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On Convexity in Product of Riemannian Manifolds

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Abstract. In this paper, the concept of convexity and starshapedness in the cartesian product of two complete, simple connected smooth Riemannian manifolds without conjugate points are studied in terms of the same concepts in the components of product. We also discuss some of their properties in the cartesian product of Riemannian manifolds without conjugate points. Results obtained in this paper may inspire future research in convex analysis and related optimization fields.

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1. Introduction

Convexity and starshapedness play an important role in optimization theory, convex analysis , Minkowski space and fractal mathematics [4, 7–9, 11–14, 16]. In [15], Pandey introduced an interesting form of a Riemannian metric g and connection $\bigtriangledown$ on the cartesian product $M_1 \times M_2$ of two C^∞ Riemannian manifolds M_1 and M_2. The main result in [2] is that the product $M_1 \times M_2$ of two Riemannian manifolds is free from conjugate (rep.focal) points under the metric and connection given in [15] if and only if both M_1 and M_2 are free from conjugate (resp. focal) points under their own metrics and connections. In [3], there are some interesting results in product of two C^∞ complete, simple connected smooth Riemannian manifolds without conjugate points. The main aim of this paper is studying the convexity and starshapedness in the cartesian product of two complete, simple connected smooth Riemannian manifolds without conjugate points.

2. Preliminaries

In this section, we recall some definitions and properties , which are used further in this paper. We refer to [18] for the standard material on differential geometry.

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Let N be a C^{∞} n-dimensional Riemannian manifold, and T_zN be the tangent space to N at z. Also, assume that $\mu_z(x_1, x_2)$ is a positive inner product on the tangent space T_zN ($x_1, x_2 \in T_zN$), which is given for each point of N. Then, a C^{∞} map $\mu: z \longrightarrow \mu_z$, which assigns a positive inner product μ_z to T_zN for each point z of N is called a Riemannian metric.

The length of a piecewise C^1 curve $\eta: [a_1, a_2] \longrightarrow N$ which is defined as follows:

$$L(\eta) = \int_{a_1}^{a_2} \|\acute{\eta}(x)\| dx.$$

We define $d(z_1, z_2) = inf \{L(\eta): \eta$ is a piecewise C^1 curve joining z_1 to $z_2\}$ for any points $z_1, z_2 \in N$. $\bigtriangledown_X Y, X, Y \in N$ is a unique determined Riemannian connection which called Levi-Civita connection on every Riemannian manifolds. Furthermore, a smooth path η is a geodesic if and only if its tangent vector is a parallel vector field along the path η, i.e , η satisfies the equation $\bigtriangledown_{\acute{\eta}(t)} \acute{\eta}(t) = 0$. Every path η is joining $z_1, z_2 \in N$ where $L(\eta) = d(z_1, z_2)$ is a minimal geodesic.

Finally, assume that (N, η) is a complete n-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$. Let $x_1, x_2 \in N$ and $\eta: [0, 1] \longrightarrow N$ be a geodesic joining the points x_1 and x_2 , which means that $\eta_{x_1, x_2}(0) = x_2$ and $\eta_{x_1, x_2}(1) = x_1$.

Definition 1 (see[10])**.** *A subset B in a Riemannian manifold N is convex if for each pair points $p, q \in N$, there is a unique minimal geodesic segment from p to q and this segment is in B.*

When dealing with a subset $B \subset W$, where W is a C^{∞} complete, simply connected n-dimensional Riemannian manifold without conjugate points, the word "a unique minimal geodesic segment" should be replaced by "the geodesic segment".

The following theorem was proved in [1] :

Theorem 1. *Let $A \subset W$ be an open convex subset. Then,*

(i) The closure of A ($\bar{A}$) is also convex.

(ii) The interior of A ($Int(A)$) is also convex.

The following theorem gives the relationship between global supporting and convexity:

Theorem 2 (see [5])**.** *Let $A \subset W$ be an open subset whose boundary A is a smooth hypersurface of W. Then, A is convex if and only if A is globally supported at each boundary point .*

Definition 2 (see [17])**.** *A subset S in a Riemannian manifold N is starshaped if there is a point $p \in S$ such that for all $q \in S$ there is a unique minimal geodesic segment γ_{pq} from p to q and this segment is in S. In such a case, the set S is starshaped with respect to p or p sees S via S.*

Remark 1. *(i) The subset of S consisting of all points like p is called the kernel of S ($kerS$).*

(ii) In W, a subset S is starshaped if there is a point $p \in S$such that for all $q \in S$, the geodesic segment γ_{pq} joining p and q is contained in S.

Theorem 3 (see [6]). *Let $S \subset W$ be an open starshaped subset with respect to some point $p \in S$, then the closure $\bar{S}$ is also starshaped with respect to the same point p .*

Definition 3 (see [5]). *Let A be an open subset of W whose boundary ∂A is a smooth hypersurface of W. A is called globally supported at $p \in \partial A$ if A is contained in one side of the tangent geodesic hypersurface S_p at $p \in \partial A$.*

Let N_1 and N_2 be two complete Riemannian manifolds with Riemannian metrics g_1 and g_2 and Riemannian connections $\bigtriangledown^1$ and $\bigtriangledown^2$, respectively. A Riemannian metric g on $N_1 \times N_2$ was defined as follows (see[19])

$$g(X,Y) = g((X_1,X_2),(Y_1,Y_2)) = g_1(X_1,Y_1) + g_2(X_2,Y_2)$$

where $X_i, Y_i \in \mathfrak{J}(N_i)$ and $\mathfrak{J}$ denotes the set of all vector fields on $N_i, i = 1,2$. Similarly, a Riemannian connection $\bigtriangledown$ on $N_1 \times N_2$ wwas given by [19]

$$\bigtriangledown_X Y = \bigtriangledown_{(X_1,X_2)}(Y_1,Y_2) = (\bigtriangledown^1_{X_1} Y_1, \bigtriangledown^2_{X_2} Y_2).$$

If $\gamma : [0,\lambda] \to N_1 \times N_2$ is a smooth curve in $N_1 \times N_2$, then the natural projections $\gamma_1 : [0,\lambda] \to N_1$ and $\gamma_2 : [0,\lambda] \to N_2$ of γ on both N_1 and N_2, respectively, are smooth curves. Moreover, γ is a geodesic in $N_1 \times N_2$ if and only if both γ_1 and γ_2 are geodesics in N_1 and N_2 , respectively. Which means $\bigtriangledown_{\dot{\gamma}}\gamma = \bigtriangledown_{(\dot{\gamma_1},\dot{\gamma_2})}(\dot{\gamma_1},\dot{\gamma_2}) = (\bigtriangledown^1_{\dot{\gamma_1}}\gamma_1, \bigtriangledown^2_{\dot{\gamma_2}}\dot{\gamma_2})$, where $\dot{\gamma}$ is the velocity vector field along the curve γ. Consequently, $\bigtriangledown_{\dot{\gamma}}\dot{\gamma} = 0$ if and only if $\bigtriangledown^i_{\dot{\gamma_i}}\dot{\gamma_i} = 0$ for $i = 1,2$, see [3].

Let W_1 and W_2 be C^∞ complete, simply connected Riemannian manifolds without conjugate points,then $W_1 \times W_2$ is also a C^∞ complete, simply connected Riemannian manifold without conjugate points. Notice that $dim(W_1 \times W_2) = dim(W_1) + dim(W_2)$. Consequently, each pair of different points (p_1, p_2) and (q_1, q_2) in $W_1 \times W_2$ are joined by a unique geodesic γ. This segment when naturally projected on W_1 and W_2 yields two geodesic segments $\gamma_i \subset W_i$ joining p_i and $q_i, i = 1,2$ each one is unique in its own manifolds.The natural projection will be denoted by $\eta_i : W_1 \times W_2 \to W_i$ where $\eta_i(p_1, p_2) = p_i$, $i = 1,2$ see[3].

The following propositions were proved in [3] :

Proposition 1. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be subsets of W_1 and W_2. Then, $A_1 \times A_2 \subset W_1 \times W_2$ is convex if and only if both A_1 and A_2 are convex.*

Proposition 2. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be two subsets. Then,*

(i) $A_1 \times A_2 \subset W_1 \times W_2$ *is starshaped if and only if both* A_1 *and* A_2 *are starshaped.*

(ii) $ker(A_1 \times A_2) = (kerA_1) \times (kerA_2)$

3. Convexity in Riemannian Manifolds Product

In this section, we study some properties of convexity in Riemannian manifolds product.

Proposition 3. *The intersection of any number of product convex subsets is convex subset.*

Proof. Let A_1, A_2, B_1, and B_2 be convex subsets, then $A_1 \times A_2$, and $B_1 \times B_2$ are convex subsets. We know that $(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2)$. Since $A_1 \cap B_1$, and $A_2 \cap B_2$ are convex, then $(A_1 \cap B_1) \times (A_2 \cap B_2)$ is also convex. Therefore, the proof is complete. □

Remark 2. *The above proposition is not true in general for the union of subsets of $W_1 \times W_2$.*

Theorem 4. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be an open convex subsets, then the closure $\overline{A_1 \times A_2}$ is also convex.*

Proof. Assume that both $A_1 \subset W_1$ and $A_2 \subset W_2$ are convex subsets. Then, $\bar{A_1} \subset W_1$ and $\bar{A_2} \subset W_2$ are convex, which means that $\bar{A_1} \times \bar{A_2}$ is convex. Then, $\overline{A_1 \times A_2}$ is convex. □

Theorem 5. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be convex subsets, then the interior of $A_1 \times A_2$ $(Int(A_1) \times Int(A_2))$ is also convex.*

Proof. The proof is direct in the light of Theorem 1. □

Notice that if $A_i \subset W_i$, $i = 1,2$ is an open subset such that $\overline{A_1 \times A_2}$ is convex, then A_i, $i = 1,2$ is not necessarily convex. The following example indicates this claim.

Example 1. *Let $A_1 = S^1 = \{(x,y) : x^2 + y^2 \leq 1\} \setminus \{(0,0)\}$ and $A_2 = [0,1] \setminus \{\frac{1}{2}\}$. Clearly $\overline{A_1 \times A_2}$ is a convex subset of $\mathbb{R}^3$ while A_1 and A_2 are non-convex.*

The relationship between global supporting and convexity in the cartesian product of Riemannian manifolds without conjugate points is given in the following theorem:

Theorem 6. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be open subsets whose boundary ∂A_1 and ∂A_2 are smooth hypersurface, respectively. Then, $A = A_1 \times A_2 \subset W_1 \times W_2$ is convex if and only if A_1 and A_2 are globally supported at each boundary point.*

Proof. Let A_1 and A_2 be globally supported at each boundary point, then by using Theorem 2 we have that A_1 and A_2 are convex which implies that $A_1 \times A_2$ is convex. Now, let $A = A_1 \times A_2$ be a convex, then A_1 and A_2 are convex, by using Theorem 2, A_1 and A_2 are globally supported at each boundary point. □

Corollary 1. *Let $A_1 \subset W_1$ and $A_2 \subset W_2$ be open subsets whose boundary ∂A_1 and ∂A_2 are smooth hypersurface of W_1 and W_2, respectively. Then, $A = A_1 \times A_2$ is convex if and only if every maximal tangent geodesic of ∂A_1 and ∂A_2 have an empty intersection with A_1 and A_2.*

4. Starshapedness in Riemannian Manifolds Product

In this section, we aim to establish some properties of starhapedness in Riemannian manifolds product.

Theorem 7. *Let A be a non-empty closed subset of W. If ∂A is starshaped, then $ker(\partial A) \subset kerA$.*

Proof. Let ∂A be starshaped with respect to x, i.e., $x \in ker(\partial A)$. Suppose that x is not in $kerA$, i.e., there is a point $y \in A$ such that $\gamma_{[xy]}$ is not contained in A. Since A is closed, there is a point $y_1 \in \partial A \cap \gamma_{[xy]}$ such that $\gamma_{(xy_1)} \cap A = \phi$. Thus, x does not see y_1 via ∂A. This contradicts the fact that ∂A is starshaped with respect to x. □

Therefore, we can state the following result as well.

Theorem 8. *Let A_1 be a non-empty closed subset of W_1, and A_2 be a non-empty closed subset of W_2. If ∂A_1 and ∂A_2 are starshaped,then $ker(\partial A_1 \times \partial A_2) \subset ker(A_1 \times A_2)$.*

Proof. Since

$$\begin{aligned} ker(\partial A_1 \times \partial A_2) =& ker(\partial A_1) \times ker(\partial A_2) \\ &\subset kerA_1 \times kerA_2 \\ =& ker(A_1 \times A_2). \end{aligned}$$

Then, $ker(\partial A_1 \times \partial A_2) \subset ker(A_1 \times A_2)$. □

Theorem 9. *Let $S = S_1 \times S_2$ be an open starshaped subset with respect to some point $p = (p_1, p_2) \in S$, then the closure $\bar{S} = \overline{S_1 \times S_2}$ is also starshaped with respect to the same point $p = (p_1, p_2)$.*

Proof. Suppose that $S = S_1 \times S_2$ is starshaped with respect to the point $p = (p_1, p_2)$, then S_1 and S_2 are starshaped with respect to the points p_1, and p_2 , respectively. This implies, by using Theorem 3 , to $\bar{S_1}$ and $\bar{S_1}$ are starshaped with respect to the points p_1 and p_2, respectively. Then, $\bar{S_1} \times \bar{S_2} = \overline{S_1 \times S_2}$ is starshaped with respect to the point $p = (p_1, p_2)$. □

Corollary 2. *Let $S = S_1 \times S_2 \subset W_1 \times W_2$ be an open starshaped subset. Then, the kernel of S is contained the kernel of $\bar{S}(ker(S_1 \times S_2) \subset ker(\overline{S_1 \times S_2}))$.*

Proof. Let $p = (p_1, p_2) \in kerS$, i.e., $p \in ker(S_1 \times S_2)$, then S is starshaped with respect to $p = (p_1, p_2)$. By Theorem 9, $\bar{S} = \overline{S_1 \times S_2}$ is also starshaped with respect to $p = (p_1, p_2)$, which implies that $p \in ker\bar{S}$. Hence, $ker(S_1 \times S_2) \subset ker(\overline{S_1 \times S_2})$. □

The relation between $ker(\overline{S_1 \times S_2})$ and $ker(S_1 \times S_2)$ for any arbitrary open starshaped subset $S = S_1 \times S_2 \subset W_1 \times W_2$ is given in the following theorem:

Theorem 10. *Let $S = S_1 \times S_2 \subset W_1 \times W_2$ be an open starshaped subset such that ∂S is a smooth hypersurface. Then, $ker(\overline{S_1 \times S_2}) = \overline{ker(S_1 \times S_2)}$.*

Proof. Since

$$\begin{aligned} ker(\overline{S_1 \times S_2}) =& ker(\bar{S_1} \times \bar{S_2}) \\ =& ker(\bar{S_1}) \times ker(\bar{S_2}) \\ =& \overline{ker(S_1)} \times \overline{ker(S_2)} \\ =& \overline{kerS_1 \times kerS_2} \end{aligned}$$

$$=\overline{ker(S_1 \times S_2)}.$$

Then, $ker(\overline{S_1 \times S_2}) = \overline{ker(S_1 \times S_2)}$. □

Corollary 3. *Let $S = S_1 \times S_2 \subset W_1 \times W_2$ be a closed starshaped such that S is smooth hypersurface. Then, $kerS$ is a closed subset.*

Proof. The proof is direct from Theorem 10 since $S = S_1 \times S_2$ is closed if and only if $S = \bar{S}$ which implies that $kerS = \overline{kerS}$. □

5. Concluding Remarks

(i) All results in this paper are valid in the cartesian product of Euclidean as well as hyperbolic spaces as examples of manifold without conjugate points. Moreover, these results are valid in the case of cartesian product of manifolds without focal points as every manifold without focal points has no conjugate points.

(ii) The results will be more interesting in the cartesian product of general Riemannian manifolds.

(iii) The study has been established in this paper could be considered as a base of a study of other concepts such as local convexity and so on.

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References

[1] M. Beltagy. *Immersions into manifolds without conjugate points*. PhD thesis, University of Durham, England, 1982.

[2] M. Beltagy. On the geometry of the cartesian product of manifolds. *Bulletin of the Calcutta Mathematical Society*, 81(4):315–320, 1989.

[3] M. Beltagy. Convex and starshaped subset in manifolds product. *Communications Faculty of Sciences University of Ankara Series A*, 41(1-2):35–44, 1992.

[4] M. Beltagy. Sufficient conditions for convexity in manifolds without focal points. *Commentationes Mathematicae Universitatis Carolinae*, 34(3):443–449, 1993.

[5] M. Beltagy and A. El-Araby. Convexity in special types of riemannian manifolds. *Bulletin of the Calcutta Mathematical Society*, 94(3):153–162, 2002.

[6] M. Beltagy and A. El-Araby. Starshaped sets in riemannian manifolds without conjugate points. *Far East Journal of Mathematical Sciences*, 6(2):187–196, 2002.

[7] V. Boltyanski, H. Martini, and P.S. Soltan. *Excursions into Combinatorial Geometry*. Springer, Berlin, Germany, 1997.

[8] L. Danzer, B. Grünbaum, and V. Klee. *Helly's theorem and its relatives*. American Mathematical Society, Providence, RI, 1963.

[9] R. J. Dwilewicz. A short history of convexity. *Differential Geometry - Dynamical Systems*, 11:112–129, 2009.

[10] S. Hosseini and M. Pouryayevali. On the metric projection onto prox-regular subsets of riemannian manifolds. *Proceedings of the American Mathematical Society*, 141(1):233–244, 2013.

[11] M. A. Jiménez, G. R. Garzón, and A. R. Lizana. *Optimality Conditions in Vector Optimization*. Bentham Science Publishers, 2010.

[12] A. Kiliçman and W Saleh. A note on starshaped sets in2-dimensional manifolds without conjugate points. *Journal of Function Spaces*, 2014:Article ID 675735, 3 pages, 2014.

[13] H. Martini and K. Swanepoel. Generalized convexity notions and combinatorial geometry. *Congressus Numerantium*, 164:65–93, 2003.

[14] H. Martini and K.J. Swanepoel. The geometry of minkowski spaces-a survey. part ii. *Expositiones mathematicae*, 22(2):93–144, 2004.

[15] H.B. Pandey. Cartesian product of two manifolds. *Indian Journal of Pure and Applied Mathematics*, 12(1):55–60, 1981.

[16] W. Saleh and A. Kiliçman. On generalized s-convex functions on fractal sets. *JP Journal of Geometry and Topology*, 17(1):63–82, 2015.

[17] G. Santhanam. Isoperimetric upper bounds for the first eigenvalue. *Proceedings-Mathematical Sciences*, 122(3):375–384, 2012.

[18] C. Udriste. *Convex functions and optimization methods on Riemannian manifolds*, volume 297. Springer Science & Business Media, 1994.

[19] F. M. Valentine. *Convex Subsets*. McGraw-Hill Series Higher Math., McGraw-Hill, New York, 1964.