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# Certain Results for a Subclass of Meromorphic Multivalent Functions Associated with the Wright Function

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**Abstract.** In this paper, we introduce a new subclass of meromorphic multivalent functions associated with Wright generalized hypergeometric function and obtain new results for this class by the application of Briot-Bouquet differential subordination.

**Key Words and Phrases**: Analytic functions, Wright generalized hypergeometric function, The Briot-Bouquet differential subordination.

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## 1. Introduction

Let $\Sigma_p$ denote the class of meromorphic function of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, .....\}), \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathbb{D} := \{z \in \mathscr{C} \,|\, 0 < |z| < 1\} = \mathscr{U} \setminus \{0\},$$

where $\mathscr{U} := \{z \in \mathscr{C} \,|\, |z| < 1\}$. Also, we denote $\Sigma = \Sigma_1$.

If $f(z)$ and $F(z)$ are analytic in $\mathscr{U}$, we say that $f(z)$ is subordinate to a function $F(z)$ written symbolically as $f \prec F$ or $f(z) \prec F(z), (z \in \mathscr{U})$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic in $\mathscr{U}$ with

$$w(0) = 0, \ \ |w(z)| < 1 \ \ (z \in \mathscr{U}),$$

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such that

$$f(z) = F(w(z)) \quad (z \in \mathscr{U}).$$

In particular, if the function $F(z)$ is univalent in $\mathscr{U}$, then we have the following equivalence [cf. 7]:

$$f(z) \prec F(z)(z \in \mathscr{U}) \iff f(0) = F(0) \text{ and } f(\mathscr{U}) \subset F(\mathscr{U}).$$

For functions $f(z) \in \Sigma_p$ given by (1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}, \tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} =: (g * f)(z) \quad (p \in \mathbb{N}; z \in \mathbb{D}). \tag{3}$$

Let $l, s \in \mathbb{N}$. For positive real parameters $\alpha_j, A_j \ (j = 1, \ldots, q)$ ; $\beta_j, B_j > 0 \ (j = 1, \ldots, s)$ , with

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0,$$

the Fox-Wright function ${}_l\psi_s$ is defined by [see 8]

$${}_l\psi_s[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] = \sum_{n=1}^{\infty} \frac{\Pi_{j=1}^{l} \Gamma(\alpha_j + nA_j) z^n}{\Pi_{j=1}^{s} \Gamma(\beta_j + nB_j) n!} \quad (z \in \mathscr{U}). \tag{4}$$

In particular, when $A_i = B_j = 1 \ (i = 1, ..., l; j = 1, ..., s)$ , we have the following relationship:

$${}_lF_s(\alpha_1, ..., \alpha_l; \beta_1, .., \beta_s; z) = \Omega \, {}_l\psi_s[(\alpha_1, 1)_{1,l}; (\beta_j, 1)_{1,s}; z] \quad (l \leq s+1; z \in \mathscr{U}) \tag{5}$$

where

$$\Omega := \frac{\Gamma(\beta_1)...\Gamma(\beta_s)}{\Gamma(\alpha_1)...(\alpha_l)}. \tag{6}$$

Let

$$\phi_p[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] = \Omega z^{-p} \ {}_l\psi_s[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] \quad (z \in \mathbb{D}). \tag{7}$$

Due to Dziok and Raina [2] (see also [1] and [3]) we consider a linear operator

$$\theta_p^{l,s}\{(\alpha_1, A_1)\} f(z) = \theta_p[(\alpha_1, A_1), ..., (\alpha_l, A_l); (\beta_1, B_1), ..., (\beta_s, B_s)] : \Sigma_p \longrightarrow \Sigma_p$$

defined by the following Hadmard product

$$\theta_p^{l,s}\{(\alpha_1, A_1)\} f(z) := \phi_p[(\alpha_j, A_j)_{1,l}; (b_j, \beta_j)_{1,s}; z] * f(z). \tag{8}$$

If $f \in \Sigma_p$ is given by the equation (1), then we have

$$\theta_p^{l,s}\left\{(\alpha_1,A_1)\right\}f(z) = z^{-p} + \Omega\sum_{n=1}^{\infty}\frac{\Pi_{j=1}^{l}\Gamma(\alpha_j+nA_j)z^{n-p}}{\Pi_{j=1}^{s}\Gamma(\beta_j+nB_j)n!}a_n\,(z\in\mathbb{D})\,. \tag{9}$$

In particular, for $A_i = B_j = 1 (i = 1,...,l, j = 1,...,s)$, we get the linear operator

$$\mathscr{H}_p[\alpha_1]f(z) = z^{-p} + \sum_{n=1}^{\infty}\frac{\Pi_{j=1}^{l}(\alpha_j)_n}{\Pi_{j=1}^{s}(\beta_j)_n n!}a_n z^{n-p} \quad (z\in\mathscr{U}), \tag{10}$$

studied by Liu and Srivastava [6]. Obviously, for $l = 2, s = p = 1$ and $\alpha_2 = 1$, we get

$$\mathscr{L}(\alpha_1,\beta_1)f(z) = z^{-1} + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n}{(\beta_1)_n}a_n z^{n-1} \quad (z\in\mathscr{U}).$$

It is easy to verify that

$$z\left[\theta_p^{l,s}\left\{(\alpha_1,A_1)\right\}f(z)\right]' = \frac{\alpha_1}{A_1}\theta_p^{l,s}\left\{(\alpha_1+1,A_1)\right\}f(z) - \left(\frac{\alpha_1}{A_1}+p\right)\theta_p^{l,s}\left\{(\alpha_1,A_1)\right\}f(z) \tag{11}$$

Also, for $-1 \le B < A \le 1$ we denote by

$$V((\alpha_1,A_1);A,B) = V((\alpha_1,A_1),...,(\alpha_l,A_l);A,B)$$

the class of functions $f \in \Sigma_p$ which satisfy the following condition:

$$\left(\frac{\alpha_1}{A_1}+p\right) - \left(\frac{\alpha_1}{A_1}\right)\frac{\theta_p^{l,s}\left(\alpha_1+1,A_1\right)f(z)}{\theta_p^{l,s}\left(\alpha_1,A_1\right)f(z)} \prec p\frac{1+Az}{1+Bz}. \tag{12}$$

Let $h$ and $q$ be analytic functions in $\mathscr{U}$ with $h(0) = q(0) = p$ and let $q$ be univalent convex function. The first-order differential subordination

$$h(z) + \frac{zh'(z)}{\beta h(z)+\gamma} \prec q(z), \tag{13}$$

is called the Briot-Bouquet differential subordination. This particular differential subordination has a surprising number of important applications in the theory of analytic functions (for details see [7]). In this paper we present one more application of the Briot-Bouquet differential subordination.

## 2. Main result

To prove our main results we need the following lemmas:

**Lemma 1** ([7], see also [4]). *Let $\beta,\gamma \in \mathscr{C}$ and suppose $q(z)$ is convex univalent in $\mathscr{U}$ with $q(0)=p$ and*

$$Re\left\{\beta q(z)+\gamma\right\}>0 \quad (z\in\mathscr{U})$$

*If $h(z)$ is analytic in $\mathscr{U}$ with $h(0)=p$, and:*

$$h(z)+\frac{zh'(z)}{\beta h(z)+\gamma}\prec q(z) \quad (z\in\mathscr{U}), \tag{14}$$

*then*

$$h(z)\prec q(z).$$

**Lemma 2** ([7]). *Let the function $w(z)$ be (nonconstant) analytic in $\mathscr{U}$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathscr{U}$, then*

$$z_0 w'(z_0)=kw(z_0), \tag{15}$$

*where $k$ is real and $k\geq 1$.*

Making use of Lemma 1, we get the following theorem:

**Theorem 1.** *If*

$$\alpha_1(1+B)>pA_1(A-B),$$

*then*

$$V((\alpha_1+m,A_1);A,B)\subset V((\alpha_1,A_1);A,B) \quad (m\in\mathbb{N}).$$

*Proof.* Obviously, it is sufficient to prove the theorem for $m=1$. Let a function $f$ belong to the class $V((\alpha_1+1,A_1);A,B)$ or equivalently

$$\frac{-z\left[\theta_p^{l,s}(\alpha_1+1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}\prec p\frac{1+Az}{1+Bz}. \tag{16}$$

Then the function

$$h(z)=\frac{-z\left[\theta_p^{l,s}(\alpha_1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1,A_1)f(z)}, \tag{17}$$

is analytic in $\mathscr{U}$ and $h(0)=p$. Using equation (11) the equation (17) can be rewritten as

$$-h(z)+\left(\frac{\alpha_1}{A_1}+p\right)=\frac{\alpha_1}{A_1}\,\frac{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}{\theta_p^{l,s}(\alpha_1,A_1)f(z)}. \tag{18}$$

Taking the logarithmic derivative of equation (18), we get

$$\frac{-zh'(z)}{\left(\frac{\alpha_1}{A_1}+p\right)-h(z)}=\frac{z\left[\theta_p^{l,s}(\alpha_1+1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}-\frac{z\left[\theta_p^{l,s}(\alpha_1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1,A_1)f(z)}. \tag{19}$$

Using (17) in the above equation we have,

$$\frac{-zh'(z)}{\left(\frac{\alpha_1}{A_1}+p\right)-h(z)}=\frac{z\left[\theta_p^{l,s}(\alpha_1+1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}+h(z), \tag{20}$$

$$h(z)+\frac{zh'(z)}{\left(\frac{\alpha_1}{A_1}+p\right)-h(z)}=-\frac{z\left[\theta_p^{l,s}(\alpha_1+1,A_1)f(z)\right]'}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}. \tag{21}$$

Thus by (16) we have

$$h(z)+\frac{zh'(z)}{\left(\frac{\alpha_1}{A_1}+p\right)-h(z)}\prec p\frac{1+Az}{1+Bz}. \tag{22}$$

Lemma 1 now yields

$$h(z)\prec p\frac{1+Az}{1+Bz}.$$

Thus, by (17) and (11) we conclude that $f(z)\in V((\alpha_1,A_1);A,B)$. This completes the proof of the Theorem 1.

Using Lemma 2 we now show the following sufficient conditions for functions to belong to the class $V((\alpha_1,A_1);A,B)$.

**Theorem 2.** *Let $m\in\mathbb{N}$ and*

$$\alpha_1(1+B)>pA_1(A-B),\ 2\left(\alpha_1+m-1\right)B^2\le A_1p[(A-B)(2B+1)]. \tag{23}$$

*If a function $f\in\Sigma_p$ satisfies the inequality*

$$\left(\frac{\alpha_1+m}{A_1}\right)\left|\frac{\theta_p^{l,s}(\alpha_1+m+1;A_1)f(z)}{\theta_p^{l,s}(\alpha_1+m;A_1)f(z)}-1\right| < \frac{A-B-\frac{\alpha_1}{pA_1}B}{A-B+\frac{\alpha_1}{pA_1}(1-B)} + \frac{B+p(A-B)}{1+B},\ (z\in U), \tag{24}$$

*then $f\in V((\alpha_1,A_1);A,B)$.*

*Proof.* It is sufficient to consider the case $m=1$. Let a function $f$ belong to the class $\Sigma_p$. On putting

$$h(z)=p\frac{1+Aw(z)}{1+Bw(z)}\quad(z\in U). \tag{25}$$

in (21), we obtain

$$\begin{aligned}\left(\frac{\alpha_1+1}{A_1}+p\right)-\left(\frac{\alpha_1+1}{A_1}\right)\frac{\theta_p^{l,s}(\alpha_1+2,A_1)f(z)}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)} &= \frac{(A-B-\frac{\alpha_1}{pA_1}B)zw'(z)}{\frac{\alpha_1}{pA_1}+\{\frac{\alpha_1}{pA_1}B+B-A\}w(z)} \\ &+ \frac{Bzw'(z)}{1+Bw(z)}+p\frac{1+Aw(z)}{1+Bw(z)}\end{aligned}$$

Consequently, we have

$$F(z)=w(z)\left\{\frac{zw'(z)}{w(z)}\left(\frac{A-B-\frac{\alpha_1}{pA_1}B}{\frac{\alpha_1}{pA_1}+\{\frac{\alpha_1}{pA_1}B+B-A\}w(z)}+\frac{B}{1+Bw(z)}\right)+\frac{p(A-B)}{1+Bw(z)}\right\}, \tag{26}$$

where

$$F(z)=\left(\frac{\alpha_1+1}{A_1}+p\right)-\left(\frac{\alpha_1+1}{A_1}\right)\frac{\left[\theta_p^{l,s}(\alpha_1+2,A_1)f(z)\right]}{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}-p.$$

By (12), (17) and (25), it is sufficient to verify that $w$ is analytic in $\mathscr{U}$ and

$$|w(z)|<1 \quad (z\in\mathscr{U}).$$

Now, suppose that there exists a point $z_0\in\mathscr{U}$ such that

$$\left|w(z_0)\right|=1,\ |w(z)|<1 \quad (|z|<\left|z_0\right|).$$

Then, applying Lemma 2, we can write

$$z_0w'(z_0)=kw(z_0), \quad w(z_0)=e^{i\theta} \quad (k\geq 1).$$

Combining these with (26), we obtain

$$\begin{aligned}\left|F(z_0)\right| &\geq kRe\left(\frac{A-B-\frac{\alpha_1}{pA_1}B}{\frac{\alpha_1}{pA_1}+\{\frac{\alpha_1}{pA_1}B+B-A\}e^{i\theta}}+\frac{B}{1+Be^{i\theta}}\right)+\frac{p(A-B)}{1+B} \\ &\geq k\left(\frac{A-B-\frac{\alpha_1}{pA_1}B}{\frac{\alpha_1}{pA_1}+A-B-\frac{\alpha_1}{pA_1}B}+\frac{B}{1+B}\right)+\frac{p(A-B)}{1+B} \\ &\geq \frac{A-B-\frac{\alpha_1}{pA_1}B}{A-B+\frac{\alpha_1}{pA_1}(1-B)}+\frac{B+p(A-B)}{1+B}.\end{aligned}$$

Since this results contradicts (24), we conclude that $w$ is the analytic function in $\mathscr{U}$ and $|w(z)|<1$ $(z\in\mathscr{U})$, which completes the proof of the Theorem 2.

Putting $p=1,\ A=1-\alpha,$ and $B=0$ in Theorem 2, we obtain the following result.

**Corollary 1.** *Let $m \in \mathbb{N}$, $0 \leq \alpha < 1$ and $\alpha_1 > A_1(1-\alpha)$. If a function $f \in \Sigma$ satisfies the following inequality:*

$$\left|\frac{\theta_p^{l,s}(\alpha_1+m+1;A_1)f(z)}{\theta_p^{l,s}(\alpha_1+m;A_1)f(z)} - 1\right| < \frac{(1-\alpha)\left(2+\frac{\alpha_1}{A_1}-\alpha\right)}{\frac{\alpha_1+m}{A_1}\left(\frac{\alpha_1}{A_1}+1-\alpha\right)},$$

*then*

$$\left|\frac{\alpha_1}{A_1}\left(1-\frac{\theta_p^{l,s}(\alpha_1+1,A_1)f(z)}{\theta_p^{l,s}(\alpha_1+,A_1)f(z)}\right)\right| < 1-\alpha.$$

Putting $A_i = B_j = 1(i = 1,...,l, j = 1,...,s)$ in Corollary 1, we obtain the following result.

**Corollary 2.** *Let $m \in \mathbb{N}$, $0 \leq \alpha < 1$ and $\alpha_1 + \alpha > 1$. If a function $f \in \Sigma$ satisfies the following inequality:*

$$\left|\frac{\mathscr{H}(\alpha_1+m+1)f(z)}{\mathscr{H}(\alpha_1+m)f(z)} - 1\right| < \frac{(1-\alpha)\left(2+\alpha_1-\alpha\right)}{\left(\alpha_1+m\right)\left(\alpha_1+1-\alpha\right)},$$

*then*

$$\left|\alpha_1+1-\alpha_1\frac{\mathscr{H}(\alpha_1+1)f(z)}{\mathscr{H}(\alpha_1)f(z)}\right| < 1-\alpha.$$

Putting $l = 2$, $s = p = A_1 = A_2 = B_1 = 1$, and $\alpha_2 = 1$ in Theorems 1 and 2, we get the following two results:

**Corollary 3.** *Let $m \in \mathbb{N}$, $\alpha_1(1+B) > (A-B)$. If a function $f \in \Sigma$ satisfies the following condition:*

$$(\alpha_1+m+1)-(\alpha_1+m)\frac{\mathscr{L}(\alpha_1+m+1,\beta_1)f(z)}{\mathscr{L}(\alpha_1+m,\beta_1)f(z)} \prec \frac{1+Az}{1+Bz},$$

*then*

$$(\alpha_1+1)-\alpha_1\frac{\mathscr{L}(\alpha_1+1,\beta_1)f(z)}{\mathscr{L}(\alpha_1,\beta_1)f(z)} \prec \frac{1+Az}{1+Bz}.$$

**Corollary 4.** *Let $m \in \mathbb{N}$, $\alpha_1(1+B) > A-B$ and $2\left(\alpha_1+m-1\right)B^2 \leq (A-B)(2B+1)$, If a function $f \in \Sigma$ satisfies the inequality:*

$$\left(\alpha_1+m\right)\left|\frac{\mathscr{L}(\alpha_1+m+1,\beta_1)f(z)}{\mathscr{L}(\alpha_1+m,\beta_1)f(z)} - 1\right| < \frac{A-B-\alpha_1 B}{A-B+\alpha_1(1-B)} + \frac{A}{1+B} \quad (z \in U),$$

*then*

$$\alpha_1+1-\alpha_1\frac{\mathscr{L}(\alpha_1+1,\beta_1)f(z)}{\mathscr{L}(\alpha_1,\beta_1)f(z)} \prec \frac{1+Az}{1+Bz}.$$

Putting $\alpha_1 = \beta_1 = m = 1$ in Corollary 4 we obtain the sufficient conditions for starlikeness.

**Corollary 5.** *Let* $2B^2 \leq (A-B)(2B+1)$. *If a function* $f \in \Sigma$ *satisfies the inequality:*

$$|\frac{z^2 f''(z) + 4zf'(z) + 2f(z)}{2\left(zf'(z) + 2f(z)\right)}| < \frac{A-2B}{1+A-2B} + \frac{A}{1+B} \quad (z \in \mathscr{U}),$$

*then*

$$\frac{-zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

*i.e., the function* $f$ *is starlike in* $\mathscr{U}$.

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