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Separation Axioms On Topological Spaces - A Unified Version

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Abstract. In this paper, a new kind of sets called generalized ψ-closed (briefly $g\psi$-closed) sets are introduced and studied in a topological space by using the concept of operation on topological space. The class of all $g\psi$-closed sets is strictly larger than the class of all ψ-closed sets. Some of their properties are investigated here. Finally, some characterizations of ψ_g-regular and ψ_g-normal spaces have been given.

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1. Introduction

It is observed from literature that there has been a considerable work on different relatively weak forms of separation axioms, like regularity and normality axioms in particular; several other neighbouring forms of them have also been studied in many papers. For instance, p-regular [7], p-normal [18], s-regular [10], s-normal [11] δp-normal [6], β-regular [1] and β-normal [12] are some of the variant forms of regularity and normality properties, that have been investigated by different researchers as separate entities. Recently, Noiri and Roy [15] has also introduced the concept of μ_g-regularity and μ_g-normality by using the concept of generalized topology towards such an unified version. As can be observed, all these variations have been effected by using different types of operators like int, $intcl$, $intcl_\delta$, $clint$, $intclint$, $clintcl$, where int and cl respectively stand for interior and closure operators, and cl_δ denotes the δ-closure operator. The concept of a generalized type of operator, called operation on the power set $\mathscr{P}(X)$ of a topological space (X,τ) was introduced by [3]. It turns out from the investigations that by judicious use of the notion of 'operation', one can

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give generalized definitions of regularity and normality axioms from which the definitions of different varied forms of such properties and many known results thereon follow as particular consequences.

2. Main Results

2.1. Properties of $g\psi$-closed Sets

We now begin by recalling a few definitions and observe that many of the existing relevant definitions considered in various papers turn out to be special cases of the ones given below.

Definition 1 ([3]). *Let (X,τ) be a topological space. A mapping $\psi : \mathscr{P}(X) \to \mathscr{P}(X)$ is called an operation on $\mathscr{P}(X)$, where $\mathscr{P}(X)$ denotes as usual the power set of X, if for each $A \in \mathscr{P}(X)\setminus\{\varnothing\}$, $intA \subseteq \psi(A)$ and $\psi(\varnothing)=\varnothing$.*

The set of all operations on a space X will be denoted by $\mathscr{O}(X)$.

Remark 1. *It is easy to check that some examples of operations on a space X are the well known operators viz. int, $intcl$, $intcl_\delta$, $clint$, $intclint$, $clintcl$.*

Definition 2 ([3]). *Let ψ denote an operation on a space (X,τ). Then a subset A of X is called ψ-open if $A \subseteq \psi(A)$. Complements of ψ-open sets will be called ψ-closed sets. The family of all ψ-open (resp. ψ-closed) subsets of X is denoted by $\psi\mathscr{O}(X)$ (resp. $\psi\mathscr{C}(X)$).*

Remark 2. *It is clear that if ψ stands for any of the operators int, $intcl$, $intcl_\delta$, $clint$, $intclint$, $clintcl$, then ψ-openness of a subset A of X coincides with respectively the openness, preopenness, δ-preopenness, semi-openness, α-openness and β-openness of A [see 5, 14, 19, 13, 11, 12].*

Definition 3 ([3]). *Let (X,τ) be a topological space, $\psi \in \mathscr{O}(X)$ and $A \subseteq X$. Then the intersection of all ψ-closed sets containing A is called the ψ-closure of A, denoted by ψ-clA; alternately, ψ-clA is the smallest ψ-closed set containing A. The union of all ψ-open subsets of G is the ψ-interior of G, denoted by ψ-$intG$.*

It is known from [8] that $x \in \psi - clA$ iff $A \cap U \neq \varnothing$, for all U with $x \in U \in \psi\mathscr{O}(X)$ and $x \in \psi - intG$ iff $\exists\ x \in U \in \psi\mathscr{O}(X)$ such that $x \in U \subseteq G$. In [8], it is also shown that $X \setminus \psi - clG = \psi - int(X \setminus G)$.

Remark 3. *Obviously if one takes interior as the operation ψ, then ψ-closure becomes equivalent to the usual closure. Similarly, ψ-closure becomes pcl, pcl_δ, scl, α-cl, β-cl, if ψ is taken to stand for the operators $intcl$, $intcl_\delta$, $clint$, $intclint$ and $clintcl$ respectively [see 14, 19, 13, 11, 12, for details].*

Definition 4. *Let ψ be an operation on a topological space (X,τ). Then $A \subseteq X$ is called a generalized ψ-closed set (or simply $g\psi$-closed set) if $\psi - cl(A) \subseteq U$ whenever $A \subseteq U \in \tau$. The complement of a $g\psi$-closed set is called a generalized ψ-open (or simply $g\psi$-open) set.*

Remark 4. *(i) Let ψ be an operation on a topological space (X,τ). Then every $g\psi$-closed set reduces to a g-closed [9] (resp. gp-closed [17], gs-closed [2], αg-closed [13], $g\delta p$-closed [6], gsp-closed [4]) set if one takes ψ to be int (resp. $intcl$, $clint$, $intclint$, $intcl_\delta$, $clintcl$).*

(ii) For an operation ψ on a topological space (X,τ), every ψ-closed set is a $g\psi$-closed set. In fact, if A is a ψ-closed with $A \subseteq U \in \tau$, then $A = \psi - cl(A) \subseteq U$, so that A is $g\psi$-closed. That the converse is false as shown by the following example.

Example 1. *Let $X = \{a,b,c,d\}$ and $\tau = \{\varnothing, X, \{a\}, \{a,b\}, \{a,b,c\}\}$. Consider the map $\psi : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by*

$$\begin{aligned}\psi(\{a\}) =& \psi(\{a,c\}) = \psi(\{a,d\}) = \psi(\{a,b\}) = \psi(\{a,b,c\}) = \psi(\{a,b,d\}) = \psi(\{a,c,d\})\\ =& \psi(X) = X,\\ \psi(\{b\}) =& \psi(\{c\}) = \psi(\{d\}) = \psi(\{c,d\}) = \psi(\{b,c\}) = \psi(\{b,d\}) = \psi(\{b,c,d\}) = \varnothing \text{ and}\\ \psi(\{\varnothing\}) =& \varnothing.\end{aligned}$$

Then ψ is an operation on the topological space (X,τ). It is easy to check that $\{a,d\}$ is $g\psi$-closed but not ψ-closed.

The next example shows that the union (intersection) of two $g\psi$-closed sets is not in general $g\psi$-closed.

Example 2. *(a) Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, \{a\}, \{a,b\}, X\}$. Then (X,τ) is a topological space. Consider the mapping $\psi : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $\psi(\varnothing) = \psi(\{b\}) = \psi(\{c\}) = \varnothing$, $\psi(\{a\}) = \{a\}$, $\psi(\{b,c\}) = \{b,c\}$, $\psi(\{a,c\}) = \{a,c\}$, $\psi(\{a,b\}) = \{a,b\}$ and $\psi(X) = X$. Then ψ is an operation on the topological space (X,τ). It can be easily verified that $A = \{a\}$ and $B = \{b\}$ are two $g\psi$-closed sets but their union $A \cup B = \{a,b\}$ is not a $g\psi$-open set.*

(b)Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, \{a\}, X\}$. Then (X,τ) is a topological space. Consider the mapping $\psi : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $\psi(\varnothing) = \varnothing$, $\psi(\{a\}) = \{a\}$,
$\psi(\{b\}) = \psi(\{c\}) = \psi(\{b,c\}) = \varnothing$, $\psi(\{a,b\}) = \psi(\{a,c\}) = \{a\}$ and $\psi(X) = X$ on the space X. Then ψ is an operation on the topological space (X,τ). It is easy to verify that $A = \{a,c\}$ and $B = \{a,b\}$ are two $g\psi$-closed sets in (X,τ) but $A \cap B = \{a\}$ is not $g\psi$-closed.

Theorem 1. *Let ψ be an operation on a topological space (X,τ). If A is $g\psi$-closed, then $\psi - cl(A) \setminus A$ does not contain any non-empty closed set.*

Proof. Let F be a closed subset of X such that $F \subseteq \psi - cl(A) \setminus A$, where A is $g\psi$-closed. Then $X \setminus F$ is open, $A \subseteq X \setminus F$ and A is $g\psi$-closed, so $\psi - cl(A) \subseteq X \setminus F$ and thus $F \subseteq X \setminus \psi - cl(A)$. Thus $F \subseteq (X \setminus \psi - cl(A)) \cap \psi - cl(A) = \varnothing$ and hence $F = \varnothing$.

Corollary 1. *Let ψ be an operation on a topological space (X,τ) and $A \subseteq X$ be a $g\psi$-closed set. Then A is ψ-closed iff $\psi - cl(A) \setminus A$ is closed.*

Proof. Let A be a $g\psi$-closed set. If A is ψ-closed, $\psi - cl(A) \setminus A = \emptyset$, and thus $\psi - cl(A) \setminus A$ becomes a closed set.

Conversely, let $\psi - cl(A) \setminus A$ be a closed set, where A is $g\psi$-closed. Then by Theorem 1, $\psi - cl(A) \setminus A$ does not contain any non-empty closed set. Since $\psi - cl(A) \setminus A$ is a closed subset of itself, $\psi - cl(A) \setminus A = \emptyset$ and hence A is ψ-closed.

Theorem 2. *A subset A of a topological space (X, τ) with an operation ψ on it is $g\psi$-closed iff $cl(\{x\}) \cap A \neq \emptyset$ for every $x \in \psi - cl(A)$.*

Proof. Let A be a $g\psi$-closed set in X and suppose if possible that there exists an $x \in \psi - cl(A)$ such that $cl(\{x\}) \cap A = \emptyset$. Therefore $A \subseteq X \setminus cl(\{x\})$, and so $\psi - cl(A) \subseteq X \setminus cl(\{x\})$. Hence $x \notin \psi - cl(A)$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any open set containing A. Let $x \in \psi - cl(A)$. Then by hypothesis $cl(\{x\}) \cap A \neq \emptyset$, so there exists $z \in cl(\{x\}) \cap A$ and so $z \in A \subseteq U$. Thus $\{x\} \cap U \neq \emptyset$. Hence $x \in U$, which implies that $\psi - cl(A) \subseteq U$.

Theorem 3. *Let ψ be an operation on a topological space (X, τ) and $A \subseteq B \subseteq \psi - cl(A)$, where A is $g\psi$-closed. Then B is $g\psi$-closed.*

Proof. Let $B \subseteq U \in \tau$. Since A is $g\psi$-closed and $A \subseteq U$, $\psi - cl(A) \subseteq U$. Now, $B \subseteq \psi - cl(A)$ implies $\psi - cl(B) \subseteq \psi - cl(A)$ and hence $\psi - cl(B) \subseteq U$.

Theorem 4. *Let (X, τ) be a topological space and ψ be an operation on X. Then A is $g\psi$-open iff $F \subseteq \psi - int(A)$ whenever $F \subseteq A$ and F is closed.*

Proof. Let A be a $g\psi$-open set and $F \subseteq A$, where F is closed. Then $X \setminus A$ is a $g\psi$-closed set contained in the open set $X \setminus F$. Hence $\psi - cl(X \setminus A) \subseteq X \setminus F$, i.e., $X \setminus \psi - int(A) \subseteq X \setminus F$. So $F \subseteq \psi - int(A)$.

Conversely, suppose that $F \subseteq \psi - int(A)$ for any closed set F whenever $F \subseteq A$. Let $X \setminus A \subseteq U$, where $U \in \tau$. Then $X \setminus U \subseteq A$ and $X \setminus U$ is closed. By assumption, $X \setminus U \subseteq \psi - int(A)$ and hence $\psi - cl(X \setminus A) = X \setminus \psi - int(A) \subseteq U$. Hence $X \setminus A$ is $g\psi$-closed and hence A is $g\psi$-open.

Theorem 5. *Let ψ be an operation on a topological space (X, τ). Then the following are equivalent:*

(i) Every open set of X is ψ-closed.

(ii) Every subset of X is $g\psi$-closed.

Proof. (i) $\Rightarrow$ (ii) : Let $A \subseteq U \in \tau$. Then by (i), U is ψ-closed so $\psi - cl(A) \subseteq \psi - cl(U) = U$. Thus A is $g\psi$-closed.

(ii) $\Rightarrow$ (i) : Let $U \in \tau$. Then by (ii), U is $g\psi$-closed and hence $\psi - cl(U) \subseteq U$, showing U to be ψ-closed.

Theorem 6. *Let ψ be an operation on a topological space (X,τ). If A is an open and $g\psi$-closed subset of X, then A is ψ-closed.*

Proof. Similar to the proof of Theorem 5((ii) $\Rightarrow$ (i)).

Theorem 7. *Let ψ be an operation on a topological space (X,τ). If a subset A of X is $g\psi$-open, then $U=X$ whenever U is open and $\psi-int(A)\cup(X\setminus A)\subseteq U$.*

Proof. Let $U\in\tau$ and $\psi-int(A)\cup(X\setminus A)\subseteq U$ for a $g\psi$-open set A. Then $X\setminus U\subseteq(X\setminus\psi-int(A))\cap A$, i.e., $X\setminus U\subseteq\psi-cl(X\setminus A)\setminus(X\setminus A)$. Since $X\setminus A$ is $g\psi$-closed, by Theorem 1, $X\setminus U=\varnothing$ and hence $U=X$.

Theorem 8. *For a T_1 topological space (X,τ) with an operation ψ on it, every $g\psi$-closed set is ψ-closed.*

Proof. Let A be a $g\psi$-closed subset of a T_1-topological space (X,τ) and $x\in\psi-cl(A)$. Then by T_1-ness of X, $\{x\}$ is a closed set. Thus by Theorem 1, $x\notin\psi-cl(A)\setminus A$. Since $x\in\psi-cl(A)$, then $x\in A$. This shows that $\psi-cl(A)\subseteq A$ or equivalently that $\psi-cl(A)=A$.

2.2. Properties of ψ_g-regular and ψ_g-normal Spaces

Definition 5. *Let (X,τ) be a topological space and ψ be an operation on X. Then (X,τ) is said to be ψ_g-regular if for each closed set F of X not containing x there exist disjoint ψ-open sets U and V such that $x\in U$, $F\subseteq V$.*

Remark 5. *Let ψ be an operation on a space (X,τ). Then every ψ_g-regular space reduces to a regular [5] (resp. p-regular [7], s-regular [10], β-regular [1]) space if one takes ψ to be int (resp. $intcl$, $clint$, $clintcl$).*

Theorem 9. *Let ψ be an operation on a topological space (X,τ). Then the following statements are equivalent:*

(i) *X is ψ_g-regular.*

(ii) *For each $x\in X$ and each $U\in\tau$ with $x\in U$, there exists $V\in\psi\mathcal{O}(X)$ such that $x\in V\subseteq\psi-cl(V)\subseteq U$.*

(iii) *For each closed set F of X, $\cap\{\psi-cl(V):F\subseteq V\in\psi\mathcal{O}(X)\}=F$.*

(iv) *For each $A\subseteq X$ and each $U\in\tau$ with $A\cap U\neq\varnothing$, there exists $V\in\psi\mathcal{O}(X)$ such that $A\cap V\neq\varnothing$ and $\psi-cl(V)\subseteq U$.*

(v) *For each non-empty subset A of X and each closed subset F of X with $A\cap F=\varnothing$, there exist $V,W\in\psi\mathcal{O}(X)$ such that $A\cap V\neq\varnothing$, $F\subseteq W$ and $W\cap V=\varnothing$.*

(vi) *For each closed set F and $x\notin F$, there exist $U\in\psi\mathcal{O}(X)$ and a $g\psi$-open set V such that $x\in U$, $F\subseteq V$ and $U\cap V=\varnothing$.*

(vii) For each $A \subseteq X$ and each closed set F with $A \cap F = \emptyset$, there exist $U \in \psi\mathcal{O}(X)$ and a $g\psi$-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

Proof. (i) ⇒ (ii) : Let $x \notin (X \setminus U)$, where $U \in \tau$. Then there exist disjoint $G, V \in \psi\mathcal{O}(X)$ such that $(X \setminus U) \subseteq G$ and $x \in V$. Thus $V \subseteq X \setminus G$ and so $x \in V \subseteq \psi - cl(V) \subseteq X \setminus G \subseteq U$.

(ii) ⇒ (iii) : Let $X \setminus F \in \tau$ with $x \in X \setminus F$. Then by (ii), there exists $U \in \psi\mathcal{O}(X)$ such that $x \in U \subseteq \psi - cl(U) \subseteq (X \setminus F)$. So $F \subseteq X \setminus \psi - cl(U) = V$ (say) $\in \psi\mathcal{O}(X)$ and $U \cap V = \emptyset$. Then $x \notin \psi - cl(V)$. Thus $F \supseteq \cap\{\psi - cl(V) : F \subseteq V \in \psi\mathcal{O}(X)\}$.

(iii) ⇒ (iv) : Let A be a subset of X such that $U \in \tau$ with $A \cap U \neq \emptyset$. Let $x \in A \cap U$. Then $x \notin (X \setminus U)$. Hence by (iii), there exists $W \in \psi\mathcal{O}(X)$ such that $X \setminus U \subseteq W$ and $x \notin \psi - cl(W)$. Put $V = X \setminus \psi - cl(W)$ which is a ψ-open set containing x and hence $A \cap V \neq \emptyset$. Now $V \subseteq X \setminus W$ and so $\psi - cl(V) \subseteq X \setminus W \subseteq U$.

(iv) ⇒ (v) : Let F be a set as in the hypothesis of (v). Then $X \setminus F \in \tau$ with $A \cap (X \setminus F) \neq \emptyset$ and hence by (iv), there exists $V \in \psi\mathcal{O}(X)$ such that $A \cap V \neq \emptyset$ and $\psi - cl(V) \subseteq X \setminus F$. If we put $W = X \setminus \psi - cl(V)$, then $F \subseteq W$ and $W \cap V = \emptyset$.

(v) ⇒ (i) : Let F be a closed set not containing x. Then $F \cap \{x\} = \emptyset$. Thus by (v), there exist $V, W \in \psi\mathcal{O}(X)$ such that $x \in V$, $F \subseteq W$ and $W \cap V = \emptyset$.

(i) ⇒ (vi) : Trivial.

(vi) ⇒ (vii) : Let $A \subseteq X$ and F be a closed set with $A \cap F = \emptyset$. Then for $a \in A$, $a \notin F$, and hence by (vi), there exist $U \in \psi\mathcal{O}(X)$ and a $g\psi$-open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. So $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(vii) ⇒ (i) : Let $x \notin F$, where F is closed in X. Since $\{x\} \cap F = \emptyset$, by (vii) there exist $U \in \psi\mathcal{O}(X)$ and a $g\psi$-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Then $F \subseteq \psi - int(W) = V$ (say) (by Theorem 4) and hence $V \cap U = \emptyset$.

Definition 6. *Let ψ be an operation on a topological space (X, τ). Then (X, τ) is said to be ψ_g-normal if for any two disjoint closed sets A and B there exist two disjoint ψ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Remark 6. *Let ψ be an operation on a space (X, τ). Then every ψ_g-normal space reduces to a normal [5] (resp. pre-normal [16] or p-normal [18], s-normal [11], δp-normal [6], β-normal [12]) space if one takes ψ to be int (resp. $intcl$, $clint$, $intcl_\delta$, $clintcl$).*

Theorem 10. *Let ψ be an operation on a topological space (X, τ). Then the following statements are equivalent:*

(i) X is ψ_g-normal.

(ii) For any pair of disjoint closed sets A and B of X, there exist disjoint $g\psi$-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(iii) For each closed set A and each open set B containing A, there exists a $g\psi$-open set U such that $A \subseteq U \subseteq \psi - cl(U) \subseteq B$.

(iv) For each closed set A and each g-open set B containing A, there exists a ψ-open set U such that $A \subseteq U \subseteq \psi - cl(U) \subseteq int(B)$.

(v) *For each closed set A and each g-open set B containing A, there exists a* $g\psi$*-open set G such that* $A \subseteq G \subseteq \psi - cl(G) \subseteq int(B)$.

(vi) *For each g-closed set A and each open set B containing A, there exists a* ψ*-open set U such that* $cl(A) \subseteq U \subseteq \psi - cl(U) \subseteq B$.

(vii) *For each g-closed set A and each open set B containing A, there exists a* $g\psi$*-open set G such that* $cl(A) \subseteq G \subseteq \psi - cl(G) \subseteq B$.

Proof. (i) ⇒ (ii) : Let A and B be a pair of disjoint closed sets of X. Then by (i) there exist disjoint ψ-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$. Then the rest follows from Remark 4(ii).

(ii) ⇒ (iii) : Let A be a closed set and B be an open set containing A. Then A and $X \setminus B$ are two disjoint closed sets. Hence by (ii) there exist disjoint $g\psi$-open sets U and V of X such that $A \subseteq U$ and $B^c \subseteq V$. Since V is $g\psi$-open and $X \setminus B$ is a closed set with $X \setminus B \subseteq V$, by Theorem 4, $X \setminus B \subseteq \psi - int(V)$. Hence $\psi - cl(X \setminus V) = X \setminus \psi - int(V) \subseteq B$. Thus $A \subseteq U \subseteq \psi - cl(U) \subseteq \psi - cl(X \setminus V) \subseteq B$.

(iii) ⇒ (i) : Let A and B be two disjoint closed subsets of X. Then A is a closed set and B^c is an open set containing A. Thus by (iii), there exists a $g\psi$-open set U such that $A \subseteq U \subseteq \psi - cl(U) \subseteq B^c$. Thus by Theorem 4, $A \subseteq \psi - int(U)$, $B \subseteq X \setminus \psi - cl(U)$, where $\psi - int(U)$ and $X \setminus \psi - cl(U)$ are two disjoint ψ-open sets.

(iv) ⇒ (v) ⇒ (ii) : Obvious.

(vi) ⇒ (vii) ⇒ (iii) : Obvious.

(iii) ⇒ (v) : Let A be a closed set and B be a g-open set c ontaining A. Since B is g-open and A is closed, by Theorem 4.2 of [9] $A \subseteq int(B)$. Thus by (iii), there exists a $g\psi$-open set G such that $A \subseteq G \subseteq \psi - cl(G) \subseteq int(B)$.

(v) ⇒ (vi) : Let A be a g-closed subset of X and B be an open set containing A. Then $cl(A) \subseteq B$, where B is g-open. Thus there exists a $g\psi$-open set G such that $cl(A) \subseteq G \subseteq \psi - cl(G) \subseteq B$. Since G is $g\psi$-open and $cl(A) \subseteq G$, by Theorem 4, $cl(A) \subseteq \psi - int(G)$. Put $U = \psi - int(G)$. Then U is ψ-open and $cl(A) \subseteq U \subseteq \psi - cl(U) = \psi - cl(\psi - int(G)) \subseteq \psi - cl(G) \subseteq B$.

(vi) ⇒ (iv) : Let A be a closed set and B be a g-open set containing A. Then by Theorem 4.2 of [9], $cl(A) = A \subseteq int(B)$, where A is g-closed (as A is closed) and $int(B)$ is open. Thus by (vi), there exists a ψ-open set U such that $cl(A) = A \subseteq U \subseteq \psi - cl(U) \subseteq int(B)$.

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