Y, \sigma)$ *and* $g : (Y, \sigma) \to (Z, \upsilon)$ *be two maps. Then,*

(1) if $g \circ f$ *is supra b-open and* f *is continuous surjective, then* g *is a supra b-open map.*

(2) if $g \circ f$ is open and g is supra b-continuous injective, then f is a supra b-open map.

Proof.

(1) Let A be an open set in Y. Then, $f^{-1}(A)$ is an open set in X. Since $g \circ f$ is a supra b-open map, then $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ (because f is surjective) is a supra b-open set in Z. Therefore, g is a supra b-open map.

(2) Let A be an open set in X. Then, $g(f(A))$ is an open set in Z. Therefore, $g^{-1}(g(f(A))) = f(A)$ (because g is injective) is a supra b-open set in Y. Hence, f is a supra b-open map.

Theorem 13. *Let (X,τ) and (Y,σ) be two topological spaces and $f : (X,\tau) \to (Y,\sigma)$ be a bijective map. Then the following are equivalent:*

(1) f is a supra b-open map;

(2) f is a supra b-closed map;

(3) f^{-1} is a supra b-continuous map.

Proof.

- (1) $\Longrightarrow$ (2): Suppose B is a closed set in X. Then $X - B$ is an open set in X and by (1), $f(X-B)$ is a supra b-open set in Y. Since f is bijective, then $f(X-B) = Y - f(B)$.Hence, $f(B)$ is a supra b-closed set in Y. Therefore, f is a supra b-closed map.
- (2) $\Longrightarrow$ (3): Let f is a supra b-closed map and B be closed set in X. Since f is bijective, then $(f^{-1})^{-1}(B) = f(B)$ which is a supra b-closed set in Y. Therefore, by Theorem 7, f is a supra b-continuous map.
- (3) $\Longrightarrow$ (1): Let A be an open set in X. Since f^{-1} is a supra b-continuous map, then $(f^{-1})^{-1}(A) = f(A)$ is a supra b-open set in Y. Hence, f is a supra b-open map.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.,* **12** (1983), 77- - 90.

[2] D. Andrijevic', On b-open sets, *Mat. Vesnik*, **48** (1996), 59- - 64.

[3] R. Devi, S. Sampathkumar and M. Caldas, On supra α-open sets and $s\alpha$-continuous maps, General Mathematics, **16** (2) (2008), 77 - - 84.

[4] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36- - 41.

[5] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mapping, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47- -53.

[6] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supra topological spaces, *Indian J. Pure and Appl. Math.*, **14** (4) (1983), 502- -510.

[7] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961- -970.