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On Characterizations of New Separation Axioms and Topological Properties

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Abstract. In this paper, we introduce new separations axioms $\Lambda_r - R_0$, $\Lambda_r - R_1$ and $\Lambda_r - D_k$, and study their properties.

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1. Introduction

Caldas and Jafari [1] introduced the notions of $\Lambda_\delta - R_0$ and $\Lambda_\delta - R_1$ topological spaces. In this paper, we define Λ_r-open sets, that is, if (X, τ) is a topological space and $A \subset X$. Then Λ_r-kernel of A is defined by

$$\Lambda_r - ker(A) = \cap \left(G/G \in \Lambda_r O(X, \tau) \text{ and } A \subset G\right).$$

Then we introduce some Λ_r-separation axioms, we call these axioms as $\Lambda_r - R_0, \Lambda_r - R_1$ and study the properties of these axioms. We also define Λ_r-difference sets and utilize them to define the $\Lambda_r - D_k$, $k = 0, 1, 2$ axioms.

Throughout the paper (X, τ) (or simply X) will always denote a topological space. Let (X, τ) be a topological space and $S \subset X$. Then S is called regularly-open if $S = Int(clS)$. The complement $S^c (= X \setminus S)$ of a regularly-open set S is called the regularly-closed set. The family of all regularly-open sets(resp. regularly-closed sets) will be denoted by $RO(X, \tau)$ (resp. $RC(X, \tau)$). A subset S of a topological space (X, τ) is called Λ_r-set if $S = \Lambda_r(S)$ where

$$\Lambda_r(S) = \cap\{G/G \in RO(X, \tau) \text{ and } S \subseteq G\}.$$

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The collection of all Λ_r-sets is denoted by $\Lambda_r(X,\tau)$.

Throughout this paper, we let A be a subset of a space (X,τ). Then A is called a Λ_r-closed set if $A = T \cap C$ where T is a Λ_r-set and C is a closed set. The complement of a Λ_r-closed set is called Λ_r-open. The collection of all Λ_r-open sets is denoted by $\Lambda_r O(X,\tau)$. The collection of all Λ_r-closed sets is denoted by $\Lambda_r C(X,\tau)$. A point $x \in X$ is called a Λ_r-cluster point of A if for every Λ_r-open set U containing x, $A \cap U \neq \emptyset$. The set of all Λ_r-cluster points of A is called the Λ_r-closure of A and is denoted by $\Lambda_r - cl(A)$.

2. $\Lambda_r - R_0$ Spaces

Definition 1. *The topological space (X,τ) is said to be $\Lambda_r - R_0$ if for each Λ_r-open set G, $x \in G \Rightarrow \Lambda_r - cl(\{x\}) \subseteq G$.*

Theorem 1. *For a topological space (X,τ), the following statements are equivalent:*

(1) (X,τ) is $\Lambda_r - R_0$,

(2) For any Λ_r-closed set F and a point $x \notin F$, $\exists U \in \Lambda_r O(X,\tau)$ such that $x \notin U$ and $F \subseteq U$,

(3) For any Λ_r-closed set F and a point $x \notin F$, $\Lambda_r - cl(\{x\}) \cap F = \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let F be a Λ_r-closed set and $x \notin F$. Then F^c is Λ_r-open and $x \in F^c$. Since X is $\Lambda_r - R_0$, $\Lambda_r - cl(\{x\}) \subseteq F^c$ and hence $F \subseteq X - (\Lambda_r - cl(\{x\}))$. Thus $X - (\Lambda_r - cl(\{x\}))$ is a Λ_r-open set containing F and $x \notin X - (\Lambda_r - cl(\{x\}))$.

(2) $\Rightarrow$ (3) Let F be a Λ_r-closed set and $x \notin F$. Then $\exists U \in \Lambda_r O(X,\tau)$ such that $x \notin U$ and $F \subseteq U$.

Claim: $U \cap \Lambda_r - cl(\{x\}) = \emptyset$. For, if $U \cap \Lambda_r - cl(\{x\}) \neq \emptyset$, then $\exists$ a point y in X such that $y \in U$ and $y \in \Lambda_r - cl(\{x\})$. That implies y is a Λ_r-cluster point of $\{x\}$. That implies for every Λ_r-open set G containing y, $G \cap \{x\} \neq \emptyset$. That is, $x \in G$. Here U is a Λ_r-open set containing y. Hence $x \in U$, which is a contradiction. Therefore $U \cap \Lambda_r - cl(\{x\}) = \emptyset$ and hence $F \cap \Lambda_r - cl(\{x\}) = \emptyset$.

(3) $\Rightarrow$ (1) Let G be a Λ_r-open set and $x \in G$. Then G^c is Λ_r-closed and $x \notin G^c$. By (3), $\Lambda_r - cl(\{x\}) \cap G^c = \emptyset$ and hence $\Lambda_r - cl(\{x\}) \subseteq G$. Therefore (X,τ) is $\Lambda_r - R_0$.

Theorem 2. *A space (X,τ) is $\Lambda_r - R_0$ iff for each pair x, y of distinct points in X, $\Lambda_r - cl(\{x\}) \cap \Lambda_r - cl(\{y\}) = \emptyset$ or $\{x, y\} \subseteq \Lambda_r - cl(\{x\}) \cap \Lambda_r - cl(\{y\})$.*

Proof. Let (X,τ) be a $\Lambda_r - R_0$ space. Let $x, y \in X$ such that $x \neq y$. Then we have

Case(i): Suppose $\Lambda_r - cl(\{x\}) \cap \Lambda_r - cl(\{y\}) \neq \emptyset$. If $\{x, y\}$ is not a subset of $\Lambda_r - cl(\{x\}) \cap \Lambda_r - cl(\{y\})$ and $x \notin \Lambda_r - cl(\{y\})$, then $x \in X - (\Lambda_r - cl(\{y\}))$ and

$X-(\Lambda_r-cl(\{y\}))$ is Λ_r-open. Since (X,τ) is Λ_r-R_0, $\Lambda_r-cl(\{x\})\subseteq X-(\Lambda_r-cl(\{y\}))$. Therefore $\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})=\emptyset$. This is a contradiction. Hence $\{x,y\}\subseteq\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})$.

Case(ii): Suppose $\{x,y\}$ is not a subset of $\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})$ and let $x\notin\Lambda_r-cl(\{y\})$. Then $x\in X-(\Lambda_r-cl(\{y\}))$ and $X-(\Lambda_r-cl(\{y\}))$ is Λ_r-open. Since (X,τ) is Λ_r-R_0, $\Lambda_r-cl(\{x\})\subseteq X-(\Lambda_r-cl(\{y\}))$ and hence $\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})=\emptyset$.

Conversely, let U be a Λ_r-open set and $x\in U$. Suppose $\Lambda_r-cl(\{x\})$ is not a subset of U. Then $\exists$ a point $y\in\Lambda_r-cl(\{x\})$ such that $y\notin U$. That implies $y\in X-U$ and $X-U$ is Λ_r-closed. Since $\Lambda_r-cl(\{y\})$ is the smallest Λ_r-closed set containing y, $\Lambda_r-cl(\{y\})\subseteq X-U$ and hence $\Lambda_r-cl(\{y\})\cap U=\emptyset$. Since $x\in U$, $x\notin\Lambda_r-cl(\{y\})$ and hence $\{x,y\}$ is not a subset of $\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})$. Also $y\in\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})$. That implies $\Lambda_r-cl(\{x\})\cap\Lambda_r-cl(\{y\})\neq\emptyset$. So by this contradiction, $\Lambda_r-cl(\{x\})\subseteq U$ and hence (X,τ) is Λ_r-R_0.

Theorem 3. *For any points x and y in a topological space (X,τ), the following are equivalent:*

(1) $\Lambda_r-ker(\{x\})\neq\Lambda_r-ker(\{y\})$,

(2) $\Lambda_r-cl(\{x\})\neq\Lambda_r-cl(\{y\})$.

Proof. (1) $\Rightarrow$ (2) Suppose $\Lambda_r-ker(\{x\})\neq\Lambda_r-ker(\{y\})$. Then $\exists$ a point z in X such that $z\in\Lambda_r-ker(\{x\})$ and $z\notin\Lambda_r-ker(\{y\})\Rightarrow\Lambda_r-cl(\{z\})\cap\{x\}\neq\emptyset$ and $\Lambda_r-cl(\{z\})\cap\{y\}=\emptyset\Rightarrow x\in\Lambda_r-cl(\{z\})$ and $y\notin\Lambda_r-cl(\{z\})\Rightarrow\Lambda_r-cl(\{x\})\subseteq\Lambda_r-cl(\{z\})$ and $y\notin\Lambda_r-cl(\{z\})\Rightarrow y\notin\Lambda_r-cl(\{x\})\Rightarrow\Lambda_r-cl(\{x\})\neq\Lambda_r-cl(\{y\})$.

(2) $\Rightarrow$ (1) Suppose $\Lambda_r-cl(\{x\})\neq\Lambda_r-cl(\{y\})$. Then $\exists$ a point z in X such that $z\in\Lambda_r-cl(\{x\})$ and $z\notin\Lambda_r-cl(\{y\})$. That implies $\exists$ a Λ_r-open set V containing z such that $xv\in V$ and $y\notin V$. That is, V is a Λ_r-open set containing x but not y. If $y\in\Lambda_r-ker(\{x\})$, then $x\in\Lambda_r-cl(\{y\})$. That implies for every Λ_r-open set G containing x, $G\cap\{y\}\neq\emptyset$. That is, $y\in G$. By this contradiction, $y\notin\Lambda_r-ker(\{x\})$ and hence $\Lambda_r-ker(\{x\})\neq\Lambda_r-ker(\{y\})$.

Theorem 4. *For a topological space (X,τ), the following are equivalent:*

(1) (X,τ) *is* Λ_r-R_0,

(2) For any non-empty set A and $G\in\Lambda_rO(X,\tau)$ such that $A\cap G\neq\emptyset$, $\exists F\in\Lambda_rC(X,\tau)$ such that $A\cap F\neq\emptyset$ and $F\subseteq G$,

(3) For any $G\in\Lambda_rO(X,\tau)$, $G=\cup\{F/F\in\Lambda_rC(X,\tau)$ and $F\subseteq G\}$,

(4) For any $F\in\Lambda_rC(X,\tau)$, $F=\cap\{G/G\in\Lambda_rO(X,\tau)$ and $F\subseteq G\}$,

(5) For any $x\in X$, $\Lambda_r-cl(\{x\})\subseteq\Lambda_r-ker(\{x\})$,

(6) For any $x, y \in X$, $y \in \Lambda_r - cl(\{x\}) \Longleftrightarrow x \in \Lambda_r - cl(\{y\})$.

Proof. (1) $\Rightarrow$ (2) Let A be any nonempty subset of X and G be a Λ_r-open set such that $A \cap G \neq \emptyset$. Let $x \in A \cap G$. Since (X, τ) is $\Lambda_r - R_0$, $x \in G \Rightarrow \Lambda_r - cl(\{x\}) \subseteq G$. Since $x \in A$, $\Lambda_r - cl(\{x\}) \cap A \neq \emptyset$. Thus $\Lambda_r - cl(\{x\})$ is a Λ_r-closed set contained in G and $A \cap \Lambda_r - cl(\{x\}) \neq \emptyset$.

(2) $\Rightarrow$ (3) Let $G \in \Lambda_r O(X, \tau)$ and $x \in G$. Then by (2), $\exists F \in \Lambda_r C(X, \tau)$ such that $\{x\} \cap F \neq \emptyset$ and $F \subseteq G$. That implies $x \in F$ where $F \in \Lambda_r C(X, \tau)$ and $F \subseteq G$ and hence
$x \in \cup\{F/F \in \Lambda_r C(X, \tau)$ and $F \subseteq G\}$. Therefore $G \subseteq \cup\{F/F \in \Lambda_r C(X, \tau)$ and $F \subseteq G\}$. Also $\cup\{F/F \in \Lambda_r C(X, \tau)$ and $F \subseteq G\} \subseteq G$. Hence $G = \cup\{F/F \in \Lambda_r C(X, \tau)$ and $F \subseteq G\}$.

(3) $\Rightarrow$ (4) Let $F \in \Lambda_r C(X, \tau)$. Then $F^c \in \Lambda_r O(X, \tau)$. By (3), $F^c = \cup\{G^c/G^c \in \Lambda_r C(X, \tau)$ and $G^c \subseteq F^c\}$. That implies $F = \cap\{G/G \in \Lambda_r O(X, \tau)$ and $F \subseteq G\}$.

(4) $\Rightarrow$ (5) Let $y \notin \Lambda_r - ker(\{x\})$. Then $x \notin \Lambda_r - cl(\{y\})$. That implies $\exists$ a Λ_r-open set V containing x such that $V \cap \{y\} = \emptyset$ and hence $\Lambda_r - cl(\{y\}) \cap V = \emptyset$. By (4),
$\Lambda_r - cl(\{y\}) = \cap\{G/G \in \Lambda_r O(X, \tau)$ and $\Lambda_r - cl(\{y\}) \subseteq G\}$. Since $x \in V$, $x \notin \Lambda_r - cl(\{y\})$ and hence $\exists G \in \Lambda_r O(X, \tau)$ such that $\Lambda_r - cl(\{y\}) \subseteq G$ and $x \notin G$. Therefore $\Lambda_r - cl(\{x\}) \cap G = \emptyset$ and hence $y \notin \Lambda_r - cl(\{x\})$. Therefore $\Lambda_r - cl(\{x\}) \subseteq \Lambda_r - ker(\{x\})$.

(5) $\Rightarrow$ (6) If $y \in \Lambda_r - cl(\{x\})$, then $y \in \Lambda_r - ker(\{x\})$ by (5). That implies $x \in \Lambda_r - cl(\{y\})$. Similarly, if $x \in \Lambda_r - cl(\{y\})$, then by (5), $x \in \Lambda_r - ker(\{y\})$ and hence $y \in \Lambda_r - cl(\{x\})$. Thus $x \in \Lambda_r - cl(\{y\}) \Longleftrightarrow y \in \Lambda_r - cl(\{x\})$.

(6) $\Rightarrow$ (1) Let $G \in \Lambda_r O(X, \tau)$ and $x \in G$. If $y \notin G$, then $y \in X - G$ and hence
$\Lambda_r - cl(\{y\}) \subseteq X - G$ since $\Lambda_r - cl(\{y\})$ is the smallest Λ_r-closed set containing y. Therefore $\Lambda_r - cl(\{y\}) \cap G = \emptyset$ and hence $x \notin \Lambda_r - cl(\{y\})$. By (6), $y \notin \Lambda_r - cl(\{x\})$. Therefore $\Lambda_r - cl(\{x\}) \subseteq G$ and hence (X, τ) is $\Lambda_r - R_0$.

Corollary 1. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is $\Lambda_r - R_0$,

(2) For any $x \in X$, $\Lambda_r - cl(\{x\}) = \Lambda_r - ker(\{x\})$.

Theorem 5. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is $\Lambda_r - R_0$

(2) If F is Λ_r-closed, then $F = \Lambda_r - ker(F)$

(3) If F is Λ_r-closed and $x \in F$, then $\Lambda_r - ker(\{x\}) \subseteq F$

(4) If $x \in X$, then $\Lambda_r - ker(\{x\}) \subseteq \Lambda_r - cl(\{x\})$.

Proof. (1) $\Rightarrow$ (2) Let F be a Λ_r-closed set and $x \notin F$. Then $X - F$ is Λ_r-open and $x \in X - F$. By (1), $\Lambda_r - cl(\{x\}) \subseteq X - F$ and hence $\Lambda_r - cl(\{x\}) \cap F = \emptyset$. That implies $x \notin \Lambda_r - ker(F)$. Therefore $\Lambda_r - ker(F) \subseteq F$. Also $F \subset \Lambda_r - ker(F)$. Hence $F = \Lambda_r - ker(F)$.

(2) $\Rightarrow$ (3) Let F be a Λ_r-closed set and $x \in F$. Then$\Lambda_r - ker(\{x\}) \subseteq \Lambda_r - ker(F) = F$.

(3) $\Rightarrow$ (4) Since $x \in \Lambda_r - cl(\{x\})$ and $\Lambda_r - cl(\{x\})$ is Λ_r-closed, by (3),
$\Lambda_r - ker(\{x\}) \subseteq \Lambda_r - cl(\{x\})$.

(4) $\Rightarrow$ (1) Since $x \in \Lambda_r - cl(\{y\}) \Leftrightarrow y \in \Lambda_r - cl(\{x\})$, (X, τ) is $\Lambda_r - R_0$.

The following Examples 1 and 2 show that $\Lambda_r - T_0$ and $\Lambda_r - R_0$ are independent.

Example 1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then $\Lambda_r O(X, \tau) = \{X, \emptyset, \{a\}, \{b, c\}\}$. Here (X, τ) is $\Lambda_r - R_0$ but it is not $\Lambda_r - T_0$.*

Example 2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Then $\Lambda_r O(X, \tau) = \{X, \emptyset, \{a\}, \{a, b\}\}$. Here (X, τ) is $\Lambda_r - T_0$ but it is not $\Lambda_r - R_0$.*

3. $\Lambda_r - R_1$ Spaces

Definition 2. *A space (X, τ) is $\Lambda_r - R_1$ if for each $x, y \in X$ with $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$, $\exists \Lambda_r$-open sets U and V such that $\Lambda_r - cl(\{x\}) \subseteq U$, $\Lambda_r - cl(\{y\}) \subseteq V$ and $U \cap V = \emptyset$.*

Proposition 1. *If (X, τ) is $\Lambda_r - R_1$, then (X, τ) is $\Lambda_r - R_0$.*

Proof. Let (X, τ) be $\Lambda_r - R_1$. Let U be Λ_r-open in X and $x \in U$. For each $y \in X - U$, $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$. Then $\exists$ disjoint Λ_r-open sets U_y and V_y such that $\Lambda_r - cl(\{x\}) \subseteq U_y$ and $\Lambda_r - cl(\{y\}) \subseteq V_y$. Take $V = \cup\{V_y / y \in X - U\}$. Then V is Λ_r-open, $X - U \subseteq V$ and $x \notin V$. Therefore $\Lambda_r - cl(\{x\}) \subseteq X - V \subseteq U$ and hence (X, τ) is $\Lambda_r - R_0$.

Theorem 6. *If (X, τ) is $\Lambda_r - T_2$, then (X, τ) is $\Lambda_r - R_1$.*

Proof. Let $x, y \in X$ such that $x \neq y$ and $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$. Since (X, τ) is $\Lambda_r - T_2$, $\exists \Lambda_r$-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. That is, $\{x\} \subseteq U$ and $\{y\} \subseteq V$. Since (X, τ) is $\Lambda_r - T_2$, it is $\Lambda_r - T_1$. Therefore for every $x \in X, \{x\} = \Lambda_r - cl(\{x\})$. Thus $\Lambda_r - cl(\{x\}) \subseteq U$, $\Lambda_r - cl(\{y\}) \subseteq V$ and $U \cap V = \emptyset$. Hence (X, τ) is $\Lambda_r - R_1$.

Remark 1. *The converse of the above theorem need not be true. For example, let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then*

$$\Lambda_r O(X, \tau) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\{b, c, d\}, \{a, d\}\}.$$

Here (X, τ) is $\Lambda_r - R_1$ but not $\Lambda_r - T_2$.

Note that $\Lambda_r - T_0$ and $\Lambda_r - R_1$ are independent as in the following examples 3 and 4.

Example 3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. *Then* $\Lambda_r O(X, \tau) = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. *Here* (X, τ) *is* $\Lambda_r - T_0$ *but it is not* $\Lambda_r - R_1$.

Example 4. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. *Then* $\Lambda_r O(X, \tau) = \{X, \emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. *Here* (X, τ) *is* $\Lambda_r - R_1$ *but it is not* $\Lambda_r - T_0$.

Theorem 7. *For a space* (X, τ)*, the following statements are equivalent:*

(1) (X, τ) *is* $\Lambda_r - R_1$,

(2) If $x, y \in X$ *such that* $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$*, then* $\exists \Lambda_r$*-closed sets* F_1 *and* F_2 *such that* $x \in F_1$, $y \notin F_1$, $x \notin F_2$, $y \in F_2$ *and* $X = F_1 \cup F_2$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in X$ such that $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$. Then by (1), $\exists$ disjoint Λ_r-open sets U and V such that $\Lambda_r - cl(\{x\}) \subseteq U$ and $\Lambda_r - cl(\{y\}) \subseteq V$. Take $F_1 = X - V$ and $F_2 = X - U$. Then F_1 and F_2 are Λ_r-closed sets such that $x \in F_1$, $y \notin F_1$, $x \notin F_2$, $y \in F_2$ and $X = F_1 \cup F_2$.

(2) $\rightarrow$ (1) Let $x, y \in X$ such that $\Lambda_r - cl(\{x\}) \neq \Lambda_r - cl(\{y\})$. Then by (2) $\exists \Lambda_r$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $x \notin F_2$, $y \in F_2$ and $X = F_1 \cup F_2$. Take $U = X - F_2$ and $V = X - F_1$. Then U and V are Λ_r-open sets, $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore (X, τ) is $\Lambda_r - T_2$ and hence (X, τ) is $\Lambda_r - R_1$.

4. $\Lambda_r - D_k$ Spaces

Definition 3. *Let* (X, τ) *be a topological space and* A *be a subset of* X. *Then* A *is called* Λ_r*-difference set (shortly* Λ_r*-D set) if* $\exists U, V \in \Lambda_r O(X, \tau)$ *such that* $U \neq X$ *and* $A = U - V$. *The collection of all* Λ_r*-difference sets of* (X, τ) *is denoted by* $\Lambda_r D(X, \tau)$.

Remark 2. *Every* Λ_r*-open set* A *different from* X *is a* Λ_r*-D set if* $U = A$ *and* $V = \emptyset$. *But the converse need not be true. For example, let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \emptyset, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. *Then* $\Lambda_r O(X, \tau) = \{X, \emptyset, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$ *and* $\Lambda_r D(X, \tau) = \{\emptyset, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{c\}, \{a\}\}$. *Here* $\{a\}$ *is a* Λ_r*-D set but not* Λ_r*-open set.*

Definition 4. *A space* (X, τ) *is called*

(1) $\Lambda_r - D_0$ *if for* $x, y \in X$, $x \neq y$, $\exists$ *a* Λ_r*-D set containing one of* x *and* y *but not the other*

(2) $\Lambda_r - D_1$ *if for* $x, y \in X$, $x \neq y$, $\exists \Lambda_r$*-D sets* U *and* V *in* X *such that* $x \in U$, $y \notin U$ *and* $y \in V$, $x \notin V$

(3) $\Lambda_r - D_2$ *if for* $x, y \in X$, $x \neq y$, $\exists \Lambda_r - D$ *sets* U *and* V *in* X *such that* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$.

Theorem 8. *A space* (X, τ) *is* $\Lambda_r - D_0$ *iff it is* $\Lambda_r - T_0$.

Proof. Suppose (X,τ) is $\Lambda_r - D_0$. Let $x, y \in X$ such that $x \neq y$. Then $\exists$ an Λ_r-D set A containing one of x and y but not the other, say $x \in A$ but $y \notin A$. Since A is a Λ_r-D set, $A = U - V$ where $U \neq X$ and $U, V \in \Lambda_r O(X,\tau)$. Since $x \in A$, $x \in U$ and $x \notin V$. For $y \notin A$, we have two cases (a) $y \notin U$

(b) $y \in U$ and $y \in V$. In case (a), $x \in U$ but $y \notin U$. In case (b), $y \in V$ but $x \notin V$. Hence (X,τ) is $\Lambda_r - T_0$. Conversely, suppose (X,τ) is $\Lambda_r - T_0$. Let $x, y \in X$ such that $x \neq y$. Then $\exists$ an Λ_r-open set U containing one of x and y but not the other, say $x \in U$ but $y \notin U$. Then $U \neq X$ and hence U is a Λ_r-D set. Therefore U is a Λ_r-D set containing x but not y. Hence (X,τ) is $\Lambda_r - D_0$.

The following examples 5 and 6 show that $\Lambda_r - R_0$ and $\Lambda_r - D_0$ are independent.

Example 5. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{a, c\}\}$. Then $\Lambda_r O(X,\tau) = \{X, \emptyset, \{b\}, \{a, c\}\}$. Here (X,τ) is $\Lambda_r - R_0$ but it is not $\Lambda_r - D_0$.*

Example 6. *Let $X = \{a, b\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then $\Lambda_r O(X,\tau) = \{X, \emptyset, \{a\}\}$. Here (X,τ) is $\Lambda_r - D_0$ but it is not $\Lambda_r - R_0$.*

Remark 3. *Examples 7 and 8 below show that $\Lambda_r - R_1$ and $\Lambda_r - D_0$ are independent.*

Example 7. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c, d\}\}$. Then $\Lambda_r O(X,\tau) = \{X, \emptyset, \{a\}, \{b, c, d\}\}$. Here (X,τ) is $\Lambda_r - R_1$ but it is not $\Lambda_r - D_0$.*

Example 8. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{a, b\}\}$. Then $\Lambda_r O(X,\tau) = \{X, \emptyset, \{b\}, \{a, b\}\}$. Here (X,τ) is $\Lambda_r - D_0$ but it is not $\Lambda_r - R_1$.*

Theorem 9. *A space (X,τ) is $\Lambda_r - D_1$ iff it is $\Lambda_r - D_2$.*

Proof. Suppose (X,τ) is $\Lambda_r - D_1$. Then for each pair of distinct points $x, y \in X$, we have Λ_r-D sets A and B such that $x \in A$, $y \notin A$ and $y \in B$, $x \notin B$. Let $A = U_1 - V_1$ and $B = U_2 - V_2$. Then U_1, V_1, U_2 and V_2 are Λ_r-open sets, $U_1 \neq X$ and $U_2 \neq X$. For $x \notin B$, we have two cases (i) $x \notin U_2$ (ii) $x \in U_2$ and $x \in V_2$.

Case(i) $x \notin U_2$ then since $y \notin A$, either $y \notin U_1$ or ($y \in U_1$ and $y \in V_1$).

If $y \notin U_1$, from $y \in B = U_2 - V_2$, it follows that $y \in U_2 - (V_2 \cup U_1)$. From $x \in A = U_1 - V_1$ and $x \notin U_2$, $x \in U_1 - (V_1 \cup U_2)$. Also $(U_2 - (V_2 \cup U_1)) \cap (U_1 - (V_1 \cup U_2)) = \emptyset$. If $y \in U_1$ and $y \in V_1$, then $x \in U_1 - V_1$. That implies $(U_1 - V_1) \cap V_1 = \emptyset$.

Case(ii) $x \in U_2$ and $x \in V_2$ then we have $y \in B = U_2 - V_2$, $x \in V_2$ and $(U_2 - V_2) \cap V_2 = \emptyset$. Hence (X,τ) is $\Lambda_r - D_2$.

Conversely, suppose (X,τ) is $\Lambda_r - D_2$. Let $x, y \in X$ such that $x \neq y$. Then $\exists \Lambda_r - D$ sets A and B such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$. Therefore $x \in A$, $y \notin A$ and $y \in B$, $x \notin B$. Hence (X,τ) is $\Lambda_r - D_1$.

Corollary 2. *If* (X,τ) *is* $\Lambda_r - D_1$*, then it is* $\Lambda_r - T_0$.

Remark 4. *The converse of the above corollary need not be true. For example, let* $X = \{a,b,c\}$ *and* $\tau = \{X,\emptyset,\{c\},\{a,c\}\}$*. Then* $\Lambda_r O(X,\tau) = \{X,\emptyset,\{c\},\{a,c\}\}$ *and* $\Lambda_r D(X,\tau) = \{\emptyset,\{a\},\{c\},\{a,c\}\}$*. Here* (X,τ) *is* $\Lambda_r - T_0$ *but not* $\Lambda_r - D_1$.

Remark 5. *Examples 9 and 10 below show that* $\Lambda_r - R_0$ *and* $\Lambda_r - D_2$ *are independent.*

Example 9. *Let* $X = \{a,b,c,d\}$ *and* $\tau = \{X,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$*. Then* $\Lambda_r O(X,\tau) = \{X,\emptyset,\{a\},\{a,d\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$*. Here* (X,τ) *is* $\Lambda_r - R_0$ *but it is not* $\Lambda_r - D_2$.

Example 10. *Let* $X = \{a,b,c\}$ *and* $\tau = \{X,\emptyset,\{b\},\{c\},\{b,c\},\{a,c\}\}$*. Then* $\Lambda_r O(X,\tau) = \{X,\emptyset,\{b\},\{c\},\{b,c\},\{a,c\}\}$*. Here* (X,τ) *is* $\Lambda_r - D_2$ *but it is not* $\Lambda_r - R_1$.

Similar to the previous cases the examples 11 and 12 show that $\Lambda_r - R_1$ and $\Lambda_r - D_1$ are independent.

Example 11. *Let* $X = \{a,b,c,d\}$ *and* $\tau = \{X,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$*. Then* $\Lambda_r O(X,\tau) = \{X,\emptyset,\{a\},\{a,d\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$*. Here* (X,τ) *is* $\Lambda_r - R_1$ *but it is not* $\Lambda_r - D_1$.

Example 12. *Let* $X = \{a,b,c\}$ *and* $\tau = \{X,\emptyset,\{a\},\{c\},\{a,c\},\{b,c\}\}$*. Then* $\Lambda_r O(X,\tau) = \{X,\emptyset,\{a\},\{c\},\{a,c\},\{b,c\}\}$*. Here* (X,τ) *is* $\Lambda_r - D_1$ *but it is not* $\Lambda_r - R_1$.

References

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