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Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution

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Abstract. This paper introduces a transmuted modified Weibull distribution as an important competitive model which contains eleven life time distributions as special cases. We generalized the three parameter modified Weibull distribution using the quadratic rank transmutation map studied by Shaw et al. [12] to develop a transmuted modified Weibull distribution. The properties of the transmuted modified Weibull distribution are discussed. Least square estimation is used to evaluate the parameters. Explicit expressions are derived for the quantiles. We derive the moments and examine the order statistics. We propose the method of maximum likelihood for estimating the model parameters and obtain the observed information matrix. This model is capable of modeling of various shapes of aging and failure criteria.
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1. Introduction

The Weibull distribution is a life time probability distribution used in the reliability engineering discipline. We introduce the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. [1]. More recently Aryal et al. [2] introduced the transmuted extreme value distribution. In this article, we introduce and study several mathematical properties of new reliability model referred to as the transmuted modified Weibull distribution. This paper focuses on all the properties of this model and presents the graphical analysis of the subject distribution. This paper presents the relationship between shape parameter and other properties such as non-reliability function, reliability function, instantaneous failure rate, cumulative instantaneous failure rate models. Recently Ammar et al. [10] proposed the modified Weibull distribution.

$$F_{MW}(t) = 1 - \exp(-\alpha t - \eta t^{\beta}). \tag{1}$$

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This article defined the family of transmuted modified Weibull distributions. The main feature of this model is that a transmuted parameter λ is introduced in the subject distribution which provides greater flexibility in the form of new distributions. Using the quadratic rank transmutation map studied by Shaw et al. [12], we develop the four parameter transmuted modified Weibull distribution $TMWD(\alpha,\beta,\eta,\lambda,t)$. We provide a comprehensive description of mathematical properties of the subject distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research. These distributions have several attractive properties for more details we refer to [5, 8, 4, 6, 7, 13].

The article is organized as follows, In Section 2, we present the flexibility of the subject distribution and some special sub-models. In Section 3, we demonstrate the reliability functions of the subject model. A range of mathematical properties are considered in Section 4-5. These include quantile functions, moment estimation, moment generating function and least square estimation. In Section 6, the minimum, maximum and median order statistics models are discussed. We also demonstrate the joint density functions $g(t_1,t_n)$ of the transmuted modified Weibull distribution. In Section 7, we demonstrate the maximum likelihood estimates (MLE_S) of the unknown parameters and the asymptotic confidence intervals of the unknown parameters. However, some of these quantities could not be evaluated in closed form and therefore special cases were used to express them. In Section 8, we fit the TMW model to illustrate its usefulness. In Section 9, concluding remarks are addressed.

2. Transmuted Modified Weibull Distribution

A random variable T is said to have transmuted Modified Weibull probability distribution with parameters $\alpha,\beta,\eta>0$ and $-1\le\lambda\le 1$. It can be used to represent the failure probability density function is given by

$$f_{TMW}(t)=(\alpha+\beta\eta t^{\beta-1})\exp(-\alpha t-\eta t^{\beta})(1-\lambda+2\lambda\exp(-\alpha t-\eta t^{\beta}))\quad t>0 \tag{2}$$

Where β and η are the shape parameters representing the different patterns of the transmuted modified Weibull distribution and are positive, α is a scale parameter representing the characteristic life and is also positive, λ is the transmuted parameter. The restrictions in equation (2) on the values of α,β,η and λ are always the same.

The transmuted modified Weibull distribution is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the transmuted modified Weibull distribution is explained in Table 1. The subject distribution includes as special cases the transmuted modified Exponential (TME), transmuted Linear Failure Rate (TLFR), transmuted Weibull (TW), transmuted Rayleigh (TR) and transmuted Exponential distributions. Figure 1 shows the transmuted modified Weibull distribution that approaches to eleven different lifetime distributions when its parameters are changed. The cumulative distribution function of the transmuted modified Weibull distribution is denoted by $F_{TMW}(t)$ and is defined as

$$F_{TMW}(t)=(1-\exp(-\alpha t-\eta t^{\beta}))(1+\lambda\exp(-\alpha t-\eta t^{\beta})) \tag{3}$$

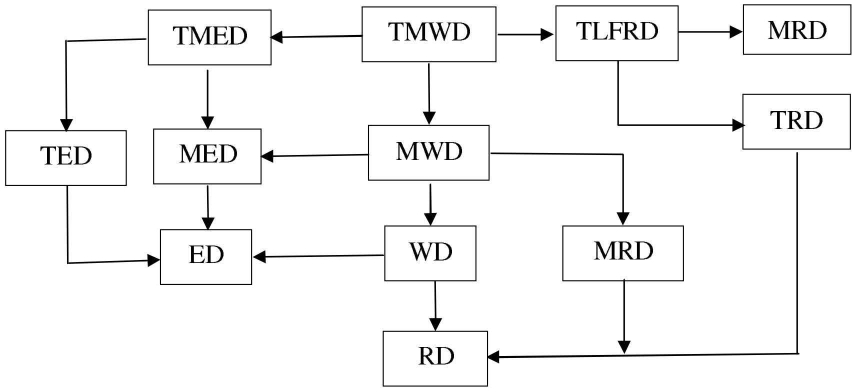


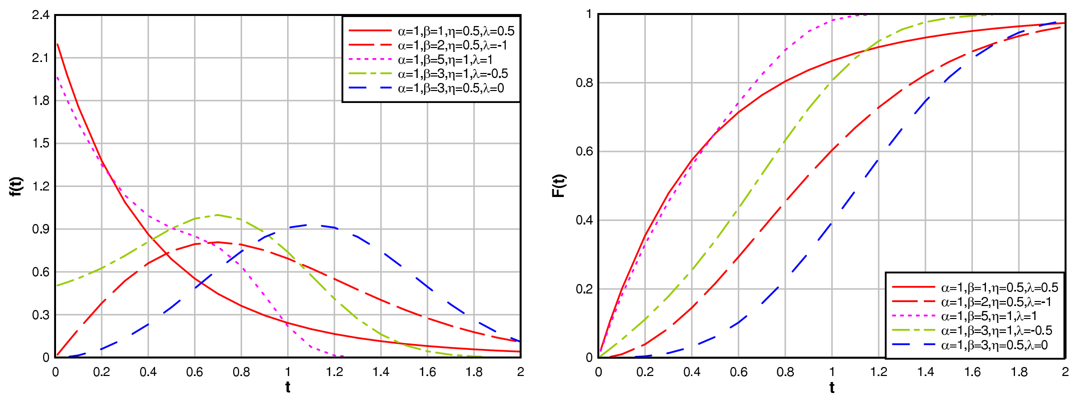
Figure 1: Sub Models of Transmuted Modified Weibull Distribution

Table 1: Modified and Transmuted Modified type distributions: T=Transmuted; M=Modified; W=Weibull; E=Exponential; R=Rayleigh

$S.No$	$Distribution$	TMW	TMW	TMW	TMW
		α	β	η	λ
1	TME	α	1	η	λ
2	TMR	α	2	η	λ
3	MW	α	β	η	0
4	MR	α	2	η	0
5	ME	α	1	η	0
6	TW	0	β	η	λ
7	TR	0	2	η	λ
8	TE	0	1	η	λ
9	W	0	β	η	0
10	R	0	2	η	0
11	E	0	1	η	0

Figure 2a shows the diverse shape of the transmuted modified Weibull PDF with different choice of parameters. The beauty of the subject distribution and its sub models are explained in Table 1.

When the CDF of the $TMWD(\alpha,\beta,\eta,\lambda,t)$ distribution has zero value then it represents no failure components. It is clear from the Figure 2b that two curves intersect at the point of (0.4, 0.574997) and two curves approximately intersect at the point (1.1, 0.885533) the characteristic point for the transmuted modified Weibull CDF.



(a) Probability Density Function (b) Cumulative Distribution Function

Figure 2: Transmuted Modified Weibull PDF & CDF

3. Reliability Analysis

The transmuted modified Weibull distribution can be a useful characterization of life time data analysis. The reliability function (RF) of the transmuted modified Weibull distribution is denoted by $R_{TMW}(t)$ also known as the survivor function and is defined as
$R_{TMW}(t) = 1 - F_{TMW}(t)$

$$R_{TMW}(t) = 1 - (1 - \exp(-\alpha t - \eta t^{\beta}))(1 + \lambda \exp(-\alpha t - \eta t^{\beta})) \tag{4}$$

It is important to note that $R_{TMW}(t) + F_{TMW}(t) = 1$. Figure 3a illustrates the reliability pattern of the transmuted modified Weibull distribution with different choice of parameters. One of the characteristic in reliability analysis is the hazard rate function defined by

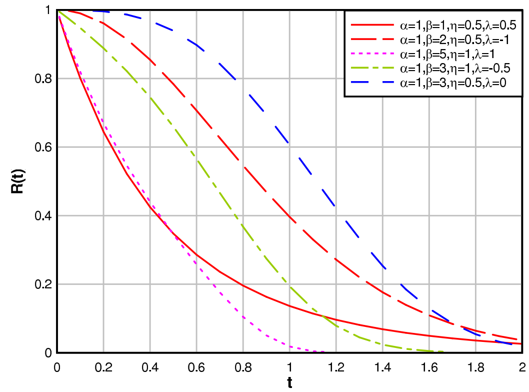
$$h_{TMW}(t) = \frac{f_{TMW}(t)}{1 - F_{TMW}(t)} \tag{5}$$

The hazard function (HF) of the transmuted modified Weibull distribution also known as instantaneous failure rate denoted by $h_{TMW}(t)$ and is defined as $\frac{f_{TMW}(t)}{R_{TMW}(t)}$

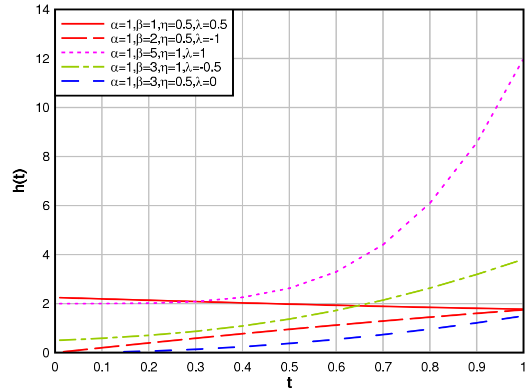
$$h_{TMW}(t) = \frac{(\alpha + \beta \eta t^{\beta-1}) \exp(-\alpha t - \eta t^{\beta})(1 - \lambda + 2\lambda \exp(-\alpha t - \eta t^{\beta}))}{1 - (1 - \exp(-\alpha t - \eta t^{\beta}))(1 + \lambda \exp(-\alpha t - \eta t^{\beta}))} \tag{6}$$

It is important to note that the units for $hTMW(t)$ is the probability of failure per unit of time, distance or cycles.

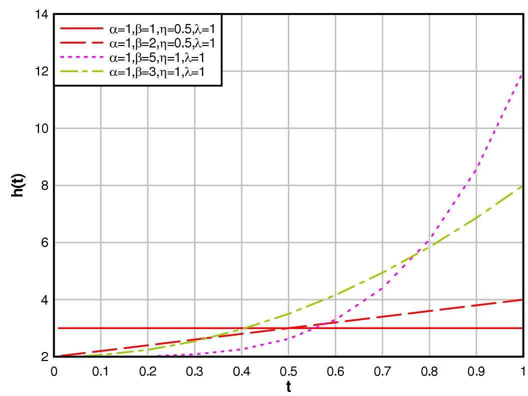
Figures 3b and 3c shows the transmuted modified Weibull instantaneous failure rate patterns. These failure rates are defined with different choices of parameters.



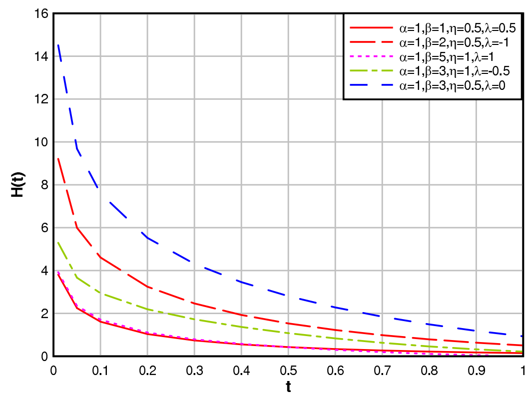
(a) Reliability Function



(b) Hazard Function (wear out period)



(c) Hazard Function (useful life period)



(d) Cumulative Hazard Function

Figure 3: Transmuted Modified Weibull Reliability and Hazard Functions

In both cases when $\beta = 5$ the distribution has the strictly increasing HR. when $\beta = 1$, the HR is steadily increasing in Figure 3b which represents failure life between early failures and wear out periods. When $\beta = 1$, the HR is constant in Figure 3c which represents failure life in useful life period. When $\beta > 1$, the HF is continually increasing which represents failures occurs after the useful life periods and approaches to the wear-out failures. The HR of the TMWD as given in equation (6) becomes identical with the HR of transmuted modified Rayleigh distribution for $\beta = 2$ and for $\beta = 1$ it coincides with the transmuted modified Exponential distribution. So the transmuted modified Weibull distribution is a very flexible reliability model.

The Cumulative hazard function of the transmuted modified Weibull distribution is denoted by $H_{TMW}(t)$ and is defined as

$$H_{TMW}(t) = -ln|(1-\exp(-\alpha t - \eta t^{\beta}))(1+\lambda \exp(-\alpha t - \eta t^{\beta}))| \tag{7}$$

It is important to note that the units for $H_{TMW}(t)$ is the cumulative probability of failure per unit of time, distance or cycles. Figure 3d shows the transmuted modified Weibull cumulative hazard failure rates with different choices of parameters. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

Theorem 1. *The hazard rate function of the transmuted modified Weibull distribution has the following properties*

(i) If $\beta = 1$ *the failure rate is same as the* $TMED(\alpha, \eta, \lambda, t)$

(ii) If $\beta = 2$ *the failure rate is same as the* $TLFRD(\alpha, \eta, \lambda, t)$

(iii) If $\alpha = 0$ *the failure rate is same as the* $TWD(\beta, \eta, \lambda, t)$

(iv) If $\alpha = 0, \beta = 1$ *the failure rate is same as the* $TED(\eta, \lambda, t)$

(v) If $\alpha = 0, \beta = 2$ *the failure rate is same as the* $TRD(\eta, \lambda, t)$

(vi) If $\lambda = 0$ *the failure rate is same as the* $MWD(\alpha, \beta, \eta, t)$

(vii) If $\lambda = 0, \beta = 1$ *the failure rate is same as the* $MED(\alpha, \eta, t)$

(viii) If $\lambda = 0, \beta = 2$ *the failure rate is same as the* $MRD(\alpha, \eta, t)$

Proof. The hazard function (HF) of the transmuted modified Weibull distribution is given in equation (6) has the special cases with different choice of parameters

(i) If $\beta = 1$ the failure rate is same as the $TMED(\alpha, \eta, \lambda, t)$

$$h_{TME}(t) = \frac{(\alpha + \eta)\exp(-\alpha t - \eta t)(1 - \lambda + 2\lambda \exp(-\alpha t - \eta t))}{1 - (1 - \exp(-\alpha t - \eta t))(1 + \lambda \exp(-\alpha t - \eta t))}$$

(ii) If $\beta = 2$ the failure rate is same as the $TLFRD(\alpha, \eta, \lambda, t)$

$$h_{TLFR}(t) = \frac{(\alpha + 2\eta t)\exp(-\alpha t - \eta t^2)(1 - \lambda + 2\lambda \exp(-\alpha t - \eta t^2))}{1 - (1 - \exp(-\alpha t - \eta t^2))(1 + \lambda \exp(-\alpha t - \eta t^2))}$$

(iii) If $\alpha = 0$ the failure rate is same as the $TWD(\beta, \eta, \lambda, t)$

$$h_{TW}(t) = \frac{(\beta \eta t^{\beta-1})\exp(-\eta t^{\beta})(1 - \lambda + 2\lambda \exp(-\eta t^{\beta}))}{1 - (1 - \exp(-\eta t^{\beta}))(1 + \lambda \exp(-\eta t^{\beta}))}$$

(iv) If $\alpha = 0, \beta = 1$ the failure rate is same as the $TED(\eta, \lambda, t)$

$$h_{TE}(t) = \frac{(\eta)\exp(-\eta t)(1 - \lambda + 2\lambda \exp(-\eta t))}{1 - (1 - \exp(-\eta t))(1 + \lambda \exp(-\eta t))}$$

(v) If $\alpha = 0, \beta = 2$ the failure rate is same as the $TRD(\eta, \lambda, t)$

$$h_{TR}(t) = \frac{(2\eta t)\exp(-\eta t^2)(1 - \lambda + 2\lambda \exp(-\eta t^2))}{1 - (1 - \exp(-\eta t^2))(1 + \lambda \exp(-\eta t^2))}$$

(vi) If $\lambda = 0$ the failure rate is same as the $MWD(\alpha, \beta, \eta, t)$

$$h_{MW}(t) = \frac{(\alpha + \beta \eta t^{\beta-1}) \exp(-\alpha t - \eta t^{\beta})}{1 - (1 - \exp(-\alpha t - \eta t^{\beta}))}$$

(vii) If $\lambda = 0, \beta = 1$ the failure rate is same as the $MED(\alpha, \eta, t)$

$$h_{ME}(t) = \frac{(\alpha + \eta) \exp(-\alpha t - \eta t)}{1 - (1 - \exp(-\alpha t - \eta t))}$$

(viii) If $\lambda = 0, \beta = 2$ the failure rate is same as the $MRD(\alpha, \eta, t)$

$$h_{MR}(t) = \frac{(\alpha + 2\eta t) \exp(-\alpha t - \eta t^2)}{1 - (1 - \exp(-\alpha t - \eta t^2))}$$

4. Statistical Properties

This section explain the statistical properties of the $TMWD(\alpha, \beta, \eta, \lambda, t)$

4.1. Quantile and Median

The quantile t_q of the $TMWD(\alpha, \beta, \eta, \lambda, t)$ is the real solution of the following equation

$$\eta t_q^{\beta} + \alpha t_q + \ln\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right) = 0 \tag{8}$$

The above equation (8) has no closed-form solution in t_q , so we have different cases by substituting the parametric values in the above quantile equation. So the derived special cases are

(1) The q-th quantile of the $TLFRD(\alpha, \eta, \lambda, t)$ by substituting $\beta = 2$

$$t_q = \frac{-\alpha + \sqrt{\alpha^2 - 4\eta \ln\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right)}}{2\eta}$$

(2) The q-th quantile of the $TWD(\beta, \eta, \lambda, t)$ by substituting $\alpha = 0$

$$t_q = \left(-\frac{1}{\eta} \ln\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right)\right)^{\frac{1}{\beta}}$$

(3) The q-th quantile of the $TRD(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=2$

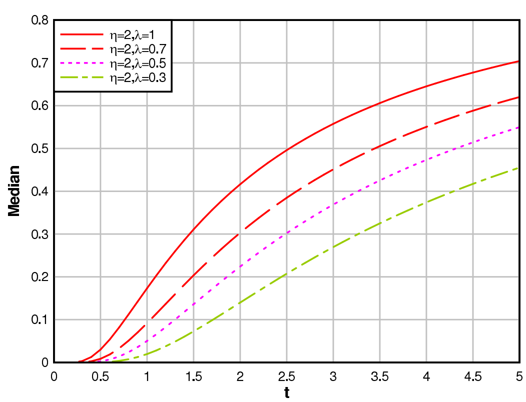
$$t_q=\sqrt{\left(-\frac{1}{\eta}\ln\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)\right)}$$

(4) The q-th quantile of the $TED(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=1$

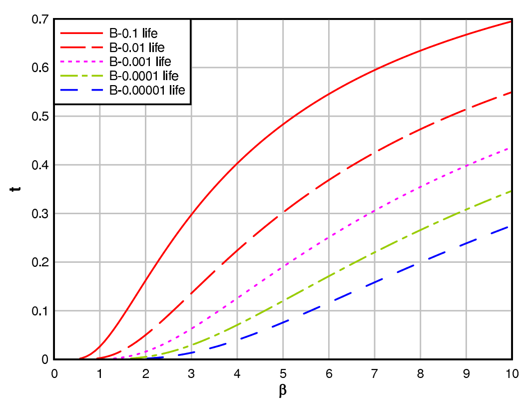
$$t_q=\left(-\frac{1}{\eta}\ln\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)\right)$$

By putting $q=0.5$ in equation (8) we can get the median of $TMWD(\alpha,\beta,\eta,\lambda,t)$.

The median life of the subject distribution is the 50th percentile. In practice, this is the life by which 50 percent of the units will be expected to have failed and so it is the life at which 50 percent of the units would be expected to still survive. Figure 4a shows the transmuted modified Weibull median life with different choice of parameters are $0.1\le\beta\le5$, $\lambda=0.3,0.5,0.7,1$ and the value of $\eta=2$. It is important to note that as the λ increases the pattern of the median life increases. Figure 4b shows the multiple patterns of the subject distribution for B-life with different choice of parameters. Here the $B-0.1$ life shows the maximum values of percentiles life and $B-0.00001$ life shows the minimum values of percentiles life. So as the percentile decreases the pattern of B-lives decreases. All of these B-lives are of increasing patterns. Here as $\beta\to\infty$ then these B-lives are also increasing. The subject distribution for percentile life is applying in those situations where the B-lives are of increasing order.



(a) Median



(b) B-lives

Figure 4: Transmuted Modified Weibull Quantiles

4.2. Random Number Generation

The random number as t of the $TMWD(\alpha,\beta,\eta,\lambda,t)$ is defined by the following equation

$$(1-\exp(-\alpha t-\eta t^{\beta}))(1+\lambda\exp(-\alpha t-\eta t^{\beta}))=\xi,\qquad \text{where}\qquad \xi\sim U(0,1)$$

$$\eta t^{\beta}+\alpha t+\ln\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda\xi}}{2\lambda}\right)=0 \tag{9}$$

The above equation is not in closed form solution in t, Using ξ a random number uniformly distributed from zero to one, we have solved the above equation $F(t)=\xi$ to obtain a random number in t.

4.3. Moments

The following theorem gives the k^{th} moment of the $TMWD(\alpha,\beta,\eta,\lambda,t)$

Theorem 2. *If T has the $TMWD(\alpha,\beta,\eta,\lambda,t)$ with $|\lambda|\geq 1$, then the k^{th} moment of T say μ_k is given as follows*

$$\mu_k=\begin{cases}\sum_{i=0}^{\infty}\frac{(-1)^i\eta^i}{i!}\left[(1-\lambda)\left(\frac{\Gamma(i\beta+k+1)}{\alpha^{i\beta+k}}+\frac{\beta\eta\Gamma(\beta(i+1)+k)}{\alpha^{\beta(i+1)+k}}\right)\right] & \text{if } \alpha,\beta,\eta>0\\ +2\lambda\sum_{i=0}^{\infty}\frac{(-1)^i(2\eta)^i}{i!}\left[\left(\frac{\alpha\Gamma(i\beta+k+1)}{(2\alpha)^{i\beta+k+1}}+\frac{\beta\eta\Gamma(\beta(i+1)+k)}{(2\alpha)^{\beta(i+1)+k}}\right)\right] & \\ \eta^{\frac{-k}{\beta}}\Gamma\left(1+\frac{k}{\beta}\right)\left((1-\lambda)+\lambda 2^{\frac{-k}{\beta}}\right) & \text{if } \alpha=0\\ \alpha^{-k}\Gamma(1+k)((1-\lambda)+\lambda 2^{-k}) & \text{if } \beta=0\end{cases} \tag{10}$$

Based on the above results given in Theorem 2, the coefficient of variation, coefficient of skewness and coefficient of kurtosis of $TMWD$ can be obtained according to the following relation

$$CV_{TMW}=\sqrt{\frac{\mu_2}{\mu_1}-1} \tag{11}$$

$$CS_{TMW}=\frac{\mu_3-3\mu_2\mu_1+2\mu_1^3}{(\mu_2-\mu_1^2)^{\frac{3}{2}}} \tag{12}$$

$$CK_{TMW}=\frac{\mu_4-4\mu_3\mu_1+6\mu_2\mu_1^2-3\mu_1^4}{(\mu_2-\mu_1^2)^2} \tag{13}$$

The relationship between β and the mean life is shown in Figure 5. From this analysis it is clear that as $\beta\rightarrow\infty$ the mean life of the subject distribution is also increasing. So β and the mean life has the positive proportion. The relationship between β and the variance life is shown in Figure 6a. From our calculation it is clear that as $\beta\rightarrow\infty$ the variance life of the subject distribution is decreasing. So β and the variance life has the negative proportion. The relationship between β and the $C.V_{TMW}$ life is shown in Figure 6b.

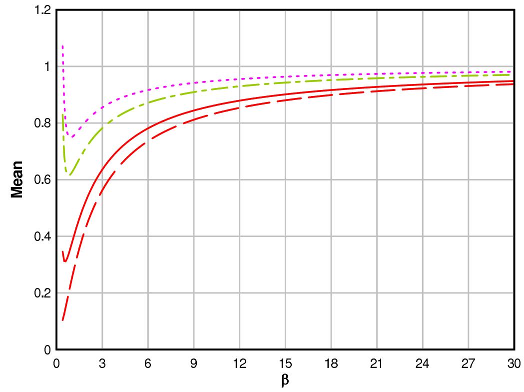
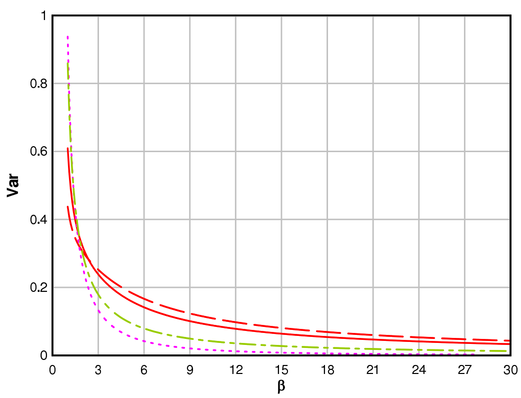
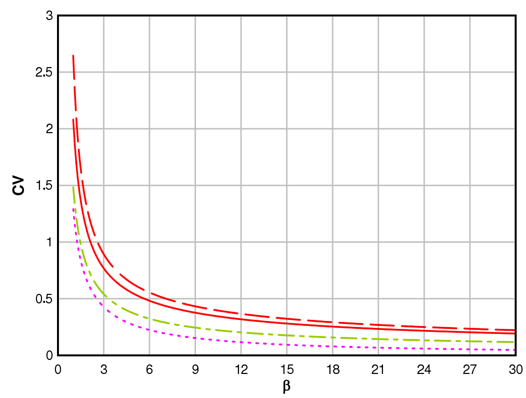


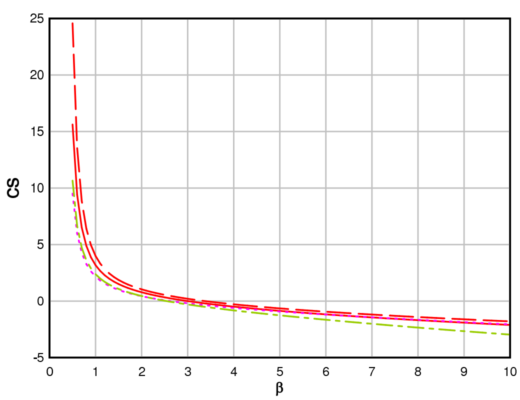
Figure 5: β vs Mean



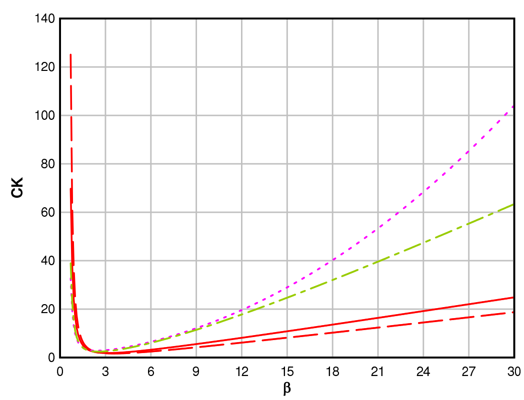
(a) β vs Var_{TMW}



(b) β vs CV_{TMW}



(c) β vs CS_{TMW}



(d) β vs CK_{TMW}

Figure 6: Transmuted Modified Weibull β vs. Coefficients

This relationship shows that it becomes asymptotic decreasing as $\beta \to \infty$. The $C.V_{TMW}$ life is used to measure the consistency of the data. Here $C.S_{TMW}$ is the quantity used to measure the skewness of the distribution. The relationship between β and $C.S_{TMW}$ is shown in Figure 6c. From our calculations it is clear that $C.S_{TMW}$ becomes asymptotically decreasing as $\beta \to \infty$. Here $C.K_{TMW}$ is the quantity used to measure the kurtosis of the distribution. The relationship between β and $C.K_{TMW}$ is shown in Figure 6d. From our calculations it is clear that as $\beta \to \infty$ the value of $C.K_{TMW}$ becomes asymptotically decreasing.

4.4. Moment Generating Function

The following theorem gives the moment generating function (mgf) of $TMWD(\alpha,\beta,\eta,\lambda,t)$

Theorem 3. *If T has the $TMWD(\alpha,\beta,\eta,\lambda,t)$ with $|\lambda| \geq 1$, then the moment generating function of T say $M_x(t)$ is given as follows*

$$M_x(t) = \begin{cases} \sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda) \left(\frac{\alpha\Gamma(i\beta+1)}{(\alpha-t)^{i\beta+1}} + \frac{\beta\eta\Gamma(\beta(i+1))}{(\alpha-t)^{\beta(i+1)}} \right) \right] & \text{if } \alpha,\beta,\eta > 0 \\ +2\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha\Gamma(i\beta+1)}{(2\alpha-t)^{i\beta+1}} + \frac{\beta\eta\Gamma(\beta(i+1)+k)}{(2\alpha-t)^{\beta(i+1)}} \right) \right] & \\ \sum_{i=0}^{\infty} \frac{t^i}{i!} \eta^{\frac{-i}{\beta}} \Gamma\left(1+\frac{i}{\beta}\right) \left((1-\lambda) + \lambda 2^{\frac{-i}{\beta}} \right) & \text{if } \alpha = 0 \\ \alpha \left(\frac{1-\lambda}{\alpha-t} + \frac{2\lambda}{2^{\alpha-t}} \right) & \text{if } \beta = 0 \end{cases} \tag{14}$$

The proof of this theorem is provided in Appendix. Based on the above results given in Theorem 3, the measure of central tendency, measure of dispersion, coefficient of variation, coefficient of skewness and coefficient of kurtosis of $TMWD(\alpha,\beta,\eta,\lambda,t)$ can be obtained according to the above relation in Theorem 2.

5. Least Square Estimation

Let $T_1, T_2, \ldots, T_n$ be a random sample of $TMWD(\alpha,\beta,\eta,\lambda,t)$ transmuted modified Weibull distribution with cdf $F_{TMW}(t)$, and suppose that $T_{(i)}, i = 1,2,\ldots,n$ denote the ordered sample. For sample of size n, we have

$$E(F(T_{(i)})) = \frac{i}{n+1}$$

The least square estimators (LSES) are obtained by minimizing

$$Q(\alpha,\beta,\eta,\lambda) = \sum_{i=0}^{n} \left(F(T_{(i)}) - \frac{i}{n+1} \right)^2 \tag{15}$$

By using (3) and (15) we have the following equation

$$Q(\alpha,\beta,\eta,\lambda) = \sum_{i=0}^{n} \left((1-\exp(-\alpha t - \eta t^{\beta}))(1+\lambda \exp(-\alpha t - \eta t^{\beta})) - \frac{i}{n+1} \right)^2 \tag{16}$$

To minimize equation (16) with respect to α, β, η and λ , we differentiate with respect to these parameters, which leads to the following equations

$$\sum_{i=0}^{n}((1-\exp(-\alpha t-\eta t^{\beta}))\left(1+\lambda\exp(-\alpha t-\eta t^{\beta}))-\frac{i}{n+1}\right)(t)$$
$$\times\,((1+\lambda)\exp(-\alpha t-\eta t^{\beta}))-2\lambda(1-\exp(-\alpha t-\eta t^{\beta}))=0$$

$$\sum_{i=0}^{n}((1-\exp(-\alpha t-\eta t^{\beta}))\left(1+\lambda\exp(-\alpha t-\eta t^{\beta}))-\frac{i}{n+1}\right)\eta t^{\beta}\ln(t)$$
$$\times\,((1+\lambda)\exp(-\alpha t-\eta t^{\beta}))-2\lambda(1-\exp(-\alpha t-\eta t^{\beta}))=0$$

$$\sum_{i=0}^{n}((1-\exp(-\alpha t-\eta t^{\beta}))\left(1+\lambda\exp(-\alpha t-\eta t^{\beta}))-\frac{i}{n+1}\right)t^{\beta}$$
$$\times\,((1+\lambda)\exp(-\alpha t-\eta t^{\beta}))-2\lambda(1-\exp(-\alpha t-\eta t^{\beta}))=0$$

$$\sum_{i=0}^{n}((1-\exp(-\alpha t-\eta t^{\beta}))\left(1+\lambda\exp(-\alpha t-\eta t^{\beta}))-\frac{i}{n+1}\right)$$
$$\times\,(\exp(-\alpha t-\eta t^{\beta}))-2(1-\exp(-\alpha t-\eta t^{\beta}))=0$$

6. Order Statistics

Let $T_{1:n} \leq T_{2:n} \leq \ldots \leq T_{n:n}$ be the order statistics from the continuous distribution, then the pdf of $T_{r:n}$ $1 \leq r \leq n$ is given by

$$f_{r:n}(t)=C_{r:n}(F(t))^{r-1}(1-F(t))^{n-r}f(t), \quad t>0 \tag{17}$$

The joint pdf of $T_{r:n}$ and $T_{s:n}$ $1 \leq r \leq s \leq n$, is given by

$$f_{r:s:n}(t,u)=C_{r:s:n}(F(t))^{r-1}(F(u)-F(t))^{s-r-1}(1-F(t))^{n-s}f(t)f(u), \tag{18}$$

for $0 \leq t \leq u \leq \infty$ and where $C_{r:n}=\frac{n!}{(r-1)!(n-r)!}$ and $C_{r:s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

6.1. Distribution of Minimum and Maximum

Let $t_1, t_2, \ldots t_n$ be n given random variables. Here we define $T_1 = Min(t_1, t_2, \ldots t_n)$ and $T_n = Max(t_1, t_2, \ldots t_n)$ ordered random variables. We find the transmuted modified Weibull distribution for the minimum and maximum observations and its sub models when its parameters are changed [9, 11].

Theorem 4. *Let $t_1, t_2, \dots t_n$ are independently identically distributed ordered random variables from the transmuted modified Weibull distribution having Ist order and nth order probability density function is given by*

$$f_{1:n}(t) = n(1-(1-\exp(-\alpha t-\eta t^{\beta}))(1+\lambda\exp(-\alpha t-\eta t^{\beta})))^{n-1}(\alpha + \beta\eta t^{\beta-1}) \times \exp(-\alpha t-\eta t^{\beta})(1-\lambda+2\lambda\exp(-\alpha t-\eta t^{\beta})) \quad (19)$$

$$f_{n:n}(t) = n((1-\exp(-\alpha t-\eta t^{\beta}))(1+\lambda\exp(-\alpha t-\eta t^{\beta})))^{n-1}(\alpha+\beta\eta t^{\beta-1}) \times \exp(-\alpha t-\eta t^{\beta})(1-\lambda+2\lambda\exp(-\alpha t-\eta t^{\beta})) \quad (20)$$

Proof. For the minimum and maximum order statistic of the four parameters transmuted modified Weibull distribution have different life time distributions when its parameters are changed.

Case A: Minimum order statistic

1. The minimum order statistic of the $TLFRD(\alpha,\eta,\lambda,t)$ by substituting $\beta = 2$

$$f_{1:n}(t) = n(1-(1-\exp(-\alpha t-\eta t^{2}))(1+\lambda\exp(-\alpha t-\eta t^{2})))^{n-1}(\alpha+2\eta t) \times \exp(-\alpha t-\eta t^{2})(1-\lambda+2\lambda\exp(-\alpha t-\eta t^{2}))$$

2. The minimum order statistic of the $TMED(\alpha,\eta,\lambda,t)$ by substituting $\beta = 1$

$$f_{1:n}(t) = n(1-(1-\exp(-\alpha t-\eta t))(1+\lambda\exp(-\alpha t-\eta t)))^{n-1}(\alpha+\eta) \times \exp(-\alpha t-\eta t)(1-\lambda+2\lambda\exp(-\alpha t-\eta t))$$

3. The minimum order statistic of the $TWD(\beta,\eta,\lambda,t)$ by substituting $\alpha = 0$

$$f_{1:n}(t) = n(1-(1-\exp(-\eta t^{\beta}))(1+\lambda\exp(-\eta t^{\beta})))^{n-1}(\beta\eta t^{\beta-1}) \times \exp(-\eta t^{\beta})(1-\lambda+2\lambda\exp(-\eta t^{\beta}))$$

4. The minimum order statistic of the $TRD(\eta,\lambda,t)$ by substituting $\alpha = 0, \beta = 2$

$$f_{1:n}(t) = n(1-(1-\exp(-\eta t^{2}))(1+\lambda\exp(-\eta t^{2})))^{n-1}(2\eta t) \times \exp(-\eta t^{2})(1-\lambda+2\lambda\exp(-\eta t^{2}))$$

5. The minimum order statistic of the $TED(\eta,\lambda,t)$ by substituting $\alpha = 0, \beta = 1$

$$f_{1:n}(t) = n(1-(1-\exp(-\eta t))(1+\lambda\exp(-\eta t)))^{n-1}(\eta) \times \exp(-\eta t)(1-\lambda+2\lambda\exp(-\eta t))$$

6. The minimum order statistic of the $MWD(\alpha,\beta,\eta,t)$ by substituting $\lambda = 0$,

$$f_{1:n}(t) = n(1-(1-\exp(-\alpha t-\eta t^{\beta})))^{n-1}(\alpha+\beta\eta t^{\beta-1})\exp(-\alpha t-\eta t^{\beta})$$

7. The minimum order statistic of the $MRD(\alpha,\eta,t)$ by substituting $\beta=2,\lambda=0$,

$$f_{1:n}(t)=n(1-(1-\exp(-\alpha t-\eta t^2)))^{n-1}(\alpha+2\eta t)\exp(-\alpha t-\eta t^2)$$

8. The minimum order statistic of the $MED(\alpha,\eta,t)$ by substituting $\beta=1,\lambda=0$,

$$f_{1:n}(t)=n(1-(1-\exp(-\alpha t-\eta t)))^{n-1}(\alpha+\eta)\exp(-\alpha t-\eta t)$$

Case B: Maximum order statistic

1. The maximum order statistic of the $TLFRD(\alpha,\eta,\lambda,t)$ by substituting $\beta=2$

$$\begin{aligned}f_{n:n}(t)=&n((1-\exp(-\alpha t-\eta t^2))(1+\lambda\exp(-\alpha t-\eta t^2)))^{n-1}(\alpha+2\eta t)\\&\times\exp(-\alpha t-\eta t^2)(1-\lambda+2\lambda\exp(-\alpha t-\eta t^2))\end{aligned}$$

2. The maximum order statistic of the $TMED(\alpha,\eta,\lambda,t)$ by substituting $\beta=1$

$$\begin{aligned}f_{n:n}(t)=&n((1-\exp(-\alpha t-\eta t))(1+\lambda\exp(-\alpha t-\eta t)))^{n-1}(\alpha+\eta)\\&\times\exp(-\alpha t-\eta t)(1-\lambda+2\lambda\exp(-\alpha t-\eta t))\end{aligned}$$

3. The maximum order statistic of the $TWD(\beta,\eta,\lambda,t)$ by substituting $\alpha=0$

$$\begin{aligned}f_{n:n}(t)=&n((1-\exp(-\eta t^\beta))(1+\lambda\exp(-\eta t^\beta)))^{n-1}(\beta\eta t^{\beta-1})\\&\times\exp(-\eta t^\beta)(1-\lambda+2\lambda\exp(-\eta t^\beta))\end{aligned}$$

4. The maximum order statistic of the $TRD(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=2$

$$\begin{aligned}f_{n:n}(t)=&n((1-\exp(-\eta t^2))(1+\lambda\exp(-\eta t^2)))^{n-1}(2\eta t)\\&\times\exp(-\eta t^2)(1-\lambda+2\lambda\exp(-\eta t^2))\end{aligned}$$

5. The maximum order statistic of the $TED(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=1$

$$f_{n:n}(t)=n((1-\exp(-\eta t))(1+\lambda\exp(-\eta t)))^{n-1}(\eta)\times\exp(-\eta t)(1-\lambda+2\lambda\exp(-\eta t))$$

6. The maximum order statistic of the $MWD(\alpha,\beta,\eta,t)$ by substituting $\lambda=0$,

$$f_{n:n}(t)=n((1-\exp(-\alpha t-\eta t^\beta)))^{n-1}(\alpha+\beta\eta t^{\beta-1})\exp(-\alpha t-\eta t^\beta)$$

7. The maximum order statistic of the $MRD(\alpha,\eta,t)$ by substituting $\beta=2,\lambda=0$,

$$f_{n:n}(t)=n((1-\exp(-\alpha t-\eta t^2)))^{n-1}(\alpha+2\eta t)\exp(-\alpha t-\eta t^2)$$

8. The maximum order statistic of the $MED(\alpha,\eta,t)$ by substituting $\beta=1,\lambda=0$,

$$f_{n:n}(t)=n((1-\exp(-\alpha t-\eta t)))^{n-1}(\alpha+\eta)\exp(-\alpha t-\eta t)$$

Theorem 5. *Let $t_1,t_2,\dots t_n$ are independently identically distributed ordered random variables from the transmuted modified Weibull distribution having median order T_{m+1} probability density function is given by*

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}(F(\tilde{t}))^m(1-F(\tilde{t}))^m f(\tilde{t}),\quad 0\le\tilde{t}\le\infty \tag{21}$$

Proof. Using (21) the median order statistic of the four parameters transmuted modified Weibull distribution is given below

$$\begin{aligned}g(\tilde{t})=&\frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta})))^m(\alpha+\beta\eta\tilde{t}^{\beta-1})\\&\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta})))^m\\&\times\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta})(1-\lambda+2\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{\beta}))\end{aligned} \tag{22}$$

Using (22) we have different life time distributions of median order statistic when its parameters are changed

1. The median order statistic of the $TLFRD(\alpha,\eta,\lambda,t)$ by substituting $\beta=2$

$$\begin{aligned}g(\tilde{t})=&\frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2})))^m(\alpha+2\eta\tilde{t})\\&\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2})))^m\\&\times\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2})(1-\lambda+2\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}^{2}))\end{aligned}$$

2. The median order statistic of the $TMED(\alpha,\eta,\lambda,t)$ by substituting $\beta=1$

$$\begin{aligned}g(\tilde{t})=&\frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t})))^m(\alpha+\eta)\\&\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}))(1+\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t})))^m\\&\times\exp(-\alpha\tilde{t}-\eta\tilde{t})(1-\lambda+2\lambda\exp(-\alpha\tilde{t}-\eta\tilde{t}))\end{aligned}$$

3. The median order statistic of the $TWD(\beta,\eta,\lambda,t)$ by substituting $\alpha=0$

$$\begin{aligned}g(\tilde{t})=&\frac{(2m+1)!}{m!m!}((1-\exp(-\eta\tilde{t}^{\beta}))(1+\lambda\exp(-\eta\tilde{t}^{\beta})))^m(\beta\eta\tilde{t}^{\beta-1})\\&\times(1-(1-\exp(-\eta\tilde{t}^{\beta}))(1+\lambda\exp(-\eta\tilde{t}^{\beta})))^m\\&\times\exp(-\eta\tilde{t}^{\beta})(1-\lambda+2\lambda\exp(-\eta\tilde{t}^{\beta}))\end{aligned}$$

4. The median order statistic of the $TRD(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=2$

$$\begin{aligned} g(\tilde{t}) =& \frac{(2m+1)!}{m!m!}((1-\exp(-\eta\tilde{t}^2))(1+\lambda\exp(-\eta\tilde{t}^2)))^m(2\eta\tilde{t}) \\ &\times(1-(1-\exp(-\eta\tilde{t}^2))(1+\lambda\exp(-\eta\tilde{t}^2)))^m \\ &\times\exp(-\eta\tilde{t}^2)(1-\lambda+2\lambda\exp(-\eta\tilde{t}^2)) \end{aligned}$$

5. The median order statistic of the $TED(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=1$

$$\begin{aligned} g(\tilde{t}) =& \frac{(2m+1)!}{m!m!}((1-\exp(-\eta\tilde{t}))(1+\lambda\exp(-\eta\tilde{t})))^m(\eta\tilde{t}) \\ &\times(1-(1-\exp(-\eta\tilde{t}))(1+\lambda\exp(-\eta\tilde{t})))^m \\ &\times\exp(-\eta\tilde{t})(1-\lambda+2\lambda\exp(-\eta\tilde{t})) \end{aligned}$$

6. The median order statistic of the $MWD(\alpha,\beta,\eta,t)$ by substituting $\lambda=0$,

$$\begin{aligned} g(\tilde{t}) =& \frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^\beta))^m(\alpha+\beta\eta\tilde{t}^{\beta-1}) \\ &\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^\beta))^m\times\exp(-\alpha\tilde{t}-\eta\tilde{t}^\beta) \end{aligned}$$

7. The median order statistic of the $MRD(\alpha,\eta,t)$ by substituting $\beta=2,\lambda=0$,

$$\begin{aligned} g(\tilde{t}) =& \frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^2))^m(\alpha+2\eta\tilde{t}) \\ &\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}^2))^m\times\exp(-\alpha\tilde{t}-\eta\tilde{t}^2) \end{aligned}$$

8. The median order statistic of the $MED(\alpha,\eta,t)$ by substituting $\beta=1,\lambda=0$,

$$\begin{aligned} g(\tilde{t}) =& \frac{(2m+1)!}{m!m!}((1-\exp(-\alpha\tilde{t}-\eta\tilde{t}))^m(\alpha+\eta) \\ &\times(1-(1-\exp(-\alpha\tilde{t}-\eta\tilde{t}))^m\times\exp(-\alpha\tilde{t}-\eta\tilde{t}) \end{aligned}$$

6.2. Joint Distribution of rth Order Statistic T_r and sth Order Statistic T_s

The joint pdf of T_r and T_s with $T_r=t$ and $T_s=u$, $(1\le r\le s\le n)$ in (18) by taking $r=1$ and $s=n$ in (17) the minimum and maximum joint density can be written as

$$g(t_1,t_n)=n(n-1)(F(t_n)-F(t_1))^{n-2}f(t_1)f(t_2) \tag{23}$$

Theorem 6. *Let $t_1,t_2,\ldots,t_n$ be independently identically distributed ordered random variables from the transmuted modified Weibull distribution having joint probability density function using* (2) *and* (3) *in* (23) *is given by*

$$g(t_1,t_n)=n(n-1)(\alpha+\beta\eta t_1^{\beta-1})\exp(-\alpha t_1-\eta t_1^\beta)(1-\lambda+2\lambda\exp(-\alpha t_1-\eta t_1^\beta))$$

$$\times(\alpha+\beta\eta t_n^{\beta-1})\exp(-\alpha t_n-\eta t_n^{\beta})(1-\lambda+2\lambda\exp(-\alpha t_n-\eta t_n^{\beta}))$$
$$\times[(1-\exp(-\alpha t_n-\eta t_n^{\beta}))(1+\lambda\exp(-\alpha t_n-\eta t_n^{\beta}))$$
$$-(1-\exp(-\alpha t_1-\eta t_1^{\beta}))(1+\lambda\exp(-\alpha t_1-\eta t_1^{\beta}))]^{n-2} \quad (24)$$

Proof. Using (24), the joint probability density function $g(t_1,t_n)$ of the subject distribution has different joint distributions when its parameters are changed

1. The min and max order statistic of the $TLFRD(\alpha,\eta,\lambda,t)$ by substituting $\beta=2$

$$g(t_1,t_n)=n(n-1)(\alpha+2\eta t_1)\exp(-\alpha t_1-\eta t_1^2)(1-\lambda+2\lambda\exp(-\alpha t_1-\eta t_1^2))$$
$$\times(\alpha+2\eta t_n)\exp(-\alpha t_n-\eta t_n^2)(1-\lambda+2\lambda\exp(-\alpha t_n-\eta t_n^2))$$
$$\times[(1-\exp(-\alpha t_n-\eta t_n^2))(1+\lambda\exp(-\alpha t_n-\eta t_n^2))$$
$$-(1-\exp(-\alpha t_1-\eta t_1^2))(1+\lambda\exp(-\alpha t_1-\eta t_1^2))]^{n-2}$$

2. The min and max order statistic of the $TMED(\alpha,\eta,\lambda,t)$ by substituting $\beta=1$

$$g(t_1,t_n)=n(n-1)(\alpha+\eta)\exp(-\alpha t_1-\eta t_1)(1-\lambda+2\lambda\exp(-\alpha t_1-\eta t_1))$$
$$\times(\alpha+\eta)\exp(-\alpha t_n-\eta t_n)(1-\lambda+2\lambda\exp(-\alpha t_n-\eta t_n))$$
$$\times[(1-\exp(-\alpha t_n-\eta t_n))(1+\lambda\exp(-\alpha t_n-\eta t_n))$$
$$-(1-\exp(-\alpha t_1-\eta t_1))(1+\lambda\exp(-\alpha t_1-\eta t_1))]^{n-2}$$

3. The min and mix order statistic of the $TWD(\beta,\eta,\lambda,t)$ by substituting $\alpha=0$

$$g(t_1,t_n)=n(n-1)(\beta\eta t_1^{\beta-1})\exp(-\eta t_1^{\beta})(1-\lambda+2\lambda\exp(-\eta t_1^{\beta}))$$
$$\times(\beta\eta t_n^{\beta-1})\exp(-\eta t_n^{\beta})(1-\lambda+2\lambda\exp(-\eta t_n^{\beta}))$$
$$\times[(1-\eta t_n^{\beta}))(1+\lambda\exp(-\eta t_n^{\beta}))$$
$$-(1-\exp(-\eta t_1^{\beta}))(1+\lambda\exp(-\eta t_1^{\beta}))]^{n-2}$$

4. The min and max order statistic of the $TRD(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=2$

$$g(t_1,t_n)=n(n-1)(2\eta t_1)\exp(-\eta t_1^2)(1-\lambda+2\lambda\exp(-\eta t_1^2))$$
$$\times(2\eta t_n)\exp(-\eta t_n^2)(1-\lambda+2\lambda\exp(-\eta t_n^2))\times[(1-\eta t_n^2))(1+\lambda\exp(-\eta t_n^2))$$
$$-(1-\exp(-\eta t_1^2))(1+\lambda\exp(-\eta t_1^2))]^{n-2}$$

5. The min and max order statistic of the $TED(\eta,\lambda,t)$ by substituting $\alpha=0,\beta=1$

$$g(t_1,t_n)=n(n-1)(\eta)\exp(-\eta t_1)(1-\lambda+2\lambda\exp(-\eta t_1))$$
$$\times(\eta)\exp(-\eta t_n)(1-\lambda+2\lambda\exp(-\eta t_n))\times[(1-\eta t_n))(1+\lambda\exp(-\eta t_n))$$
$$-(1-\exp(-\eta t_1))(1+\lambda\exp(-\eta t_1))]^{n-2}$$

6. The min and max order statistic of the $MWD(\alpha,\beta,\eta,t)$ by substituting $\lambda=0$,

$$\begin{aligned}g(t_1,t_n)=&n(n-1)(\beta^2\eta^2 t_1^{\beta-1}t_n^{\beta-1})\exp(-\eta t_1^{\beta}-\eta t_n^{\beta})\\&[(1-\exp(-\eta t_n^{\beta}))-(1-\exp(-\eta t_1^{\beta}))]^{n-2}\end{aligned}$$

7. The min and max order statistic of the $MRD(\alpha,\eta,t)$ by substituting $\beta=2,\lambda=0$,

$$\begin{aligned}g(t_1,t_n)=&n(n-1)(4\eta^2 t_1 t_n)\exp(-\eta t_1^2-\eta t_n^2)\\&[(1-\exp(-\eta t_n^2))-(1-\exp(-\eta t_1^2))]^{n-2}\end{aligned}$$

8. The min and max order statistic of the $MED(\alpha,\eta,t)$ by substituting $\beta=1,\lambda=0$,

$$\begin{aligned}g(t_1,t_n)=&n(n-1)(\eta^2)\exp(-\eta t_1-\eta t_n)\\&[(1-\exp(-\eta t_n))-(1-\exp(-\eta t_1))]^{n-2}\end{aligned}$$

7. Maximum Likelihood Estimation

Consider the random samples $t_1,t_2,\ldots t_n$ consisting of n observations from the transmuted modified Weibull distribution $TMWD(\alpha,\beta,\eta,\lambda,t)$ having probability density function. The likelihood function of equation (2) is given by

$$L(t_1,\ldots t_n,\alpha,\beta,\eta,\lambda,)=\prod(\alpha+\beta\eta t^{\beta-1})\exp(-\alpha t-\eta t^{\beta})(1-\lambda+2\lambda\exp(-\alpha t-\eta t^{\beta})) \tag{25}$$

By accumulation taking logarithm of equation (25), we find the log-likelihood function $Ł=\ln L$, differentiating equation (26) with respect to α,β,η and λ then equating it to zero, we obtain the estimating equations are

$$\begin{aligned}L(t_1,t_2,\ldots t_n,\alpha,\beta,\eta,\lambda,)=&\sum_{i=0}^{n}\ln(\alpha+\beta\eta t_i^{\beta-1})-\alpha\sum_{i=0}^{n}t_i-\eta\sum_{i=0}^{n}t_i^{\beta}\\&+\sum_{i=0}^{n}\ln(1-\lambda+2\lambda\exp(-\alpha t_i-\eta t_i^{\beta}))\end{aligned} \tag{26}$$

$$\frac{\partial Ł}{\partial\alpha}=\sum_{i=0}^{n}\frac{1}{(\alpha+\beta\eta t_i^{\beta-1})}-\sum_{i=0}^{n}t_i-\sum_{i=0}^{n}\frac{2\lambda t_i\exp(-\alpha t_i-\eta t_i^{\beta})}{(1-\lambda+2\lambda\exp(-\alpha t_i-\eta t_i^{\beta}))} \tag{27}$$

$$\frac{\partial Ł}{\partial\beta}=\sum_{i=0}^{n}\frac{t_i^{\beta-1}(1+\beta\ln(t_i))}{(\alpha+\beta\eta t_i^{\beta-1})}-\sum_{i=0}^{n}t_i^{\beta}\ln(t_i)-\sum_{i=0}^{n}\frac{2\lambda\exp(-\alpha t_i-\eta t_i^{\beta})t_i^{\beta}\ln(t_i)}{(1-\lambda+2\lambda\exp(-\alpha t_i-\eta t_i^{\beta}))}=0 \tag{28}$$

$$\frac{\partial Ł}{\partial\eta}=\sum_{i=0}^{n}\frac{\beta t_i^{\beta-1}}{(\alpha+\beta\eta t_i^{\beta-1})}-\sum_{i=0}^{n}t_i^{\beta}-\sum_{i=0}^{n}\frac{2\lambda\exp(-\alpha t_i-\eta t_i^{\beta})t_i^{\beta}}{(1-\lambda+2\lambda\exp(-\alpha t_i-\eta t_i^{\beta}))} \tag{29}$$

$$\frac{\partial Ł}{\partial \lambda} = \sum_{i=0}^{n} \frac{2\exp(-\alpha t_i - \eta t_i^{\beta}) - 1}{(1 - \lambda + 2\lambda \exp(-\alpha t_i - \eta t_i^{\beta}))} = 0 \tag{30}$$

By solving this nonlinear system of equations (27) -(30), these solutions will yield the ML estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\eta}$ and $\hat{\lambda}$. For the four parameters transmuted modified Weibull distribution $TMWD(\alpha,\beta,\eta,\lambda,t)$ pdf all the second order derivatives exist. Thus we have the inverse dispersion matrix is

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\eta} \\ \hat{\lambda} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ \eta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V_{11}} & \hat{V_{12}} & \hat{V_{13}} & \hat{V_{14}} \\ \hat{V_{21}} & \hat{V_{22}} & \hat{V_{23}} & \hat{V_{24}} \\ \hat{V_{31}} & \hat{V_{32}} & \hat{V_{33}} & \hat{V_{34}} \\ \hat{V_{41}} & \hat{V_{42}} & \hat{V_{43}} & \hat{V_{44}} \end{pmatrix} \right]$$

$$V^{-1} = -E \left[\begin{pmatrix} V_{11} & \dots & V_{14} \\ \dots & \dots & \dots \\ V_{41} & \dots & V_{44} \end{pmatrix} = -E \begin{pmatrix} \frac{\partial^2 Ł}{\partial \alpha^2} & \dots & \frac{\partial^2 Ł}{\partial \alpha \partial \lambda} \\ \dots & \dots & \dots \\ \frac{\partial^2 Ł}{\partial \alpha \partial \lambda} & \dots & \frac{\partial^2 Ł}{\partial \lambda^2} \end{pmatrix} \right] \tag{31}$$

Equation (31) is the variance covariance matrix of the $TMWD(\alpha,\beta,\eta,\lambda,t)$

$$V_{11} = \frac{\partial^2 Ł}{\partial \alpha^2} \; V_{12} = \frac{\partial^2 Ł}{\partial \alpha \partial \beta}$$
$$V_{22} = \frac{\partial^2 Ł}{\partial \beta^2} \; V_{13} = \frac{\partial^2 Ł}{\partial \alpha \partial \eta}$$
$$V_{33} = \frac{\partial^2 Ł}{\partial \eta^2} \; V_{14} = \frac{\partial^2 Ł}{\partial \alpha \partial \lambda}$$
$$V_{44} = \frac{\partial^2 Ł}{\partial \lambda^2} \; V_{23} = \frac{\partial^2 Ł}{\partial \beta \partial \eta}$$
$$V_{24} = \frac{\partial^2 Ł}{\partial \beta \partial \lambda} \; V_{34} = \frac{\partial^2 Ł}{\partial \eta \partial \lambda}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variance and co-variances of these ML estimators for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\eta}$ and $\hat{\lambda}$. By using (31), approximately $100(1-\alpha)\%$ confidence intervals for α, β, η and λ can be determined as

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V_{11}}} \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V_{22}}} \qquad \hat{\eta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V_{33}}} \qquad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V_{44}}}$$

Where $Z_{\frac{\alpha}{2}}$ is the upper αth percentile of the standard normal distribution.

8. Numerical Example

In this section we provide a data analysis in order to assess the goodness-of-fit of a model with respect to a maximum flood levels data to see how the new model works in practice. The

data have been obtained from Dumonceaux and Antle [3].
The analysis of least square estimates for the unknown parameters in these distributions namely: Transmuted Modified Weibull (TMW), Transmuted Weibull (TW), Transmuted Modified exponential (TME), Modified Weibull (MW) and Weibull (W) distributions by using the method of least squares are defined. The LSE(s) of the unknown parameter(s), coefficient of determination (R^2) and the corresponding Mean square error for transmuted modified Weibull families of distributions are given in Table 2.

We have provided the parametric estimate of the cumulative distribution function and the fitted functions in Figure 7. It is clear that the transmuted modified Weibull (TMW) distribution provides better fit than the other distributions. In this analysis some estimated values are negative which is not good in the LSE. Another check is to compare the respective coefficients of determination for these regression lines. We have supporting evidence that the coefficient of determination of (TMW) is 0.984714, which is higher than the coefficient of determination of (TME), (TW), (MW) and (W) distributions. Hence the data point from the transmuted modified Weibull (TMW) has better relationship and hence this distribution is good model for life time data.

Table 2: Estimated parameters of TMW, TME, TW, MW and W Distributions

$Parameters$	TMW	TME	TW	MW	W
α	0.035028	-0.45467	-	0.03065	-
β	0.1	1	0.458922	0.05	0.496531
η	0.384726	0.494294	0.254649	0.219493	0.183961
λ	-0.57901	0.380212	-0.32547	-	-
R^2	0.984714	0.953394	0.978145	0.984598	0.977871
MSE	0.003663	0.01051	0.004929	0.003473	0.004713

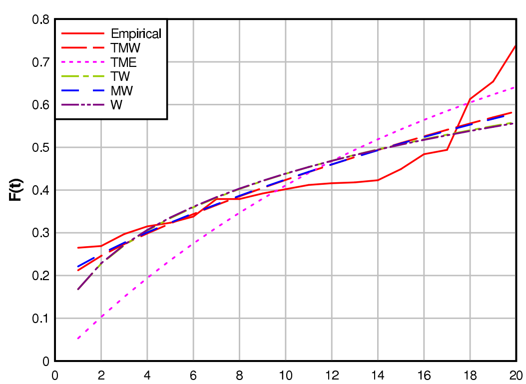


Figure 7: Empirical CDF for the fitted models

9. Concluding Remarks

In this paper we introduce a new generalization of the Weibull distribution called transmuted modified Weibull distribution and presented its theoretical properties. The new distribution is very flexible model that approaches to different life time distributions when its parameters are changed. From the instantaneous failure rate analysis it is observed that it has increasing and decreasing failure rate pattern for life time data. This model has the capability to provide consistent results from all estimation methods.

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Appendix

Proof of Theorem 2

$$\mu_k = \int_0^\infty t^k f(\alpha,\beta,\eta,\lambda,t)dt$$

By substituting (2) into the above relation we have

$$\mu_k = \int_0^\infty t^k(\alpha+\beta\eta t^{\beta-1})\exp(-\alpha t-\eta t^\beta)(1-\lambda+2\lambda\exp(-\alpha t-\eta t^\beta))dt \qquad \text{(A1)}$$

Case A: In this case $\alpha,\beta,\eta>0$ and $|\lambda|\geq 1$. The exponent quantity is $Exp(-\eta t^\beta)$

$$Exp(-\eta t^\beta) = \sum_{i=0}^\infty \frac{(-1)^i\eta^i(t)^{i\beta}}{i!} \qquad \text{(A2)}$$

Here equation (A1) takes the following form

$$\begin{aligned}\mu_k =& \sum_{i=0}^\infty \frac{(-1)^i\eta^i}{i!}\left[(1-\lambda)\left(\frac{\Gamma(i\beta+k+1)}{\alpha^{i\beta+k}}+\frac{\beta\eta\Gamma(\beta(i+1)+k)}{\alpha^{\beta(i+1)+k}}\right)\right]\\ &+2\lambda\sum_{i=0}^\infty \frac{(-1)^i(2\eta)^i}{i!}\left[\left(\frac{\alpha\Gamma(i\beta+k+1)}{2\alpha^{i\beta+k+1}}+\frac{\beta\eta\Gamma(\beta(i+1)+k)}{2\alpha^{\beta(i+1)+k}}\right)\right]\end{aligned} \qquad \text{(A3)}$$

Case B: In this case $\alpha=0$, $\beta,\eta>0$ and $|\lambda|\geq 1$.

$$\mu_k = \int_0^\infty t^k(\beta\eta t^{\beta-1})\exp(-\eta t^\beta)(1-\lambda+2\lambda\exp(-\eta t^\beta))dt$$

By substituting $w=\eta t^{(\beta)}$ then we get

$$\mu_k = \eta^{\frac{-k}{\beta}}\Gamma\left(1+\frac{k}{\beta}\right)\left((1-\lambda)+\lambda 2^{\frac{-k}{\beta}}\right) \qquad \text{(A4)}$$

Case C: In this case $\alpha>0$, $\beta=0$, $\eta=0$ and $|\lambda|\geq 1$.

$$\mu_k = \int_0^\infty t^k\alpha\exp(-\alpha t)(1-\lambda+2\lambda\exp(-\alpha t))dt$$

$$\mu_k = \alpha^{-k}\Gamma(1+k)((1-\lambda)+\lambda 2^{-k}) \qquad \text{(A5)}$$

Proof of Theorem 3

$$M_x(t) = \int_0^\infty e^{tx} f(\alpha, \beta, \eta, \lambda, t) dt$$

By substituting (2) into the above relation we have

$$M_x(t) = \int_0^\infty e^{tx}(\alpha + \beta\eta t^{\beta-1}) \exp(-\alpha t - \eta t^{\beta})(1 - \lambda + 2\lambda \exp(-\alpha t - \eta t^{\beta}))dt \tag{A6}$$

Case A: In this case $\alpha, \beta, \eta > 0$ and $|\lambda| \geq 1$. The exponent quantity is $Exp(-\eta t^{\beta})$. Here equation (A6) takes the following form

$$\begin{aligned} M_x(t) = &\sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda) \left(\frac{\alpha\Gamma(i\beta+1)}{(\alpha-t)^{i\beta+1}} + \frac{\beta\eta\Gamma(\beta(i+1))}{(\alpha-t)^{\beta(i+1)}} \right) \right] \\ &+ 2\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha\Gamma(i\beta+1)}{(2\alpha-t)^{i\beta+1}} + \frac{\beta\eta\Gamma(\beta(i+1)+k)}{(2\alpha-t)^{\beta(i+1)}} \right) \right] \end{aligned} \tag{A7}$$

Case B: In this case $\alpha = 0$, $\beta, \eta > 0$ and $|\lambda| \geq 1$.

$$M_x(t) = \int_0^\infty e^{tx}(\beta\eta t^{\beta-1}) \exp(-\eta t^{\beta})(1 - \lambda + 2\lambda \exp(-\eta t^{\beta}))dt$$

By substituting $w = \eta t^{(\beta)}$ then we get

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \eta^{\frac{-k}{\beta}} \Gamma\left(1 + \frac{i}{\beta}\right) \left((1-\lambda) + \lambda 2^{\frac{-i}{\beta}}\right) \tag{A8}$$