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On the Number of Pairs of Points in a Quadratic Equation with Rational Distance

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Abstract. In this paper is shown a solution to the number of pair of points in a quadratic equation with rational distance, this result have an important impact to solve the open problem "Points on a parabola" [3] proposed in The Center for Discrete Mathematics and Theoretical Computer Science (**DIMACS**), because it's an approach to set down basis in the problem.

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1. Introduction

Define $f : \mathfrak{R} \longrightarrow \mathfrak{R}$ by the function $f(x) = ax^2 + bx + c$; where $x > 0$ and $a, b, c \in \mathfrak{R}$ with $a \neq 0$, then the question is: how many pairs of points, so that the distance between them is a rational number?, although exist some references about quadratic equations and distances[1–3, 6, 7, 9, 10] , there is no information about this specifically question and the solution of this problem allows to start to solve the still open problem "Points on a parabola" [3], that it's about to find the maximum number of points that satisfies the condition to have a rational distance between any of them.

2. Main Result

Theorem 1. *Let $f : \mathfrak{R} \longrightarrow \mathfrak{R}$ by $f(x) = ax^2 + bx + c$; where $x \in Z^+$ and $a, b, c \in \mathfrak{R}$ with $a \neq 0 \Rightarrow$exist infinite pairs of points within the polynomial, where the distance between them is a rational number.*

By reductio ad absurdum, we suppose that the quadratic equation with form $f(x) = ax^2 + bx + c$ have finite pairs of points that satisfies the condition that its distance is a rational number.

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Select two points in the polynomial: $(r, ar^2 + br + c)$ and $(s, as^2 + bs + c)$; where $r, s \in Z^+$
Define the distance function for this case [5]

$$d = \sqrt{(s-r)^2 + (as^2 + bs + c - ar^2 - br - c)^2}$$

Cancel the constants: $c - c = 0$

$$d = \sqrt{(s-r)^2 + (as^2 + bs - ar^2 - br)^2}$$

Factorize by common factor:

$$d = \sqrt{(s-r)^2 + (a(s^2 - r^2) + b(s-r))^2}$$

Factorize by difference of squares:

$$d = \sqrt{(s-r)^2 + (a(s-r)(s+r) + b(s-r))^2}$$

Again, factorize by common factor:

$$d = \sqrt{(s-r)^2 + ((s-r)(a(s+r) + b))^2} \tag{1}$$

Now define $d = \frac{p}{q}$; where $p, q \in Z$ and $q \neq 0$, to find all points where the distance between them is a rational number. Replace d in the equation by (1).

$$\frac{p}{q} = \sqrt{(s-r)^2 + ((s-r)(a(s+r) + b))^2}$$

Squaring both sides:

$$\frac{p^2}{q^2} = (s-r)^2 + ((s-r)(a(s+r) + b))^2$$

Reorganizing the equation:

$$\left(\frac{p}{q}\right)^2 = (s-r)^2 + ((s-r)(a(s+r) + b))^2 \tag{2}$$

Without loss of generality, we will use the equation [5]

$$(5n)^2 = (-3n)^2 + (4n)^2 \tag{3}$$

Where $n \in Q$, to represent that this family of pairs of points is infinite even if it's a subset of all points that satisfies the condition to have a rational distance between them inside the quadratic equation.

Matching the equations (2) and (3)

$$\frac{p}{q} = 5n \tag{4}$$

$$s - r = -3n \tag{5}$$

$$(s - r)(a(s + r) + b) = 4n \tag{6}$$

Replace (5) in (6):

$$-3n(a(s + r) + b) = 4n$$

Divide by $3n$ in both sides:

$$a(s + r) + b = -\frac{4}{3}$$

Deduct b in both sides:

$$a(s + r) = -\frac{4}{3} - b$$

Divide by a in both sides:

$$s + r = -\frac{\frac{4}{3} + b}{a}$$

Reorganizing the equation:

$$\begin{aligned} s + r &= -\frac{\frac{4}{3} + \frac{3b}{3}}{a} \\ &= -\frac{\frac{4-3b}{3}}{a} \\ &= -\frac{4 - 3b}{3a} \end{aligned} \tag{7}$$

Define $j = -\frac{4-3b}{3a}$; where $j \in \Re$ and replace in the equation (7).

$$s + r = j \tag{8}$$

Do (5)+(8)

$$2s = j - 3n$$

Divide by 2 in both sides:

$$s = \frac{j - 3n}{2} \tag{9}$$

Now, do (8)-(5)

$$2r = j + 3n$$

Divide by 2 in both sides:

$$r = \frac{j + 3n}{2} \tag{10}$$

The equations (9) and (10) present some restriction:

$$j > 0 \tag{11}$$

$$r > 0 \tag{12}$$

$$s > 0 \tag{13}$$

Replace (10) in (12)

$$\frac{j+3n}{2} > 0$$

Multiply 2 in both sides:

$$j + 3n > 0$$

Deduct j in both sides:

$$3n > -j$$

Divide by 3 in both sides

$$n > \frac{-j}{3} \tag{14}$$

Now, replace (9) in (13)

$$\frac{j-3n}{2} > 0$$

Multiply 2 in both sides:

$$j - 3n > 0$$

Add $3n$ in both sides:

$$j > 3n$$

Divide by 3 in both sides:

$$\frac{j}{3} > n \tag{15}$$

From (14) and (15)

$$\frac{-j}{3} < n < \frac{j}{3} \tag{16}$$

Lemma 1. *The set $Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$ is countably infinite.*

Proof. Let $s \in Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$, then each s will be written in the (unique) form $\frac{p}{q}$, where $p, q \in Z^+$ and have no common divisor other than 1 [8, 11]. Now, define $f : Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right) \to Z^+ \times Z^+$ by $f\left(\frac{p}{q}\right) = (p, q)$, and let $K = \text{range } f$. For $\frac{p}{q}, \frac{u}{v} \in Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$, we find that $f\left(\frac{p}{q}\right) = f\left(\frac{u}{v}\right) \Rightarrow (p,q) = (u,v) \Rightarrow p = u$ and $q = v \Rightarrow \left(\frac{p}{q}\right) = \left(\frac{u}{v}\right)$, so f is a one-to-one function. Therefore $|Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)| = |K|$, a subset of the countable set $Z^+ \times Z^+$ (by Theorem A3.5 in [4] we know that $Z^+ \times Z^+$ is countable). From [4, Theorem A3.5] it now follows that the set $Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$ is countable.

Define $g : Z^+ \to Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$ by $g(x) = \frac{-j(x-1)}{3(x+1)}$; where $x \in Z^{+1}$ and let $L = \text{range } g$. For $c, d \in Z^+$, we find that if $g(c) = g(d) \Rightarrow c = d$, so g is a one-to-one function. Consequently $|Z^+| = |L|$, then we can notice that L is countably infinite, but $L \in Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$, therefore $Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$ is also infinite. □

Now, it's known that $n \in Q \cap \left(\frac{-j}{3}, \frac{j}{3}\right)$, then r, s could take infinities values, where the distance between them is a rational number, because they depend of n.

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