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Majorization for Certain Analytic Functions

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Abstract. In this paper two subclasses $S^{\delta}_{p,q}(\gamma, A, B)$ and $C^{\delta}_{p,q}(\gamma, A, B)$ of p-valently starlike and p-valently convex functions of complex order $\gamma \neq 0$ in the open unit disk U are introduced and for these classes several majorization problems are discussed.

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1. Introduction and Definitions

Definition 1 ([see 5]). *Let the functions $f(z)$ and $g(z)$ be analytic in the open unit disk*

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We say that $f(z)$ is majorized by $g(z)$ and write

$$f(z) \ll g(z) \tag{1}$$

if there exists a function $\phi(z)$ analytic in $\cup$, such that

$$|\phi(z)| \leq 1 \text{ and } f(z) = \phi(z)g(z). \tag{2}$$

Also, we say that $f(z)$ is subordinate to $g(z)$ and write

$$f(z) \prec g(z)$$

if there exist a function $w(z)$ analytic in U, such that

$$w(0) = 0, |w(z)| \leq |z| \text{ and } f(z) = g(w(z)).$$

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Definition 2 ([see 8]). *The fractional derivative of order δ is defined by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \qquad (0 \leqslant \delta < 1) \tag{3}$$

where $f(z)$ is an analytic function in a simply connected region of the $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 3 ([see 8]). *Under the hypotheses of definition 2, the fractional derivative of order $(n+\delta)$ is defined by*

$$D_z^{n+\delta} f(z) = \frac{d}{dz^n} D_z^{\delta} f(z). \tag{4}$$

Several majorization problems investigated by Altıntaş and Owa [1], Altıntaş et al. [2] and [3].

Let A_p denote the class of functions f normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (P \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $p-valent$ in U. Also let a function $f \in A_p$ is said to be in the class $S_{p,q}^{\delta}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma}\left(\frac{z f^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)} - p + q + \delta\right) \prec \frac{1+AZ}{1+BZ} \tag{5}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leqslant \delta < 1$, $-1 \leqslant B < A \leqslant 1$ and

$$\left|\gamma(A-B) + (p-q-\delta)B\right| \leqslant \left|p-q-\delta\right|.$$

Furthermore a function $f \in A_p$ is said to be in the class $C_{p,q}^{\delta}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma}\left(1 + \frac{z f^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta\right) \prec \frac{1+AZ}{1+BZ} \tag{6}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leqslant \delta < 1$, $-1 \leqslant B < A \leqslant 1$ and

$$\left|\gamma(A-B) + (p-q-\delta)B\right| \leqslant \left|p-q-\delta\right|.$$

We have the following relationships (from [3, 11, 2], respectively)

$$S_{p,q}^{0}(\gamma, 1, -1) = S_{p,q}(\gamma).$$
$$C_{p,q}^{0}(\gamma, 1, -1) = C_{p,q}(\gamma).$$

$$S^0_{p,0}(\gamma,1,-1)=S(\gamma) \text{ and } C^0_{p,0}(\gamma,1,-1)=C(\gamma).$$

$S(\gamma)$ and $C(\gamma)$ were considered by Nasr and Aouf in [6].

$$S^0_{p,0}(1-\alpha,1,-1)=S^*(\alpha) \text{ and } C^0_{p,0}(1-\alpha,1,-1)=C(\alpha)$$

denote respectively the class of starlike and convex functions of order $\alpha,(0\leqslant\alpha<1)$ which were introduced by Robertson in [9].

2. Majorization Problems for the Class $S^{\delta}_{p,q}(\gamma,A,B)$

We begin by proving.

Theorem 1. *Let the function $f(z)$ be in the class A_p and suppose that $g\in S^{\delta}_{p,q}(\gamma,A,B)$. If $f^{(q+\delta)}(z)$ is majorized by $g^{(q+\delta)}(z)$ in U for $q\in\mathbb{N}_o$ and $0\leqslant\delta<1$, then*

$$\left|f^{(q+\delta+1)}(z)\right|\leqslant\left|g^{(q+\delta+1)}(z)\right| \quad \left(|z|\leqslant r_1\right) \tag{7}$$

where $r_1=r_1(p,q,\delta,\gamma,A,B)$ is the smallest positive root of the equation

$$\begin{aligned}\left|\gamma(A-B)+(p-q-\delta)B\right|r^3-(p-q-\delta+2|B|)r^2-[\left|\gamma(A-B)+(p-q-\delta)B\right|\\ +2]r+p-q-\delta=0\end{aligned} \tag{8}$$

where $p\in\mathbb{N}$, $q\in\mathbb{N}_0$, $\gamma\in\mathbb{C}\setminus\{0\}$, $0\leqslant\delta<1$ and

$$\left|\gamma(A-B)+(p-q-\delta)B\right|\leqslant\left|p-q-\delta\right|.$$

Proof. Since $g\in S^{\delta}_{p,q}(\gamma,A,B)$, we obtain from (5)

$$1+\frac{1}{\gamma}\left(\frac{z\ g^{(q+\delta+1)}(z)}{g^{(q+\delta)}(z)}-p+q+\delta\right)=\frac{1+A\omega(z)}{1+B\omega(z)} \tag{9}$$

where

$$\omega(0)=0 \text{ and } |\omega(z)|\leqslant|z| \qquad (z\in U). \tag{10}$$

From (9) we readily obtain

$$\frac{z\ g^{(q+\delta+1)}(z)}{g^{(q+\delta)}(z)}=\frac{p-q-\delta+\left[\gamma(A-B)+(p-q-\delta)B\right]\omega(z)}{1+B\omega(z)}. \tag{11}$$

Using (10) in (11) we find

$$\left|g^{(q+\delta)}(z)\right|\leqslant\frac{(1+|B|\,|z|)\,|z|}{p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right|\,|z|}\left|g^{(q+\delta+1)}(z)\right|. \tag{12}$$

Since $f^{(q+\delta)}(z)$ is majorized by $g^{(q+\delta)}(z)$ from (2) we have

$$f^{(q+\delta+1)}(z) = \phi(z)g^{(q+\delta+1)}(z) + \phi'(z)g^{(q+\delta)}(z), \tag{13}$$

$\phi(z)$ is satisfies the inequality [cf. Nehari 7, p. 168]:

$$\left|\phi'(z)\right| \leqslant \frac{1-\left|\phi(z)\right|^2}{1-|z|^2} \; (z \in U) \tag{14}$$

and using (12) and (14) in (13), we get

$$\left|f^{(q+\delta+1)}(z)\right| \leqslant \left|\phi(z)\right| + \frac{1-\left|\phi(z)\right|^2}{1-|z|^2} \frac{(1+|B|\,|z|)\,|z|}{p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right|\,|z|} \left|g^{(q+\delta+1)}(z)\right| \tag{15}$$

which, upon setting

$$|z| = r, \left|\phi(z)\right| = \rho \qquad (0 \leqslant \rho \leqslant 1)$$

leads us to the inequality

$$\left|f^{(q+\delta+1)}(z)\right| \leqslant \frac{\theta(\rho)}{(1-r^2)\left[p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right| r\right]} g^{(q+\delta+1)}(z) \tag{16}$$

where

$$\theta(\rho) = -(r+|B|\,r^2)\rho^2 + (1-r^2)p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right| r]\rho + (r+|B|\,r^2) \tag{17}$$

takes on its maximum value at $\rho = 1$ with $r = r_1(p,q,\delta,\gamma,A,B)$ gives by (8) if

$$0 \leqslant \sigma \leqslant r_1(p,q,\delta,\gamma,A,B)$$

then the function $\wedge(\rho)$ defined by

$$\wedge(\rho) = -(\sigma+\sigma^2\,|B|)\rho^2 + (1-\sigma^2)\left[p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right|\sigma\right]\rho + (\sigma+\sigma^2\,|B|) \tag{18}$$

is an increasing function on the interval $0 \leqslant \rho \leqslant 1$ so that

$$\wedge(\rho) \leqslant \wedge(1) = (1-\sigma^2)\left[p-q-\delta-\left|\gamma(A-B)+(p-q-\delta)B\right|\sigma\right]$$
$$(0 \leqslant \rho \leqslant 1; 0 \leqslant \sigma \leqslant r_1(p,q,\delta,\gamma,A,B)).$$

Hence, by setting $\rho = 1$ in (16), we conclude that Theorem 1 holds true for $|z| \leqslant r_1(p,q,\delta,\gamma,A,B)$ is given by (8). This completes the proof of Theorem 1.

Corollary 1 ([see 3]). *Let the function $f(z)$ be in the class A_p and suppose that $g \in S^0_{p,q}(\gamma,1,-1)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in U, then*

$$\left|f^{(q+1)}(z)\right| \leqslant \left|g^{(q+1)}(z)\right| \qquad (|z| \leqslant R_1)$$

where

$$R_1 = R_1(p,q,\delta) = \frac{k - \sqrt{k^2 - 4(p-q)\left|2\gamma - p + q\right|}}{2\left|2\gamma - p + q\right|} \tag{19}$$

$$(k = p - q + 2 + \left|2\gamma - p + q\right|, p \in \mathbb{N}, q \in \mathbb{N}_{0,} \gamma \in \mathbb{C} \setminus \{0\}).$$

Proof. If we set $\delta = 0, A = 1, B = -1$ in Theorem 1, then

$$\left|f^{(q+1)}(z)\right| \leqslant \left|g^{(q+1)}(z)\right| \qquad |z| \leqslant R_1$$

where $R_1 = R_1(p,q,\delta)$ is the smallest positive root of the equation

$$\left|2\gamma - p + q\right| r^3 - (p - q + 2)r^2 - [\left|2\gamma - p + q\right| + 2]r + p - q = 0$$

$r = -1$ is the root of the above equation and we obtain

$$\left|2\gamma - p + q\right| r^2 - (\left|2\gamma - p + q\right| + p - q + 2)r + p - q = 0. \tag{20}$$

and the positive root of the equation (20) is $R_1 = R_1(p,q,\delta)$.

Corollary 2 ([see 2]). *Let the function $f(z)$ be in the class A_1 and suppose that $g \in S^0_{1,0}(\gamma, 1, -1)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$\left|f^{'}(z)\right| \leqslant \left|g^{'}(z)\right| \qquad (|z| \leqslant R_2)$$

where

$$R_2 = R_2(\gamma) = \frac{3 + \left|2\gamma - 1\right| - \sqrt{9 + 2\left|2\gamma - 1\right| + \left|2\gamma - 1\right|^2}}{2\left|2\gamma - 1\right|}$$

Corollary 3 ([see 5]). *Let $f(z)$ be in the class A_1 and suppose that $g(z) \in S^0_{1,0}(1, 1, -1)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$\left|f^{'}(z)\right| \leqslant \left|g'(z)\right| \qquad (|z| \leqslant R_3)$$

where $R_3 = 2 - \sqrt{3}$.

3. Majorization Problems for the Class $C^{\delta}_{p,q}(\gamma, A, B)$.

The proof Theorem 2 is based upon the following Lemmas.

Lemma 1 ([see 10, Theorem 1]). *If $f \in C^{\delta}_{p,q}(\gamma, A, B)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ then*

$$Re\left[1 + \frac{1}{\gamma}\left(\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1\right)\right] > \frac{1-A}{1-B} \tag{21}$$

Proof. If $f \in C^{\delta}_{p,q}(\gamma, A, B)$ then we have from (6)

$$1+\frac{1}{\gamma}(\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)}-p+q+\delta+1)=\frac{1+Aw(z)}{1+Bw(z)} \tag{22}$$

where $w(0)=0$ and $|w(z)| \leqslant |z|$, $(-1 \leqslant B < A \leqslant 1)$. We let

$$h(z)=\frac{1+Aw(z)}{1+Bw(z)} \tag{23}$$

and

$$h(z)=u+iv,\ |w(z)|^2=\left|\frac{h(z)-1}{A-Bh(z)}\right|^2 \leqslant 1$$

and

$$(1-B^2)u^2-2(1-AB)u+1-A^2 \leqslant 0 \tag{24}$$

from (24) implies that

$$\frac{1-A}{1-B} \leqslant Reh(z)=u \leqslant \frac{1+A}{1+B}. \tag{25}$$

The following lemma is proved in [3] for $\delta=0$.

Lemma 2. *If $f \in C^{\delta}_{p,q}(\gamma, A, B)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ then $f \in S^{\delta}_{p,q}(\frac{1}{2}\gamma, A, B)$ that is*

$$C^{\delta}_{p,q}(\gamma, A, B) \subset S^{\delta}_{p,q}(\frac{\gamma}{2}, A, B) \tag{26}$$

Proof. We know that all convex function in U is starlike of order $\frac{1}{2}$ in U, [see 4, p. 7] or, equivalently

$$Re[1+\frac{zf''(z)}{f'(z)}]>0 \Rightarrow \mathrm{Re}[\frac{zf'(z)}{f(z)}]>\frac{1}{2}. \tag{27}$$

If we let

$$\mathrm{Re}[1+\frac{zf''(z)}{f'(z)}]>\alpha \text{ for } f(z) \longrightarrow f^{(q+\delta)}(z),$$

and using Lemma 1, we have

$$\mathrm{Re}[1+\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)}-p+q+\delta+1)]>\frac{1-A}{1-B} \tag{28}$$

or

$$\mathrm{Re}[1+\frac{1}{1-\alpha}(\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)}-p+q+\delta+1]>0. \tag{29}$$

This implies that

$$1+\frac{1}{1-\alpha}(\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)}-p+q+\delta+1=\frac{1-w(z)}{1+w(z)}. \tag{30}$$

So, we have

$$1+\frac{1}{\gamma}(\frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)}-(p+q+\delta+1)=\frac{\gamma+(\gamma-2+2\alpha)w(z)}{\gamma(1+w(z))}. \tag{31}$$

On the other hand we know that

$$\frac{zf'(z)}{f(z)}>\alpha\Rightarrow \mathrm{Re}(1+\frac{1}{1-\alpha}\frac{zf'(z)}{f(z)})>0. \tag{32}$$

Similarly using (27) and (29) we obtain the following relations.

$$[1+\frac{1}{1-\alpha}(\frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)}-p+q+\delta)]>\frac{1}{2}, \tag{33}$$

$$1+\frac{1}{1-\alpha}(\frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)}-p+q+\delta)=\frac{1}{1+w(z)}, \tag{34}$$

$$1+\frac{2}{\gamma}(\frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)}-p+q+\delta)=\frac{\gamma+(\gamma-2+2\alpha)}{\gamma(1+w(z))}. \tag{35}$$

(36)

The inclusion property (26) is easily seen that from (31) and (35).

Upon replacing γ in Theorem 1 by $\frac{1}{2}\gamma$, if we apply Lemma 2 we have,

Theorem 2. *Let the function $f(z)$ be in the class A_p and suppose that $g(z)\in C^{\delta}_{p,q}(\gamma,A,B)$. If $f^{(q+\delta)}(z)$ is majorized by $g^{(q+\delta)}(z)\in U$, for $p\in\mathbb{N}$ $q\in\mathbb{N}_0$ and $0\leqslant\delta<1$ then*

$$\left|f^{(q+\delta+1)}(z)\right|\leqslant\left|g^{(q+\delta+1)}(z)\right| \qquad (|z|\le r_2) \tag{37}$$

where $r_2=r_2(p,q,\delta,\gamma,A,B)$ is the smallest positive root of the equation

$$\left|\frac{1}{2}\gamma(A-B)+(p-q-\delta)B\right|r^3-(p-q-\delta+2\,|B|)r^2-[\frac{1}{2}\gamma(A-B)+(p-q-\delta)\,|B| \\ +2]r+p-q-\delta=0$$

where $p\in\mathbb{N}$, $q\in\mathbb{N}_0$, $\gamma\in\mathbb{C}\setminus\{0\}$, $0\leqslant\delta<1$, and

$$\left|\frac{1}{2}\gamma(A-B)+(p-q-\delta)B\right|\leqslant\left|p-q-\delta\right|.$$

Corollary 4 ([see 3])**.** *Let the function $f(z)$ be in the class A_p and suppose that $g(z)\in C^{0}_{p,q}(\gamma,1,-1)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in U, then*

$$\left|f^{(q+1)}(z)\right|\leqslant\left|g^{(q+1)}(z)\right| \qquad (|z|\le R_1)$$

where

$$R_1 = R_1(p,q,\delta) = \frac{\mu - \sqrt{\mu^2 - 4(p-q)\left|\gamma - p + q\right|}}{2\left|\gamma - p + q\right|}$$

$$\mu = 2 + p - q + \left|\gamma - p + q\right|, p \in \mathbb{N}, q \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\}$$

Proof. We let $\delta = 0, A = 1, B = -1$ in Theorem 2.

Corollary 5 ([see 2]). *Let the function $f(z)$ be in the class A_1 and suppose that $g(z) \in C^0_{1.0}(1,1,-1)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$\left|f^{'}(z)\right| \leqslant \left|g^{'}(z)\right| \qquad (|z| \leqslant R_2)$$

where

$$R_2 = R_2(\gamma) = \frac{3 + \left|\gamma - 1\right| - \sqrt{9 - 2\left|\gamma - 1\right| + \left|\gamma - 1\right|^2}}{2\left|\gamma - 1\right|}$$

Proof. We let $p = 1$, $q = 0$, $\delta = 0, A = 1$, $B = -1$ in Theorem 2.

Corollary 6 ([see 5]). *Let the function $f(z)$ be in the class A_1 and suppose that $g(z) \in C^0_{1.0}(1,1,-1)$. If $f(z)$ is majorized by $g(z)$ in U, then*

$$\left|f^{'}(z)\right| \leqslant \left|g^{'}(z)\right| \qquad (|z| \leqslant \frac{1}{3})$$

Proof. We let limit for $\gamma \longrightarrow 1$ in Corollary 5 or $\gamma \longrightarrow \frac{1}{2}\gamma$ in Corollary 1.

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