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Macwilliams Identity for M-Spotty Hamming Weight Enumerator Over the Ring $\mathbb{F}_2 + v\mathbb{F}_2$

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Abstract. The m-spotty byte error control codes can correct or detect multiple spotty byte errors that are distributed in multiple bytes. These codes are successfully applied to computer memory systems that use RAM chips with b-bit Input/Output data when high-energy particles strike a particular RAM chips. In this paper, we derive a MacWilliams type identity for m-spotty Hamming weight enumerator over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$.

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1. Introduction

The theory of error correcting codes has been studied for over half a century. Also, error control codes have been extensively applied to various digital systems, such as computer and communication systems, as an essential technique to improve system reliability. In the applications of error correcting codes to computer systems, a new class of byte error control codes called m-spotty byte error control codes are very effective for correcting/detecting errors in semiconductor memory systems that employ high-density RAM chips with wide Input/Output data. These RAM chips are strongly vulnerable to $\alpha-$ particles, neutrons, and so forth. Because of these facts, in order to be able to correct multiple errors a special type of byte errors called spotty byte errors has been introduced in [6, 7].

Suzuki *et al.* [5] introduced the MacWilliams identity for the m-spotty weight enumerator of m-spotty byte error control codes and clarified that the indicated identity also includes the MacWilliams identity for the Hamming weight enumerator over the binary field $\mathbb{F}_2$ or the extension fields of the binary field $\mathbb{F}_2$. Özen and Şiap [4] generalized this result to arbitrary

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finite fields. Morever, Şiap [1] extended the definition of m-spotty weight originally introduced in [6] from binary codes to codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$. In this paper,we establish a MacWilliams type identity for the m-spotty Hamming weight enumerator over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$.

The organization of this paper is as follows: In Section 2, definition of m-spotty Hamming weight and m-spotty Hamming distance are presented. In Section 3, the MacWilliams identity for m-spotty Hamming weight enumerator over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$ is established. In Section 4, we give an example of how the weight distribution of the m-spotty byte error control code can be applied. Finally, the paper concludes in Section 5.

2. Preliminaries

Let R_2 be the commutative ring $\mathbb{F}_2 + v\mathbb{F}_2 = \{0, 1, v, 1 + v\}$, where $v^2 = v$. Any element of R_2 can be expressed as $c = a + bv$, where $a, b \in \mathbb{F}_2$. Let N positive integer. For a positive divisor b of N, C is said to be a byte error control code of length N and byte length b over R_2 if and only if C is an R_2-submodule of R_2^N. The elements of C are called codewords.

Let $v = (v_{11}, v_{12}, \ldots, v_{1b}, \ldots, v_{n1}, v_{n2}, \ldots, v_{nb}) \in R_2^{bn}$ be a vector of length $N = bn$. The first byte of v consists of the first b entries denoted by $(v_{11}, v_{12}, \ldots, v_{1b})$. Hence, the i^{th} byte of v will be denoted by $v_i = (v_{i1}, v_{i2}, \ldots, v_{ib})$. Then for any two vectors $u = (u_1, u_2, \ldots, u_N)$ and $v = (v_1, v_2, \ldots, v_N) \in R_2^N$, the inner product of u and v, denoted by $\langle u, v\rangle$, is defined as follows:

$$\langle u, v\rangle = \sum_{i=1}^{n} \langle u_i, v_i\rangle = \sum_{i=1}^{n} \left(\sum_{j=1}^{b} u_{ij} v_{ij} \right).$$

Here, $\langle u_i, v_i\rangle = \sum\limits_{j=1}^{b} u_{ij} v_{ij}$ denotes the inner product of the bytes u_i and v_i. Also, u_{ij} and v_{ij} are the j^{th} bits of u_i and v_i, respectively.

For a byte error control code of length bn over R having byte length b, the set

$$C^{\perp} = \left\{ v \in R_2^{bn} : \langle c, v\rangle = 0 \text{ for all } c \in C \right\}$$

is called the dual code of C.

First, we define the $t/b-$error as follows:

Definition 1. *[7] An error is said to be a $t/b-$error or a spotty byte error if t or fewer bits within a $b-$bit byte are in error, where $1 \leqslant t \leqslant b$.*

If there exist more than $t-$bit errors in a byte, the errors are called multiple $t/b-$errors in a byte.

Definition 2. *[6] An error is said to be an m-spotty byte error if at least one $t/b-$error is present in a byte.*

We now present the following definitions in order to define the m-spotty Hamming weight enumerator of byte error control code.

Definition 3. *[6] Let $e \in R_2^N$ be an error vector and $e_i \in R_2^b$ be the i^{th} byte of e, where $1 \leqslant i \leqslant n$. The number of $t/b-$errors in e, denoted by $w_M(e)$, is called the m-spotty Hamming weight and is defined as*

$$w_M(e) = \sum_{i=1}^{n} \left\lceil \frac{w(e_i)}{t} \right\rceil,$$

where $w(e_i)$ denotes the Hamming weight of e_i over R_2, which is equal to the number of non-zero components of e_i. Here, $\lceil x \rceil$ denotes the ceiling value of x, i.e., the number $\lceil x \rceil$ is equal to the smallest integer not less than x.

If $t = b$, the m-spotty Hamming weight equals to the Hamming weight in R_2^b. In addition, if $t = 1$, then $w_M(e) = w(e)$.

Definition 4. *[6] Let $c = (c_1, c_2, \ldots, c_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be codewords of an m-spotty byte error control code C. Here, c_i and v_i are the i^{th} bytes of c and v, respectively. Then, the m-spotty Hamming distance between c and v, denoted by $d_M(c, v)$, is defined as follows:*

$$d_M(c, v) = \sum_{i=1}^{n} \left\lceil \frac{d(c_i, v_i)}{t} \right\rceil = \sum_{i=1}^{n} \left\lceil \frac{w(c_i - v_i)}{t} \right\rceil = w_M(c - v),$$

where $d(c_i, v_i) = w(c_i - v_i)$ denotes the Hamming distance between i^{th} bytes c_i and v_i for each i over R_2.

The following theorem is originally proved for the binary field $\mathbb{F}_2$ given in the paper [6].

Theorem 1. *The m-spotty Hamming distance is a metric over R_2.*

Definition 5. *[5] Let z be an indeterminate element. The m-spotty Hamming weight enumerator of a byte error control code C is defined as follows:*

$$W_C(z) = \sum_{c \in C} z^{w_M(c)}.$$

For a codeword c, let $\alpha_j(c)$ be the number of bytes having the Hamming weight j, $0 \leqslant j \leqslant b$. The Hamming weight distribution vector $(\alpha_0(c), \alpha_1(c), \ldots, \alpha_b(c))$ is uniquely determined for the codeword c. Then, the m-spotty Hamming weight of the codeword c is defined as $w_M(c) = \sum_{j=0}^{b} \lceil j/t \rceil . \alpha_j(c)$. Let $A_{(\alpha_0, \alpha_1, \ldots, \alpha_b)}$ be the number of codewords with the Hamming weight distribution vector $(\alpha_0, \alpha_1, \ldots, \alpha_b)$. For example, let $(1v1\ 000\ 0v1\ 00v\ 010) \in R_2^{15}$ be a codeword with byte length $b = 3$. Then, the Hamming weight distribution vector of the codeword is $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, 2, 1, 1)$. Therefore, $A_{(1,2,1,1)}$ is the number of codewords with the Hamming weight distribution vector $(1, 2, 1, 1)$.

By using the parameter $A_{(\alpha_0, \alpha_1, \ldots, \alpha_b)}$, the m-spotty Hamming weight enumerator can be rewritten as follows:

$$W_C(z) = \sum_{\alpha_0 + \alpha_1 + \ldots + \alpha_b = n} A_{(\alpha_0, \alpha_1, \ldots, \alpha_b)} \prod_{j=0}^{b} \left(z^{\lceil j/t \rceil} \right)^{\alpha_j}, \tag{1}$$

where $\sum\limits_{\alpha_0 + \alpha_1 + \ldots + \alpha_b = n}$ denotes the summation over all $(\alpha_0, \alpha_1, \ldots, \alpha_b)$'s satisfying the conditions $\alpha_0, \alpha_1, \ldots, \alpha_b \geqslant 0$ and $\alpha_0 + \alpha_1 + \ldots + \alpha_b = n$.

3. The MacWilliams Identity

One of the most celebrated results in coding theory is the MacWilliams identity [2] that describes how the weight enumerator of a linear code and the weight enumerator of the dual code relate to each other. This identity has found widespread application in coding theory [3]. In this section, we obtain the MacWilliams identity for the m-spotty Hamming weight enumerator over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$.

The ring R_2 is a principal ring. As for the ideal structure we can easily find all the ideals of R_2 which are

$$\langle 0\rangle = \{0\}, \quad \langle v\rangle = \{0, v\}, \quad \langle 1+v\rangle = \{0, 1+v\}, \quad R_2.$$

As we see from the ideals, the ring $\mathbb{F}_2 + v\mathbb{F}_2$ is not a finite chain ring. Also, these ideals are the additive subgroups of R_2. In this paper, we define the character χ for this ring as follows:

Definition 6. *For $a + vb \in R_2$, we refer to the character χ defined by*

$$\chi(a + vb) = (-1)^b \tag{2}$$

Note that χ is a nontrivial character. It can be easily seen that the character χ is a group homomorphism. We get the following result by using the definition of the character χ in (2).

Lemma 1. *[3] Let $I \neq \{0\}$ be an ideal of R_2. Then,*

$$\sum_{a\in I} \chi(a) = 0.$$

The following lemma plays an important role in proving Theorem 2:

Lemma 2. *[3] Let f be a function defined on R_2^{nb}. We define*

$$\widetilde{f}(c) = \sum_{v\in R_2^{nb}} \chi(\langle c, v\rangle) f(v), \quad c \in R_2^{nb}.$$

Then, the following relation holds between $f(v)$ and $\widetilde{f}(c)$:

$$\sum_{v\in C^{\perp}} f(v) = \frac{1}{|C|}\sum_{c\in C} \widetilde{f}(c),$$

where $|C|$ denotes the size of the set C.

The following theorem holds for the m-spotty weight enumerator $W_C(z)$ of the byte error control code C and that of the dual code $C^{\perp}$, denoted by $W_{C^{\perp}}(z)$.

Theorem 2. *Let the code length in bits N be a multiple of byte length b, i.e., $N = nb$. Then, the following relation holds:*

$$W_{C^{\perp}}(z) = \frac{1}{|C|} \sum_{\alpha_0+\alpha_1+\ldots+\alpha_b=n} A_{(\alpha_0,\alpha_1,\ldots,\alpha_b)} \prod_{j=0}^{b} \left(\left(V_j^{(t)}(z)\right)\right)^{\alpha_j},$$

where

$$V_j^{(t)}(z)=\sum_{p=0}^{b}\left(\sum_{s=0}^{p}(-1)^{p-s}3^s\binom{j}{p-s}\binom{b-j}{s}\right)z^{\lceil p/t\rceil}.$$

Proof. In Lemma 2, we set $f(v)=\prod_{i=1}^{n} z^{\lceil w(v_i)/t\rceil}$, where v_i denotes the i^{th} byte of v. Let χ be the character defined in (2). Then,

$$\begin{aligned}\widetilde{f}(c) &= \sum_{v\in R_2^{nb}}\chi\left(\langle c,v\rangle\right)\prod_{i=1}^{n} z^{\lceil w(v_i)/t\rceil}\\ &=\sum_{v\in R_2^{nb}}\chi\left(\langle c_1,v_1\rangle+\langle c_2,v_2\rangle+..\langle c_n,v_n\rangle\right)\prod_{i=1}^{n} z^{\lceil w(v_i)/t\rceil}\\ &=\sum_{v_1\in R_2^b}\sum_{v_2\in R_2^b}\cdots\sum_{v_n\in R_2^b}\left(\prod_{i=1}^{n}\chi\left(\langle c_i,v_i\rangle\right)z^{\lceil w(v_i)/t\rceil}\right)\\ &=\prod_{i=1}^{n}\left(\sum_{v_i\in R_2^b}\chi\left(\langle c_i,v_i\rangle\right)z^{\lceil w(v_i)/t\rceil}\right).\end{aligned}$$

Let c_i be a fixed vector. Then, $\sum_{v_i\in R_2^b}\chi\left(\langle c_i,v_i\rangle\right)z^{\lceil w(v_i)/t\rceil}$ depends on $w(c_i)$. Assume that the Hamming weight of the fixed vector c_i is $w(c_i)=j$. For all vectors v_i having Hamming weight p, we obtain the equality as follows:

$$\sum_{w(v_i)=p}\chi\left(\langle c_i,v_i\rangle\right)z^{\lceil w(v_i)/t\rceil}=\sum_{s=0}^{p}(-1)^{p-s}3^s\binom{j}{p-s}\binom{b-j}{s}z^{\lceil p/t\rceil}.$$

Since $w(v_i)=p$, the number of non-zero components of v_i is p. The inner product of c_i and v_i shows the number of positions in which the components of neither c_i nor v_i are zero. The number of $p-s$ positions chosen from j positions is $\binom{j}{p-s}$. Morever, there are s non-zero elements of v_i left. These elements are in $b-j$ positions. The number of s non-zero positions chosen from $b-j$ positions is $3^s\binom{b-j}{s}$. Hence,

$$\sum_{v_i\in R_2^b}\chi\left(\langle c_i,v_i\rangle\right)z^{\lceil w(v_i)/t\rceil}=\sum_{p=0}^{b}\left(\sum_{s=0}^{p}(-1)^{p-s}3^s\binom{j}{p-s}\binom{b-j}{s}\right)z^{\lceil p/t\rceil}.$$

Hence, the function $\widetilde{f}(c)$ is expressed as

$$\widetilde{f}(c)=\prod_{j=0}^{b}\left(V_j^{(t)}(z)\right)^{\alpha_j}. \tag{3}$$

Substituting Eq. (3) in Lemma 2 we obtain

$$\sum_{v\in C^{\perp}}\prod_{j=0}^{b}\left(z^{\lceil j/t\rceil}\right)^{\alpha_j}=\frac{1}{|C|}\sum_{c\in C}\prod_{j=0}^{b}\left(V_j^{(t)}(z)\right)^{\alpha_j}.$$

Hence the proof is completed.

4. An illustrative Example

Let

$$G=\begin{pmatrix} 1 & 0 & v & 1 & 0 & v \\ 0 & 1 & 1 & 1+v & 0 & 1 \end{pmatrix}$$

be the generator matrix of byte error control code (or a linear code) C over $\mathbb{F}_2+v\mathbb{F}_2$ of length 6. The dual code of C is a byte error control code of length 6 and it has 256 codewords.

Let $b=3$ and $t=2$. The codewords of the byte control code C, the Hamming weight distribution vectors of the codewords of C and the corresponding $V_j^{(2)}$ expressions are shown in Table 1 for the necessary computations to be used in Theorem 2.

Table 1: Codewords and Their Corresponding Terms.

Codeword	$(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$	$V_0^{(2)}V_1^{(2)}V_2^{(2)}V_3^{(2)}$
$(0,0,0,0,0,0)$	$(2,0,0,0)$	$V_0^{(2)}V_0^{(2)}$
$(0,1,1,1+v,0,1)$	$(0,0,2,0)$	$V_2^{(2)}V_2^{(2)}$
$(0,v,v,0,0,v)$	$(0,1,1,0)$	$V_1^{(2)}V_2^{(2)}$
$(0,1+v,1+v,1+v,0,1+v)$	$(0,0,2,0)$	$V_2^{(2)}V_2^{(2)}$
$(1,0,v,1,0,v)$	$(0,0,2,0)$	$V_2^{(2)}V_2^{(2)}$
$(1,1,1+v,v,0,1+v)$	$(0,0,1,1)$	$V_2^{(2)}V_3^{(2)}$
$(1,v,0,1,0,0)$	$(0,1,1,0)$	$V_1^{(2)}V_2^{(2)}$
$(1,1+v,1,v,0,1)$	$(0,0,1,1)$	$V_2^{(2)}V_3^{(2)}$
$(v,0,v,v,0,v)$	$(0,0,2,0)$	$V_2^{(2)}V_2^{(2)}$
$(v,1,1+v,1,0,1+v)$	$(0,0,1,1)$	$V_2^{(2)}V_3^{(2)}$
$(v,v,0,v,0,0)$	$(0,1,1,0)$	$V_1^{(2)}V_2^{(2)}$
$(v,1+v,1,1,0,1)$	$(0,0,1,1)$	$V_2^{(2)}V_3^{(2)}$
$(1+v,0,0,1+v,0,0)$	$(0,2,0,0)$	$V_1^{(2)}V_1^{(2)}$
$(1+v,1,1,0,0,1)$	$(0,1,0,1)$	$V_1^{(2)}V_3^{(2)}$
$(1+v,v,v,1+v,0,v)$	$(0,0,1,1)$	$V_2^{(2)}V_3^{(2)}$
$(1+v,1+v,1+v,0,0,1+v)$	$(0,1,0,1)$	$V_1^{(2)}V_3^{(2)}$

By Eq. (1) and Table 1, we get the m-spotty Hamming weight enumerator of C as follows:

$$W_C(z)=1+8z^2+7z^3.$$

By using both Theorem 2 and Table 1, we obtain

$$W_{C^\perp}(z) = \left(V_0^{(2)}\right)^2 + 4\left(V_2^{(2)}\right)^2 + 3V_1^{(2)}V_2^{(2)} + 5V_2^{(2)}V_3^{(2)} + 2V_1^{(2)}V_3^{(2)} + \left(V_1^{(2)}\right)^2$$
$$= 1 + 4z + 85z^2 + 118z^3 + 48z^4.$$

Here $V_0^{(2)} = 1 + 36z + 27z^2$, $V_1^{(2)} = 1 + 8z - 9z^2$, $V_2^{(2)} = 1 - 4z + 3z^2$ and $V_3^{(2)} = 1 - z^2$.

5. Conclusion

In this paper, we prove a MacWilliams type identity for m-spotty Hamming weight enumerators over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ with $v^2 = v$. We conclude the paper by giving an illustration of Theorem 2. This provides the relation between the m-spotty Hamming enumerator of the code and that of the dual code.

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